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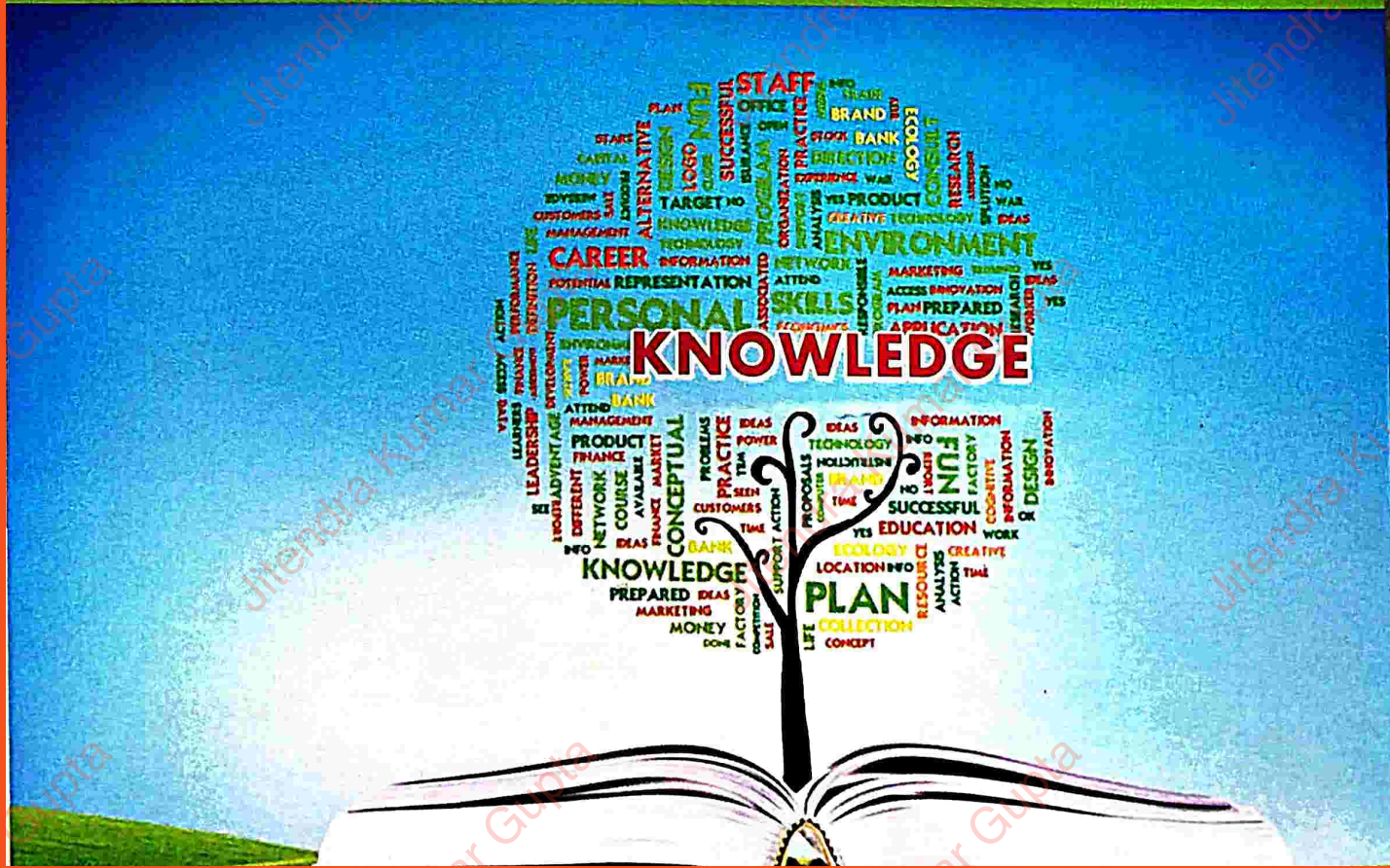

Jitendra Kumar Gupta

Krishna's

TEXT BOOK on

REAL ANALYSIS

(As per U.P. UNIFIED Syllabus)

Jitendra Kumar Gupta

For B.A. and B.Sc. Students of All Colleges Affiliated to Universities in Uttar Pradesh

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Krishna

Authors & Publishers

Jitendra Kumar Gupta

Preface

This book on **REAL ANALYSIS** has been specially written according to the latest **Unified Syllabus** to meet the requirements of the **B.A. and B.Sc. Part-III Students** of all Universities in Uttar Pradesh.

The subject matter has been discussed in such a simple way that the students will find no difficulty to understand it. The proofs of various theorems and examples have been given with minute details. Each chapter of this book contains complete theory and a fairly large number of solved examples. Sufficient problems have also been selected from various university examination papers. At the end of each chapter an exercise containing objective questions has been given.

We have tried our best to keep the book free from misprints. The authors shall be grateful to the readers who point out errors and omissions which, inspite of all care, might have been there.

The authors, in general, hope that the present book will be warmly received by the students and teachers. We shall **indeed** be very thankful to our colleagues for their recommending this book to their students.

The authors wish to express their thanks to **Mr. S.K. Rastogi**, *M.D.*, **Mr. Sugam Rastogi**, *Executive Director*, **Mrs. Kanupriya Rastogi**, *Director* and **entire team** of **KRISHNA Prakashan Media (P) Ltd., Meerut** for bringing out this book in the present nice form.

The authors will feel amply rewarded if the book serves the purpose for which it is meant. Suggestions for the improvement of the book are always welcome.

June, 2013

—Authors

Jitendra Kumar Gupta

Syllabus

Real Analysis

U.P. UNIFIED (*w.e.f.* 2013-14)

THIRD YEAR FIRST PAPER

B.A./B.Sc. Paper-I M.M. : 36 / 75

UNIT -1

Axiomatic study of real numbers, Completeness property in R, Archimedean property, Countable and uncountable sets, Neighbourhood, Interior points, Limit points, Open and closed sets, Derived sets, Dense sets, Perfect sets, Bolzano-Weierstrass theorem.

UNIT-2

Sequences of real numbers, Subsequences, Bounded and monotonic sequences, Convergent sequences, Cauchy's theorems on limit, Cauchy sequence, Cauchy's general principle of convergence, Uniform convergence of sequences and series of functions, Weierstrass M-test, Abel's and Dirichlet's tests.

UNIT -3

Sequential continuity, Boundeness and intermediate value properties of continuous functions, Uniform continuity, Meaning of sign of derivative, Darboux theorem.
Limit and continuity of functions of two variables, Taylor's theorem for functions of two variables, Maxima and minima of functions of three variables, Lagrange's method of undetermined multipliers.

UNIT -4

Riemann integral, Integrability of continuous and monotonic functions, Fundamental theorem of integral calculus, Mean value theorems of integral calculus, Improper integrals and their convergence, Comparison test, μ-test, Abel's test, Dirichlet's test, Integral as a function of a parameter and its differentiability and integrability.

UNIT -5

Definition and examples of metric spaces, Neighbourhoods, Interior points, Limit points, Open and closed sets, Subspaces, Convergent and Cauchy sequences, Completeness, Cantor's intersection theorem.

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REAL ANALYSIS

Chapters

1. Real Number System (Axiomatic Study of Real Numbers)

2. Sequences

3. Uniform Convergence of Sequences and Series of Functions

4. More About Continuity and Darboux Theorem

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5. Limt and Continuity of Functions of Two Variables

6. Maxima and Minima (Of Functions of Several Independent Variables)

7. The Riemann Integral

8. Convergence of Improper Integrals

9. Integral as a Functions of a Parameter

10. Matric Spaces

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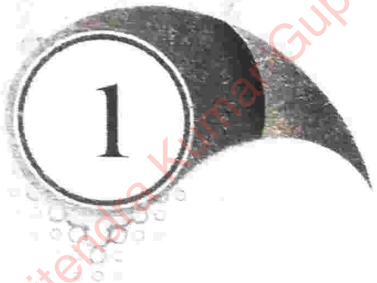
Chapter



Real Number System

(Axiomatic Study of Real Numbers)

1.1 Introduction

The real number system is the foundation on which the whole branch of mathematics known as 'Real Analysis' rests. The beauty of this mathematical system lies in the fact that for many important abstract mathematical systems the structure of real number system serves as a model. Its knowledge is absolutely necessary for any worthwhile student of mathematics. We shall here make a formal description of the real number system with axioms, definitions of some terms and statements of some theorems. There are different ways of introducing the real number system. We shall consider the real numbers as *undefined objects* satisfying certain axioms. These axioms will characterize the real number system.

We assume that the reader is familiar with the real numbers and the binary composition of addition and multiplication possessing a number of properties and the manner of dealing with inequalities involving real numbers. We shall select those properties as axioms concerning the real number system from which all the other properties of the real numbers can be proved. These axioms are divided into

three categories :

(1) *Field axioms*, (2) *Order axioms* and

(3) *Completeness axiom*.

1.2 Field Axioms

Let **R** be the set of real numbers having at least two distinct elements equipped with two fundamental algebraic operations called addition and multiplication and denoted by '+' and '.' respectively. These operations satisfy the following axioms:

A_1. The set **R** is closed with respect to addition *i.e.*, $a + b$ is a unique real number for any two real numbers a and b.

A_2. Addition is associative, *i.e.*,

$$(a + b) + c = a + (b + c) \quad \forall \quad a, b, c \in \mathbf{R}.$$

A_3. Addition is commutative *i.e.*, $a + b = b + a \quad \forall \quad a, b \in \mathbf{R}$.

A_4. There exists an element 0 in **R** such that $0 + a = a \quad \forall \quad a \in \mathbf{R}$.

A_5. To each element a in **R** there exists an element $-a$ in **R** such that $-a + a = 0$.

M_1. The set **R** is closed with respect to multiplication *i.e.*, $a \,.\, b$ is a unique real number for any two real numbers a and b.

M_2. Multiplication is associative *i.e.*,

$$a \,.\, (b \,.\, c) = (a \,.\, b) \,.\, c \quad \forall \quad a, b, c \in \mathbf{R}.$$

M_3. Multiplication is commutative *i.e.*,

$$a \,.\, b = b \,.\, a \quad \forall \quad a, b, c \in \mathbf{R}.$$

M_4. There exists an element namely $1 \neq 0$ in **R** such that

$$1 \,.\, a = a \quad \forall \quad a \in \mathbf{R}.$$

M_5. To each element $a \neq 0$ in **R** there exists an element $1/a$ in **R** such that

$$\frac{1}{a} \,.\, a = 1.$$

The real number $1 / a$ is also denoted by a^{-1}.

Distributive law: Multipliction is distributive with respect to addition *i.e.*,

$$a \,.\, (b + c) = a \,.\, b + a \,.\, c \quad \forall \quad a, b, c \in \mathbf{R}.$$

Because of the above properties the algebraic structure $(\mathbf{R}, +, .)$ is called a field. As a matter of fact any mathematical system safisfying the above exioms is called a field. Thus we may speak of the field **Q** of rational numbers or the field **C** of complex numbers.

The real number 0 is the identity element for addition and the number $-a$ is the additive inverse of the real number a and is usually called the negative of a. The real number 1 is the identity element for multiplication and the real number $1 / a$ or a^{-1} is the multiplicative inverse of the real number a and is usually called the *reciprocal* of a.

1.3 Substraction and Division in R

Definition 1: *The difference between two real numbers a and b is defined by* $a + (-b)$ *and is denoted by* $a - b$.

The operation of finding the difference is called **substraction**.

Definition 2: *The* **quotient** *of a real number a by a real number b* $(b \neq 0)$ *is defined by* $a \cdot b^{-1}$, *and is denoted by*

$$\frac{a}{b} \text{ or } a / b \text{ or } a \div b.$$

The operation of finding quotient is called **division**.

Some Remarks: 1. In general $a - b \neq b - a$ and $a / b \neq b / a$.

2. Division by 0 is not allowed.
3. Though a / b has meaning, b / a may not be defined.

1.4 Some Properties of Real Numbers

We shall state some important consequences of the field properties of real numbers.

1. There exists a *unique* identity element for addition in **R**.
2. There exists a *unique* additive inverse for each element in **R**.
3. $a + b = a + c \Rightarrow b = c$.
4. $a + b = a \Rightarrow b = 0$.
5. $a + b = 0 \Rightarrow b = -a$.
6. $-(-a) = a$.
7. There exists a unique identity element for multiplication in **R**.
8. There exists a unique multiplicative inverse for each non-zero element in **R**.
9. $a \neq 0, a \cdot b = a \cdot c \Rightarrow b = c$.
10. $a \neq 0, a \cdot b = a \Rightarrow b = 1$.
11. $a \neq 0, a \cdot b = 1 \Rightarrow b = 1 / a$.
12. $a \neq 0 \Rightarrow \dfrac{1}{(1/a)} = a$.
13. $a \cdot 0 = 0 \; \forall \; a \in \mathbf{R}$.
14. $a \neq 0, b \neq 0 \Rightarrow a \cdot b \neq 0$.
15. $a \cdot b = 0 \Leftrightarrow a = 0$ or $b = 0$.
16. $a \cdot (-b) = -(a \cdot b)$ and $(-a) \cdot b = -(a \cdot b)$.
17. $(-a) \cdot (-b) = a \cdot b$.

18. $(-1) \cdot a = -a$.

19. $-(a+b) = -a - b$.

20. $\frac{1}{a \cdot b} = \left(\frac{1}{a}\right) \cdot \left(\frac{1}{b}\right), a \neq 0, b \neq 0$.

21. If a and b are any two real numbers, then the equation $x + a = b$ has a unique solution $x = [b + (-a)] = b - a$ in $\mathbf{R}$.

22. If a, b are any real numbers and $a \neq 0$, then the equation $ax = b$ has a unique solution $x = (1/a) \cdot b = b/a$ in $\mathbf{R}$.

1.5 Integral Powers of a Real Number

Let $a \in \mathbf{R}$.

If n is a positive integer, we define $a^n = a \cdot a \cdot a \ldots a$ to n factors. In particular, $a^1 = a, a^2 = a \cdot a, a^3 = a \cdot a \cdot a = a^2 \cdot a$ and so on.

We define $a^0 = 1$.

If n is a positive integer, then $-n$ is a negative integer. For $a \neq 0$, we define $a^{-n} = (a^n)^{-1}$ where $(a^n)^{-1}$ is the multiplicative inverse of a^n in $\mathbf{R}$.

We are free to write $(a^{-1})^n$ or $(a^n)^{-1}$ in place of a^{-n}.

1.6 The Order Axioms

The order relation 'greater than' $(>)$ between pairs of real numbers satisfies the following axioms :

$\mathbf{O_1}$. For any two real numbers a, b one and only one of the following is true :

$$a > b, a = b, b > a.$$

It is known as the *law of trichotomy*.

$\mathbf{O_2}$. For $a, b, c \in \mathbf{R}$, $a > b, b > c \Rightarrow a > c$. It is known as the *law of transitivity*.

$\mathbf{O_3}$. For all real numbers a, b and c, $a > b \Rightarrow a + c > b + c$. It is known as *monotone property for addition*.

$\mathbf{O_4}$. For all real numbers a, b and c, $a > b$ and $c > 0 \Rightarrow ac > bc$. It is known as *monotone property for multiplication*.

Because of these properties the field of real numbers is an *ordered field*.

The system $\mathbf{Q}$ of all rational numbers is an ordered field while the system $\mathbf{C}$ of all complex numbers is a field which is not ordered.

1.7 Some More Definitions

We define some other relations in terms of the relation 'greater than' on the real numbers.

1. The *order relation* '**less than**' ($<$) between the real numbers a and b is defined as

   $a < b$ if $b > a$.
2. A real number a is said to be **greater than or equal** to b ($a \geq b$) if either $a > b$ or $a = b$.
3. A real number a is said to be **less than or equal** to b ($a \leq b$) if either $a < b$ or $a = b$.
4. A real number a is said to be **positive** if $a > 0$.
5. A real number a is said to be negative if $a < 0$.

   The sets of all positive real numbers and all negative real numbers are denoted by $\mathbf{R}^+$ and $\mathbf{R}^-$ respectively.

   Hence $\mathbf{R} = \mathbf{R}^+ \cup \{0\} \cup \mathbf{R}^-$

Some properties of order relation

1. For each real number a, one and only one of the following holds :

   $a > 0, a = 0, -a > 0$.
2. For each real number a, one and only one of the following holds :

   $a < 0, a = 0, -a < 0$.
3. (i) $a \in \mathbf{R}^+ \Leftrightarrow a > 0$ and $a \in \mathbf{R}^- \Leftrightarrow a < 0$.

   (ii) $a \in \mathbf{R}^+, b \in \mathbf{R}^- \Rightarrow a > b$ *i.e.* every positive number is greater than every negative number.
4. $a, b \in \mathbf{R}^+ \Rightarrow a + b \in \mathbf{R}^+$ and $ab \in \mathbf{R}^+$ *i.e.*,

   $a > 0, b > 0 \Rightarrow a + b > 0$ and $ab > 0$.
5. $a, b \in \mathbf{R}^- \Rightarrow a + b \in \mathbf{R}^-$ and $ab \in \mathbf{R}^+$.
6. $a < b$ and $b < c \Rightarrow a < c$.
7. $a < b \Leftrightarrow a + c < b + c$,

   $a < b$ and $c < 0 \Rightarrow ac > bc$.
8. $a < 0 \Leftrightarrow -a > 0, a > 0 \Leftrightarrow -a < 0$.
9. $a > b \Leftrightarrow a - b > 0, a < b \Leftrightarrow a - b < 0$.
10. $a > b \Leftrightarrow -a < -b$.
11. $a > 0 \Leftrightarrow \frac{1}{a} > 0$.

12. (i) $a > b > 0 \Rightarrow \frac{1}{b} > \frac{1}{a} > 0$.

(ii) $0 < a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$.

13. $a \neq 0 \Rightarrow a^2 > 0$. In particular $1 > 0$.

14. $a > b > 0 \Rightarrow a^2 > b^2, a < b < 0 \Rightarrow a^2 > b^2$.

The relations '$\geq$' and '$\leq$' are known as the *weak inequalities* while the relations '$>$' and '$<$' are known as the *strict inequalities*.

1.8 The Extended Real Number System

It is often convenient to extend the system of the real numbers by the addition of two elements ∞ and $-\infty$. The enlarged set is called the set of extended real numbers. We preserve the original order in **R** and define the use of ∞ and $-\infty$ in combination with real numbers by the symbols $+, -, \times, \div, <, >$ as follows :

If a is any real number, then

$$-\infty < a < \infty, a + \infty = \infty + a = -a + \infty = \infty;$$

$$a - \infty = -\infty + a = -\infty - a = -\infty;$$

$$\frac{a}{\infty} = 0; \frac{\infty}{a} = \infty \times a = a \times \infty = \begin{cases} \infty & \text{if } a > 0 \\ -\infty & \text{if } a < 0. \end{cases}$$

Further $\infty \times \infty = (-\infty) \times (-\infty) = \infty + \infty = \infty.$

$$\infty \times (-\infty) + (-\infty) \times \infty = -\infty - \infty = -\infty.$$

The following combinations remain meaningless.

$$\infty - \infty, -\infty + \infty, 0 \times \infty, \infty \times 0, \frac{\infty}{\infty}.$$

1.9 Some Important Subsets of R

1. The set of natural numbers

Inductive set: Definition: *A subset S of **R** is called an inductive set if*

(i) $1 \in S$

and *(ii)* $p \in S \Rightarrow p + 1 \in S$.

For example **R** itself is an inductive set.

The set **N** of natural numbers is the smallest inductive subset of **R**.

By inductive hypothesis, $1 \in N \Rightarrow 1 + 1 \in \mathbf{N}$ and is denoted by 2. Now $2 \in \mathbf{N} \Rightarrow 2 + 1 \in \mathbf{N}$ and is denoted by 3. Continuing in this way, we have

$$\mathbf{N} = \{1, 2, 3, 4, 5, \ldots\}$$

The set of natural numbers is also called the set of positive integers and is also denoted by $\mathbf{Z}^+$.

Principle of Mathematical Induction

A proposition $P(m)$ is true for all $m \in \mathbf{N}$ provided :

(i) $P(1)$ is true *i.e.*, the proposition is true when $m = 1$, and

(ii) $\forall\ k \in \mathbf{N}$, $P(k)$ is true implies $P(k+1)$ is true *i.e.*, if the proposition is true for any $k \in \mathbf{N}$ then it is also true for $k+1 \in \mathbf{N}$.

The natural numbers form a limited system because the algebraic operations on **N** do not satisfy the field axioms $\mathbf{A}_4$, $\mathbf{A}_5$ and $\mathbf{M}_5$.

2. Integers: The subset of **R** containing all the natural numbers, their additive inverses *i.e.*, negatives and the additive identity *i.e.*, 0, is called the set of integers and is denoted by **Z** or **I**.

Thus $\mathbf{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Hence $\mathbf{N} \subset \mathbf{Z} \subset \mathbf{R}$.

The algebraic operations on **Z** do not satisfy the field axiom $\mathbf{M}_5$.

3. Rational numbers: The set of rational numbers contains all the real numbers of the form p/q, where p, q are integers and $q \neq 0$. It is denoted by **Q**. Thus

$$\mathbf{Q} = \{p/q : p, q \in \mathbf{Z}, q \neq 0\}.$$

It should be noted that every integer is a rational number but the converse is not true. For example, 5 is an integer and we can write $5 = \frac{5}{1}$ and so 5 is also a rational number. On the other hand $\frac{2}{3}$ is a rational number but not an integer.

Hence $\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R}$.

The system $(\mathbf{Q}, +, .)$ is an ordered field.

4. Irrational numbers: Any real number which is not rational, is called an irrational number. The set $\mathbf{R} - \mathbf{Q}$ is the set of irrational numbers.

Illustrative Examples

Example 1: *Show that $\sqrt{2}$ is irrational.*

Solution: Suppose there is a rational number p/q, whre $q \neq 0$, whose square is 2. Suppose also that p and q do not have 2 as a common factor because if such a factor exists then it can be cancelled.

Now $(p/q)^2 = 2 \Rightarrow p^2 = 2q^2$.

Since q is an integer, so is q^2 and $2q^2$. Therefore p^2 is an integer divisible by 2.

Now 2 is a prime integer and 2 is a divisor of p^2 *i.e.*, pp.

$\therefore$ 2 must be a divisor of p.

[Note that if a prime number a is a divisor of bc, where b and c are integers, then a must be a divisor of b or a must be a divisor of c].

Since 2 is a factor of p, let $p = 2n \Rightarrow p^2 = 4n^2$.

Now $p^2 = 4n^2 \Rightarrow 2q^2 = 4n^2 \Rightarrow q^2 = 2n^2 \Rightarrow 2$ is a factor of q^2

$\Rightarrow$ 2 is a factor of q.

Thus we conclude that p and q have 2 as a common factor which is contrary to our hypothesis. Therefore there exists no rational number whose square is 2. Hence we cannot put $\sqrt{2}$ in the form p/q. This shows that $\sqrt{2}$ is not rational. Hence $\sqrt{2}$ is irrational.

Example 2: *Prove that no positive integer other than a square number has a square root within the set of rational number.*

Solution: Let m be a positive itneger which is not the square of any itneger. We are to prove that there exists no rational number x such that $x^2 = m$.

Suppose there is a rational number p/q whose square is m. Suppose also that $p$asnd q are positive integers relatively prime to each other because if p and q have any common factor then it can be cancelled.

Now $(p/q)^2 = m \Rightarrow p^2 = mq^2$.

Let a be any prime factor of q.

Then a is a divisor of p^2.

$\Rightarrow a$ is a divisor of $pp \Rightarrow a$ is a divisor of p.

Thus a is a prime number which is a common factor of p and q and so p and q are not relatively prime which contradicts our assumption. Therefore no prime number can be a factor of q and so q must be equal to 1. Then

$$p^2 = m$$

i.e., m is the square of the integer p. This contradicts the hypothesis that m is not the square of any itneger. Hence there exists no rational number p/q whose square is m.

Example 3: *Show that $\sqrt{8}$ is not a rational number.*

Solution: If possible, let $\sqrt{8}$ be the rational number p/q, whre $q \neq 0$ and p, q are positive integers prime to each other.

Now $2 < \sqrt{8} < 3$.

$\therefore$ $2 < p/q < 3 \Rightarrow 2q < p < 3q \Rightarrow 0 < p - 2q < q$.

Thus $p - 2q$ is a positive integer less than q,so that $(\sqrt{8})(p - 2q)$ *i.e.*,$(p/q)(p - 2q)$ is not an integer.

But $$(\sqrt{8})(p - 2q) = \frac{p}{q}(p - 2q) = \frac{p^2}{q} - 2p = \frac{p^2}{q^2} \cdot q - 2p = 8q - 2p,$$

which is an integer. Thus we arrive at a contradiction.

Hence $\sqrt{8}$ is not a rational number.

1.10 Intervals

Interval: Definition: *A subset S of $\mathbf{R}$ is called an interval if* $a, b \in S, x \in \mathbf{R}, a < x < b \Rightarrow x \in S$.

If a and b are any two real numbers such that $a \le b$, then the set $\{x \in \mathbf{R} : a < x < b\}$ is called an **open interval** and is denoted by $]\, a, b\, [$ or by (a, b). Here both the end points a and b do not belong to the interval. If to the open interval $]\, a, b\, [$, we add the real numbers a and b, we get the **closed interval** $[a, b]$.

Thus $[a, b] = \{\, x \in \mathbf{R} : a \le x \le b\}$. It should be noted that

$$[a, b] = \{a\} \cup]\, a, b\, [\cup \{b\}.$$

If $a = b$, then $]\, a, a\, [= \varnothing$ and $[a, a] = \{a\}$.

In the case of the closed interval $[a, b]$ both the end points a and b belong to the interval. Out of the two end point a and b, a is known as the **left end point** whreas b is known as the **right end point**.

The sets $\{\, x \in \mathbf{R} : a < x \le b\}$ and $\{\, x \in \mathbf{R} : a \le x < b\}$ are called the **semi-open** or **semi-closed** intervals and are denoted respectively by the symbols $]\, a, b\,]$ and $[a, b\, [$ or by the symbols $(a, b]$ and $[a, b)$. The interval $[a, b\, [$ is called **a right half open interval** and the interval $]\, a, b\,]$ is called *a* **left half open interval**.

The sets $]\, a, \infty\, [= \{\, x \in \mathbf{R} : x > a\}$

and $]-\infty, a\, [= \{\, x \in \mathbf{R} : x < a\}$

are called **open rays**.

The sets $[a, \infty\, [= \{\, x \in \mathbf{R} : x \ge a\}$

and $]-\infty, a] = \{\, x \in \mathbf{R} : x \le a\}$

are called **closed rays**.

Since $\mathbf{R} = \{\, x : x \le 0\} \cup \{\, x : x \ge 0\}$

$=]-\infty, 0] \cup [0, \infty[\,,$

therefore we can write $\mathbf{R}$ in the form of an open interval as

$$\mathbf{R} =]-\infty, \infty\, [.$$

Length of an interval: For each interval whose end points are any real numbers a and b such that $a \le b$, the length of the interval is $b - a$. Obviously the length of each of the intervals $]\, a, b\, [\,, [\, a, b\,]\,, [a, b\, [$ and $]\, a, b\,]$ is $b - a$. These intervals are called **finite intervals** because the length of each of them is finite. The intervals $]\, a, \infty[\,, [a, \infty[\,,]-\infty, a\, [\,,]-\infty, a]$ and $]-\infty, \infty\, [$ are called **infinite intervals** because the length of each of them is infinite.

1.11 Finite and Infinite Subsets of R

A subset S of **R** *is said to be finite if either it is empty or there exists a one-to-one mapping from the set* $\{1, 2, \ldots, n\}$ *onto the set S for some natural number n.*

A subset S of **R** *which is not finite is called* **infinite**.

Illustrations: 1. The sets **N**, **Z**, **Q**, **R** are infinite.

2. The set $\{e, \pi, \sqrt{2}\}$ is a finite set because from the set $\{1, 2, 3\}$ onto the set $\{e, \pi, \sqrt{2}\}$ many one-to-one mappings exist. One such mapping is $1 \to e$, $2 \to \pi, 3 \to \sqrt{2}$.
3. The set $\left\{1, \frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^{100}}\right\}$ is a finite set.
4. The set of all primes less than 10^{100} is a finite set.
5. If $a, b \in \mathbf{R}$ and $a < b$, then all the intervals $]a, b[, [a, b], [a, b[,]a, b]$, $]-\infty, a[,]-\infty, a],]a, \infty[, [a, \infty[$ are infinite subsets of **R**. It should be noted that the interval $]a, b[$ is called a finite interval in the sense that its length $b - a$ is finite though as a subset of **R** it is an infinite subset of **R**.

1.12 Absolute Value (Modulus of a Real Number)

Definition: *If* $x \in \mathbf{R}$, *then its absolute value, denoted by* $|x|$, *is defined as follows :*

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

It should be observed that $|x|$ is defined for every $x \in \mathbf{R}$. Also $a = b \Rightarrow |a| = |b|$ but $|a| = |b|$ does not necessarily imply that $a = b$. For example, $|-5| = |5| = 5$ but $-5 \neq 5$.

Also it is easy to see that $|x| = 0 \Leftrightarrow x = 0$.

From the definition of $|x|$, it is obvious that $|x| = -x$ if $x \leq 0$. For, if $x = 0$, we have $|x| = 0$ and $-0 = 0$.

Theorem 1: *For every* $x \in \mathbf{R}$,

(*i*) $|x| \geq 0$.
(*ii*) $|x| = max.\{-x, x\}$.
(*iii*) $|x| \geq x$.
(*iv*) $x \geq -|x|$.
(*v*) $|x| = |-x|$.
(*vi*) $|x|^2 = x^2 = |-x|^2$.

Proof: (i) If $x \in \mathbf{R}$, then by the law of trichotomy, one and exactly one of the following is true :

$$x > 0, x = 0, x < 0.$$

If $x \geq 0$, then by the definition of $|x|$, we have

$$|x| = x \geq 0. \qquad \ldots(1)$$

Again if $x < 0$, then by the definition of $|x|$, we have

$$|x| = -x > 0. \qquad ...(2)$$

Note that if $x < 0$ *i.e.*, x is – ive, then $-x > 0$ *i.e.*, $-x$ is + ive.

From (1) and (2), we conclude that

$$|x| \geq 0 \quad \forall\, x \in \mathbf{R}.$$

(ii) If $x \geq 0$, then by the definition of $|x|$, we have

$$|x| = x, \text{ and } x \geq -x.$$

Again if $x < 0$, then by the definition of $|x|$, we have

$$|x| = -x, \text{ and } -x > x.$$

Thus in either case, $|x|$ is the greater of the two numbers x and $-x$.

Hence $|x| = \max.\{x, -x\} \quad \forall\, x \in \mathbf{R}.$

(iii) If $x \geq 0$, then by the definition of $|x|$, we have

$$|x| = x \geq x.$$

Again if $x < 0$, then by the definition of $|x|$, we have

$|x| = -x > x$. [Note that if x is – ive, then $-x$ is + ive and so $-x > x$].

Thus in either case, we have $|x| \geq x$.

Hence $|x| \geq x \quad \forall\ x \in \mathbf{R}.$

(iv) If $x \geq 0$, then by the definition of $|x|$, we have

$$|x| = x. \text{ But } |x| \geq -|x| \text{ and so } x \geq -|x|.$$

Again if $x < 0$, then $|x| = -x$ and so

$$-|x| = -(-x) \text{ i.e. } x = -|x|.$$

Thus in either case, we have $x \geq -|x|$.

(v) We have $|x| = \max.\{x, -x\}$, as proved in (ii).

$\therefore \quad |-x| = \max.\{-x, -(-x)\} = \max.\{-x, x\} = |x|.$

The students should prove this result by direct application of the definition of $|x|$.

(vi) If $x \geq 0$, then $|x| = x$ and so $|x|^2 = x^2$.

Again if $x < 0$, then $|x| = -x$ and so $|x|^2 = (-x)^2 = x^2$.

Thus in either case, we have $|x|^2 = x^2$.

$\therefore \quad |x|^2 = x^2 \quad \forall\, x \in \mathbf{R}.$

Applying this result, we have

$$|-x|^2 = (-x)^2 = x^2.$$

Hence $|x|^2 = x^2 = |-x|^2 \quad \forall\, x \in \mathbf{R}.$

Theorem 2: *For all $x, y \in \mathbf{R}$ prove that*

(i) $|xy| = |x| \cdot |y|$. *(ii)* $|x + y| \leq |x| + |y|$. **(The triangle inequality)**

(iii) $|x - y| \geq ||x| - |y||$.

Proof: (i) We have $|xy|^2 = (xy)^2 = x^2 y^2 = |x|^2 . |y|^2 = (|x| . |y|)^2$.

$\therefore \quad |xy| = \pm (|x| . |y|)$.

But $|xy|$ and $|x| . |y|$ are both non-negative.

So rejecting the – ive sign which is inadmissible, we have

$$|xy| = |x| . |y| .$$

(ii) We have $|x + y|^2 = (x + y)^2 = x^2 + y^2 + 2xy$

$$\leq x^2 + y^2 + 2|xy| \qquad [\because |xy| \geq xy]$$

$$= |x|^2 + |y|^2 + 2|x| . |y| \qquad [\because |xy| = |x| . |y|]$$

$$= (|x| + |y|)^2 .$$

Thus $|x + y|^2 \leq (|x| + |y|)^2$. ...(1)

Since $|x + y|$ and $|x| + |y|$ are both non-negative, therefore from (1), we have

$$|x + y| \leq |x| + |y| .$$

(iii) We have $|x| = |(x - y) + y|$

$\leq |x - y| + |y|$, by the triangle inequality.

$\therefore \quad |x| - |y| \leq |x - y|$ . ...(1)

Again $|y| = |(y - x) + x|$

$\leq |y - x| + |x|$, by the triangle inequality.

$\therefore \quad |y| - |x| \leq |y - x|$

or $\quad -(|x| - |y|) \leq |x - y|$, since $|y - x| = |x - y|$. ...(2)

From (1) and (2), we see that

$$|x - y| \geq \max. \{(|x| - |y|), -(|x| - |y|)\}$$

i.e., $\quad |x - y| \geq ||x| - |y||$.

Theorem 3: *(i) If x, ε be real numbers and $\varepsilon > 0$, then*

$$|x| < \varepsilon \Leftrightarrow -\varepsilon < x < \varepsilon .$$

(ii) If x, y, ε be real numbers and $\varepsilon > 0$, then

$$|x - y| < \varepsilon \Leftrightarrow y - \varepsilon < x < y + \varepsilon .$$

Proof: (i) We have $|x| < \varepsilon \Leftrightarrow \max. \{x, -x\} < \varepsilon$

$\Leftrightarrow -x < \varepsilon$ and $x < \varepsilon$

$\Leftrightarrow x > -\varepsilon$ and $x < \varepsilon$

$\Leftrightarrow -\varepsilon < x < \varepsilon$.

(ii) We have $|x - y| < \varepsilon \Leftrightarrow \max. \{(x - y), -(x - y)\} < \varepsilon$

$\Leftrightarrow -(x - y) < \varepsilon$ and $x - y < \varepsilon$

$\Leftrightarrow x - y > -\varepsilon$ and $x - y < \varepsilon$

$\Leftrightarrow x > y - \varepsilon$ and $x < y + \varepsilon$

$\Leftrightarrow y - \varepsilon < x < y + \varepsilon$.

Illustrative Examples

Example 4: *For any real numbers x and y, show that*

$$|x - y| \leq |x| + |y|.$$

Solution: We have $|x - y| = |x + (-y)|$

$\leq |x| + |-y|$, by the triangle inequality

$= |x| + |y|$. $[\because |-y| = |y|]$

$\therefore \quad |x - y| \leq |x| + |y| \quad \forall\ x, y \in \mathbf{R}.$

Example 5: *Prove that* $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, $y \neq 0$.

Solution: We have $\left|\frac{x}{y}\right|^2 = \left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2}$

$$= \frac{|x|^2}{|y|^2}$$

$$= \left(\frac{|x|}{|y|}\right)^2.$$

$\therefore \quad \left|\frac{x}{y}\right| = \pm \frac{|x|}{|y|}$.

But $\left|\frac{x}{y}\right|$ and $\frac{|x|}{|y|}$ are both non-negative.

Therefore rejecting the – ive sign which is inadmissible, we have

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}.$$

Example 6: *If $k \in \mathbf{R}^+$ and $|x - y| < k\varepsilon\ \forall\ \varepsilon > 0$, then prove that $x = y$.*

Solution: If possible, let $x \neq y$. Then $|x - y| > 0$ and so taking $\varepsilon = \frac{1}{2k}|x - y| > 0$

and applying the given result that

$$|x - y| < k\varepsilon\ \forall\ \varepsilon > 0,$$

we have $|x - y| < k \cdot \frac{1}{2k}|x - y|$

i.e., $|x - y| < \frac{1}{2}|x - y|$

which is not possible.

Hence our initial assumption is wrong and we must have $x = y$.

Comprehensive Exercise 1

1. For any real numbers x and y, show that
$$|x + y| \geq ||x| - |y||.$$
2. If x, y, z are any real numbers, then prove that
$$|x + y + z| \leq |x| + |y| + |z|.$$
3. If $x_1, x_2, \ldots, x_n$ be any real numbers, then prove that
   (i) $|x_1 + x_2 + \ldots + x_n| \leq |x_1| + |x_2| + \ldots + |x_n|$.
   (ii) $|x_1 x_2 \ldots x_n| = |x_1| \cdot |x_2| \ldots . |x_n|$.
4. If x be any real number not equal to zero, then prove that
$$\left|\frac{1}{x}\right| = \frac{1}{|x|}.$$
5. For any $a, b \in \mathbf{R}$, prove that
   (i) $|a - b| = 0 \Leftrightarrow a = b$. (ii) $|a - b| = |b - a|$.
6. Prove that $a < x < b \Leftrightarrow \left|x - \frac{1}{2}(a + b)\right| < \frac{1}{2}(b - a)$; $a, x, b \in \mathbf{R}$.
7. Show that there is no rational number whose square is 3.
8. If $0 < \theta < 1, |x| < 1$, show that $\left|\frac{x(1 - \theta)}{1 + \theta x}\right| < 1$.

1.13 Bounded and Unbounded Subsets of Real Numbers

1. Aggregate: Definition: *A non-empty subset S of* $\mathbf{R}$ *is called an aggregate.*
For example, the set of all positive integers $\mathbf{Z}^+$ is an aggregate. If $S = \{1, 2, 3\}$, then S is an aggregate. The null set $\varnothing$ is not an aggregate.

Boundedness of subsets of R

2. Upper bound of a subset of R.

Definition: *Let S be a subset of real numbers. If there exists a real number u, such that*
$$x \leq u \quad \forall \quad x \in S,$$
then u is called an **upper bound** *of S.*
If there exists an upper bound for a set S, then the set S is said to be **bounded above.**
It is not necessary that a set S should be bounded above. If there exists no real number u such that $x \leq u \ \forall x \in S$, then the set S is said to be **not bounded above** or **unbounded above**. Thus a set S is unbounded above if however large a real number u we may take, there exists at least one $x \in S$ such that $x > u$.

For example, the set of positive integers $Z^+ = \{1, 2, 3, 4, \ldots\}$ is not bounded above while the set of negative integers

$$Z^- = \{\ldots, -3, -2, -1\}$$

is bounded above and -1 is an upper bound. The set of negative real numbers is also bounded above, 0 being an upper bound.

If u is an upper bound of a set S, then every real number greater than u is also an upper bound of S. Thus if a set S is bounded above, then the set of all such numbers that are upper bounds of S is infinite.

3. Least upper bound or supremum

Definition: *If s is an upper bound of a subset S of* **R** *and any real number less than s is not an upper bound of S, then s is called the* **least upper bound (l.u.b)** or **supremum (sup)** *of S.*

Thus if a set S is bounded above and if the set of all upper bounds of S has a smallest member, say s, then s is called the least upper bound or the supremum of the set S.

If a real number s is the supremum of a subset S of real numbers, then for every $\varepsilon > 0$, there exists a real number $x \in S$ such that

$$s - \varepsilon < x \leq s.$$

Example 7: *Prove that the greatest member of a set, if it exists, is the supremum (g.l.b.) of the set.*

Solution: Let g be the greatest member of the set S. Then $x \leq g$, $\forall\, x \in S$ and so g is an upper bound of S.

Also no number less than g can be an upper bound of S. For if y be any number less than g, there exists at least one member g of S which is greater than y.

Thus g is the least of all the upper bounds of S *i.e.*, g is the supremum of S.

4. Lower bound of a subset of R.

As we have defined an upper bound of a set, in the same way we can also define a lower bound of a set.

Definition: *Let S be a subset of real numbers. If there exists a real number v, such that*

$$x \geq v \ \forall x \in S,$$

then v is called a **lower bound** *of S.*

If there exists a lower bound for a set S, then the set S is said to be **bounded below.**

It is not necessary that a set should be bounded below. If there exists no real number v such that $x \geq v \ \forall\, x \in S$, then the set S is said to be **not bounded below** or **unbounded below**. Thus a set S is unbounded below if however small a real number v we may take, there exists at least one $x \in S$ such that $x < v$.

For example, the set of negative integers

$$Z^- = \{\ldots, -3, -2, -1\}$$

is not bounded below while the set of positive integers $Z^+ = \{1, 2, 3, \ldots\}$ is bounded below and 1 is a lower bound. The set of positive real numbers R^+ is also bounded below, 0 being a lower bound.

If v is a lower bound of a set S, then every real number smaller than v is also a lower bound of S. Thus if a set S is bounded below, then the set of all such real numbers that are lower bounds of S is infinite.

5. Greatest lower bound or infimum.

Definition: *If t is a lower bound of a subset S of* R *and any real number greater than t is not a lower bound of S, then t is called the* **greatest lower bound (g.l.b.)** *or* **infimum (inf.)** *of S.* **(Kanpur 2011)**

Thus if a set S is bounded below and if the set of all lower bounds of S has a greatest member, say t, then t is called the greatest lower bound or the infimum of the set S.

If a real number t is the infimum of a subset of S of real numbers, then for every $\varepsilon > 0$, there exists a real number $x \in S$ such that

$$t \leq x < t + \varepsilon.$$

Note: Supremum is defined only for the subsets of R which are bounded above and infimum for the subsets which are bounded below.

Exercise: *Prove that the smallest member of a set, if it exists, is the infimum (g.l.b.) of the set.*

6. Bounded subsets of real numbers.

Definition: *A subset S of real numbers is said to be* **bounded**, *if it is bounded above as well as bounded below.*

Thus a set S is bounded if and only if there exist two real numbers u, v such that

$$v \leq x \leq u \quad \forall\ x \in S$$

or $S \subset [v, u]$ *i.e.*,

S is a subset of the closed interval $[v, u]$.

It can be easily seen that a set S is bounded if and only if there exists a positive real number k such that

$$|x| < k \quad \text{for all} \quad x \in S.$$

Unbounded set of real numbers

Definition: *A subset S of* R *is said to be unbounded if it is not bounded above or not bounded below.*

A finite subset of R is always bounded. The subsets Q, Z and Z^+ of R are not bounded. The open interval] 3, 5 [is a bounded subset of R, 5 being an upper bound and 3 a lower bound.

The null set $\varnothing$ is bounded but it neither possesses supremum nor infimum.

The null set $\varnothing$ is bounded above because if u is any real number, then u is an upper bound for $\varnothing$. Obviously the condition $x \leq u$ for all $x \in \varnothing$ is vaccuously satisfied because $\varnothing$ has no elements.

Thus every real number is an upper bound for $\varnothing$. Since the set of all real numbers has no smallest member, therefore sup $\varnothing$ does not exist.

Also we observe that $\varnothing$ is bounded below. In fact every real number is a lower bound for $\varnothing$. Since the set of all real numbers has no greatest member, therefore inf $\varnothing$ does not exist.

7. Greatest and least members of a subset of R.

If the supremum of a subset S of R *is a member of S, then we say that S* **attains** *its supremum and this supremum is called the greatest member of S.*

If the infimum of a subset of S of R *is a member of S, then we say that S* **attains** *its infimum and this infimum is called the least member of S.*

The greatest member of a set S, if it exists, is the supremum of S. But the supremum of S need not be the greatest member of S unless S attains this supremum.

For example if S is the open interval

$$]\,2, 3\,[\text{ i.e., } S = \{x : 2 < x < 3\},$$

then sup $S = 3$ and inf $S = 2$. Note that 3 is an upper bound for S and no real number less than 3 is an upper bound for S. Similarly 2 is a lower bound for S and no real number greater than 2 is a lower bound for S. Since $2 \notin S$ and $3 \notin S$, therefore S has no least member and no greatest member.

On the other hand if A is the closed interval $[2, 3]$ *i.e.*, $A = \{x : 2 \le x \le 3\}$, then sup $A = 3$ and inf $A = 2$. Here both the infimum and the supremum *i.e.*, 2 and 3 are members of A and so 2 is the least member of A and 3 is the greatest member of A.

8. Bounded and unbounded intervals.

Bounded intervals: If a and b are any real numbers such that $b > a$, then the intervals $]\,a, b\,[$, $[a, b]$, $[a, b\,[$ and $]\,a, b]$ are called **bounded intervals** because each of them is a bounded subset of R.

If $S_1 =]\,a, b\,[$, then sup $S_1 = b$ and inf $S_1 = a$. Here both a and b do not belong to S_1 and so S_1 has no least member and no greatest member.

If $S_2 = [a, b]$, then sup $S_2 = b$ and inf $S_2 = a$. Here both a and b are members of S_2 and so a is the least member of S_2 and b is its greatest member.

If $S_3 = [\,a, b\,[$, then sup $S_3 = b$ and inf $S_3 = a$. Here $a \in S_3$ and is the least member of S_3 but $b \notin S_3$ and so S_3 has no greatest member.

If $S_4 =]\,a, b\,]$, then sup $S_4 = b$ and inf $S_4 = a$. Here $b \in S_4$ and is the greatest member of S_4 but $a \notin S_4$ and so S_4 has no least member.

Unbounded intervals: If a is any real number, then the intervals $]\,a, \infty\,[$, $[a, \infty\,[$, $]-\infty, a\,[$ and $]-\infty, a]$ are called **unbounded intervals** because each of them is not a bounded subset of R. The set $\mathrm{R} =]-\infty, \infty\,[$ is also an unbounded interval. It is neither bounded above nor bounded below.

If $S_1 =]a, \infty[$, then S_1 is not bounded above and so the question of sup S_1 does not arise. However S_1 is bounded below and inf $S_1 = a$. Since $a \notin S_1$, therefore S_1 has no least member.

Similarly we can discuss the cases of the other three unbounded intervals mentioned above.

9. Some important observations about supremum and infimum of a subset of R if they exist.

(i) A non-empty finite subset of R is always bounded and has its greatest member as its supremum and its smallest member as its infimum.

(ii) Supremum and infimum of a bounded subset of R are unique.

(iii) In the case of a singleton set $S = \{a\}, a \in R$, supremum and infimum coincide. In this case a is both sup S and inf S.

(iv) Supremum and infimum of a bounded set need not necessarily belong to the set.

(v) Even in the case of an infinite bounded subset of R supremum and infimum may both belong to the set. For example, if S is the closed interval $[4, 9]$, then S is an infinite subset of R and sup $S = 9$ and inf $S = 4$ are both members of S.

(vi) If s and t are the supremum and the infimum of a non-empty subset S of R, then $t \leq s$.

1.14 Some Properties of Supremum and Infimum

Theorem 1. Uniqueness of the supremum.

The supremum of a set $S \subset \mathbf{R}$, if it exists, is unique.

Proof: Let S be a non-empty subset of R which is bounded above. If possible let s_1 and s_2 be two suprema of S. Then to show that $s_1 = s_2$.

Since both s_1 and s_2 are suprema of S, therefore they are upper bounds of S.

Now s_1 is a supremum and s_2 is an upper bound of S

$\Rightarrow \quad s_1 \leq s_2$. ...(1)

Similarly s_2 is a supremum and s_1 is an upper bound of S

$\Rightarrow \quad s_2 \leq s_1$. ...(2)

From (1) and (2), we have $s_1 \not< s_2$ and $s_2 \not< s_1$. Therefore by the law of trichotomy, we have $s_1 = s_2$.

Theorem 2: Uniqueness of the infimum.

The infimum of a set $S \subset \mathbf{R}$, if it exists, is unique.

Proof: Let S be a non-empty subset of R which is bounded below. If possible let t_1 and t_2 be two infima of S. Then to show that $t_1 = t_2$.

Since both t_1 and t_2 are infima of S, therefore they are lower bounds of S.

Now t_1 is an infimum and t_2 is a lower bound of S

$$\Rightarrow t_1 \geq t_2. \qquad ...(1)$$

Similarly t_2 is an infimum and t_1 is a lower bound of S

$$\Rightarrow t_2 \geq t_1. \qquad ...(2)$$

From (1) and (2), we have $t_1 = t_2$.

Theorem 3: A characteristic property of supremum .

If S be a non-empty subset of R, *then a real number s is the supremum for S if and only if*

(*i*) $x \leq s$ *for all* $x \in S$,

and (ii) for each positive real number ε, *there exists a real number* $x \in S$ *such that* $x > s - \varepsilon$.

Proof. The '**only if part**' *i.e.*, the given conditions are **necessary** for s to be the supremum of S.

Let s be the supremum for S. Then since s is an upper bound for S, we have

$$x \leq s \quad \forall x \in S.$$

Again take any real number $\varepsilon > 0$. Then $s - \varepsilon < s$ and so $s - \varepsilon$ cannot be an upper bound for S since it is less than the supremum. Hence there must exist some $x \in S$ such that

$$x > s - \varepsilon.$$

Hence the conditions are necessary.

The '**if part**' *i.e.*, the given conditions are **sufficient** for s to be the supremum of S.

Let the conditions (i) and (ii) hold. Then to show that

$$s = \sup S$$

By condition (i) s is an upper bound for S. Now s will be the supremum of S if we show that no real number less than s can be an upper bound for S.

Let s' be any real number less than s, then $s - s' > 0$. Now if we take $\varepsilon = s - s' > 0$, then by the given condition (ii) there exists $x \in S$ such that $x > s - \varepsilon$ *i.e.*, $x > s - (s - s')$ *i.e.*, $x > s'$, showing that s' is not an upper bound for S. Thus we see that s is an upper bound for S and no real number less than s is an upper bound for S. Hence s is the supremum of S.

This completes the proof of the theorem.

Theorem 4: A characterization of infimum.

If S be a non-empty subset of R, *then a real number t is the infimum for S if and only if*

(*i*) $x \geq t$ *for all* $x \in S$,

and (ii) for each real number $\varepsilon > 0$, *there exists a real number* $x \in S$ *such that* $x < t + \varepsilon$.

Proof: The '**only if part**' *i.e.*, the given conditions are **necessary** for t to be the infimum of S.

Let t be the infimum for S. Then since t is a lower bound for S, we have

$$x \geq t \quad \forall x \in S.$$

Again take any real number $\varepsilon > 0$. Then $t + \varepsilon > t$ and so $t + \varepsilon$ cannot be a lower bound for S since it is greater than the greatest lower bound (*i.e.*, inf) t of S. Thus we see that t is a lower bound for S and no real number greater than t is a lower bound for S. Hence there must exist some $x \in S$ such that $x < t + \varepsilon$.

Hence the conditions are necessary.

The **'if part'** *i.e.*, the given conditions are **sufficient** for t to be the infimum of S.

Let the conditions (i) and (ii) hold. Then to show that $t = \inf S$.

By condition (i) t is a lower bound for S.

Now t will be the infimum (*i.e.*, g.l.b.) of S if we show that no real number greater than t can be a lower bound for S.

Let t' be any real number greater than t, then $t' - t > 0$. Now if we take $\varepsilon = t' - t > 0$, then by the given condition (ii) there exists $x \in S$ such that $x < t + \varepsilon$ *i.e.*, $x < t + (t' - t)$ *i.e.*, $x < t'$, showing that t' is not a lower bound for S. Thus we see that t is a lower bound for S and no real number greater than t is a lower bound for S. Hence t is the g.l.b. *i.e.*, the infimum of S.

This completes the proof of the theorem.

Illustrative Examples

Example 8: *Prove that the set* R^+ *of positive real numbers is not bounded above.*

Solution: The set R^+ of positive real numbers is not bounded above. For, suppose that R^+ is bounded above and u is an upper bound.

Since $1 \in R^+$ and u is an upper bound for R^+, therefore,

$$1 \leq u, \text{ which means } u > 0.$$

$\therefore$ $u + 1 > 0$ and consequently $u + 1 \in R^+$.

Thus there exists $u + 1 \in R^+$ such that $u + 1 >$ an upper bound u of R^+. But this is a contradiction because if u is an upper bound for R^+, then we must have $x \leq u$ for all x in R^+.

Hence u is not an upper bound of R^+ and so R^+ is not bounded above.

Example 9: *Let S be a non-empty bounded subset of* R *such that sup S = inf S. What can be said about the set S ?*

Solution: Let $\sup S = \inf S = u$.

Then u is an upper bound as well as a lower bound for S.

$\therefore$ $x \leq u$ for all $x \in S$ ...(1)

and $x \geq u$ for all $x \in S$. ...(2)

From (1) and (2), we have $x = u$ for all $x \in S$ *i.e.*, u is the only element in S *i.e.*, S is the singleton $\{u\}$.

$\therefore$ if $\sup S = \inf S$, then S is a singleton set.

Example 10: *If u is an upper bound of a set $S \subset \mathbf{R}$ and $u \in S$, then $u = \sup S$.*

Solution: If possible, let s be sup S.

Since s is sup S, therefore s is an upper bound for S.

$\therefore \quad u \in S \Rightarrow u \leq s.$ ...(1)

Also u is an upper bound of S and s is the l.u.b. of S

$\Rightarrow s \leq u.$ ...(2)

From (1) and (2), we have $s = u$.

Hence $\sup S = u.$

Example 11: *Show that every non-empty finite subset of* R *is bounded.*

Solution: Let S be a non-empty finite subset of R. Then there are only a finite number of elements in S and so by the properties of the order relation in R out of these elements one element $a \in S$ shall be the smallest element of S and one element $b \in S$ shall be the greatest element of S.

Thus we have $a \leq x \leq b \ \forall \ x \in S$. Hence the set S is bounded.

Example 12: *Find the supremum and infimum of the singleton*

$\{2\} \subset \mathbf{R}$.

Solution: For $\{2\} \subset \mathbf{R}$, we find that the set of upper bounds of $\{2\}$ is given by $\{x \in \mathbf{R} : x \geq 2\}$ and 2 being the least of these upper bounds, we have, $\sup\{2\} = 2$.

Again the set of lower bounds of $\{2\}$ is given by

$\{x \in \mathrm{R} : x \leq 2\}$

and 2 being the greatest of these lower bounds, we have

$\inf\{2\} = 2.$

Hence $\sup\{2\} = 2 = \leq \inf\{2\}.$

Example 13: *Find the g.l.b. and l.u.b. of the set*

$S = \{x \in \mathrm{Z} : x^2 \leq 25\}.$

Solution: Writing in tabular form, we have

$S = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}.$

Here S is a finite subset of R. The smallest member of S is -5 and so it is a lower bound for S. All real numbers less than -5 are also lower bounds for S. Thus the set of all lower bounds of the set S is the set $\{x : x \in \mathrm{R} \text{ and } x \leq -5\}$. Since -5 is the greatest of all these lower bounds of S, therefore g.l.b. of $S = -5$.

Similarly the greatest member of S is 5 and so it is an upper bound for S. Since an upper bound 5 of the set S is a member of the set S, therefore it is the supremum of S. Hence $\sup S = 5$.

Example 14: *Give examples to show that*

(*i*) *every infinite set need not be bounded.*

(*ii*) *every subset of an unbounded set is not necessarily unbounded.*

Solution: (i) Consider the infinite set $S = \{x \in R : 2 < x < 3\}$. Then S is a bounded subset of R, 2 being a lower bound and 3 an upper bound for S.

Again consider the set of integers

$$Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

The set Z is an infinite subset of R and is not bounded. It is neither bounded above nor bounded below.

(ii) The set of integers Z is unbounded but its subset $S = \{1, 2, 3\}$ is bounded. We see that 1 is a lower bound and 3 an upper bound for S.

Example 15: *Give an example each of a bounded set which contains its*

(i) g.l.b. but does not contain its l.u.b.

(ii) l.u.b. but does not contain its g.l.b.

Solution: (i) Consider the set $S = \{x : x \in R \text{ and } 2 \le x < 5\}$. We have g.l.b. of $S = 2$ which is a member of S and l.u.b. of $S = 5$ which is not a member of S.

(ii) Consider the set $A = \{x : x \in R \text{ and } 3 < x \le 7\}$. We have g.l.b. of $A = 3$ and $3 \notin A$.

Also l.u.b. of $A = 7$ and $7 \in A$.

Example 16: *A, B are sets such that $a \in A, b \in B \Rightarrow a < b$. Show that l.u.b. $A \le$ g.l.b. B.*

Solution: Let l.u.b. $A = s$ and g.l.b. $B = t$.

To show that $s \le t$.

Suppose if possible $s > t$.

Since l.u.b. $A = s$ and $t < s$, therefore there exists $x \in A$ such that $x > t$.

Now g.l.b. $B = t$ and $x > t \Rightarrow$ there exists $y \in B$ such that $y < x$.

Thus there exists $x \in A$ and $y \in B$ such that $x > y$ which is against the hypothesis that $a \in A, b \in B \Rightarrow a < b$.

Hence our initial assumption $s > t$ is wrong and we must have $s \le t$.

Example 17: *Find the supremum and infimum, if they exist, of the following sets:*

(i) $\left\{\frac{1}{n} : n \in N\right\}$ (ii) $\left\{x \in Q : x = \frac{n}{n+1}, n \in N\right\}$

(iii) $\left\{-2, -\frac{3}{2}, -\frac{4}{3}, -\frac{5}{4}, \ldots\right\}$. (iv) *the set of positive integers* Z^+

(v) $\{x : x = (-1)^n n, n \in N\}$. (vi) $\left\{1 + \frac{(-1)^n}{n} : n \in N\right\}$.

Solution: (i) Let $S = \left\{\frac{1}{n} : n \in N\right\}$

i.e. $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$.

The set S is bounded above, 1 being an upper bound for S. Note that we have $x \leq 1$ for all $x \in S$.

Since $1 \in S$, therefore any number less than 1 is not an upper bound for S.

$\therefore$ $\sup S = 1$. Also we observe that $1 \in S$.

Again 0 is a lower bound for S because $x \geq 0$ for all $x \in S$.

Any real number greater than 0 cannot be a lower bound for S as shown below:

Let $v > 0$. However small $v > 0$ may be, there exists $n \in \mathrm{N}$ such that $1/n < v$.

Thus there exists $1/n \in S$ such that $1/n < v$.

Therefore v is not a lower bound for S.

Thus 0 is a lower bound for S and no real number greater than 0 can be a lower bound for S.

$\therefore$ g.l.b. of S *i.e.*, $\inf S = 0$. Here we observe that $0 \notin S$.

(ii) Let $S = \left\{\frac{n}{n+1} : n \in \mathrm{N}\right\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$.

The set S is bounded below, $\frac{1}{2}$ being a lower bound for S. Since $\frac{1}{2} \in S$, therefore any real number greater than $\frac{1}{2}$ cannot be a lower bound for S.

$\therefore$ g.l.b. of S *i.e.*, $\inf S = \frac{1}{2}$. We observe that $\frac{1}{2} \in S$.

Since $\frac{n}{n+1} < 1 \ \forall n \in \mathrm{N}$, therefore 1 is an upper bound for S. Any real number less than 1 cannot be an upper bound for S as shown below.

Let u be any real number < 1. We shall show that there exists some $x \in S$ such that $x > u$ *i.e.*, there exists some $n \in \mathrm{N}$ such that $n/(n+1) > u$.

We have $\frac{n}{n+1} = \frac{(n+1)-1}{n+1} = 1 - \frac{1}{n+1}$.

Now $u < 1 \Rightarrow 1 - u > 0$. However small $1 - u > 0$ may be, there exists $n \in \mathrm{N}$ such that

$$\frac{1}{n} < 1 - u \Rightarrow \frac{1}{1+n} < 1 - u \qquad \left[\because \frac{1}{1+n} < \frac{1}{n}\right]$$

$$\Rightarrow 1 - \frac{1}{1+n} > u \Rightarrow \frac{n}{n+1} > u.$$

Thus there exists $n \in \mathrm{N}$ such that $\frac{n}{n+1} > u$ and hence u cannot be an upper bound for S.

Thus 1 is an upper bound for S and no real number less than 1 can be an upper bound for S.

$\therefore$ sup $S = 1$. We observe that $1 \notin S$.

(iii) Let $S = \left\{-2, -\frac{3}{2}, -\frac{4}{3}, -\frac{5}{4}, \ldots\right\}$

$= \left\{-2, -1\frac{1}{2}, -1\frac{1}{3}, -1\frac{1}{4}, \ldots\right\}$.

We have inf $S = -2 \in S$

and sup $S = -1 \notin S$.

(iv) We have $Z^+ = \{1, 2, 3, \ldots\}$.

The set Z^+ is bounded below, 1 being a lower bound for Z^+. Since $1 \in Z^+$, therefore no real number greater than 1 can be a lower bound for Z^+.

$\therefore$ inf $Z^+ = 1$.

But the set Z^+ is not bounded above.

There is no real number u such that

$x \leq u$ for all $x \in Z^+$.

Since the set Z^+ is not bounded above, therefore sup Z^+ has no meaning.

(v) Let $S = \{x : x = (-1)^n\ n, n \in N\}$

$= \{-1, 2, -3, 4, -5, 6, \ldots\}$

$= \{\ldots, -5, -3, -1, 2, 4, 6, \ldots\}$.

Here the set S is not bounded above and so sup S cannot be discussed. Also S is not bounded below and so inf S has no meaning.

(vi) Let $S = \left\{1 + \frac{(-1)^n}{n} : n \in N\right\}$

$= \left\{0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \frac{7}{6}, \frac{6}{7}, \frac{9}{8}, \frac{8}{9}, \ldots\right\}$

$= \left\{\frac{0}{1}, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{8}{9}, \ldots, \frac{2n-2}{2n-1}, \ldots\right\} \cup \left\{\frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \frac{9}{8}, \ldots, \frac{2n+1}{2n}, \ldots\right\}$.

Here we observe that the proper fractions $\frac{0}{1}, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \ldots$ are increasing and tending to 1. The improper fractions beginning with $\frac{3}{2}$ are decreasing and tending to 1.

We have inf $S = 0$ and sup $S = \frac{3}{2}$.

Comprehensive Exercise 2

1. Let $S = \{x \in R : x = n + 3, n \in N\}$. Show that S is bounded below but not above. Find the g.l.b. of S.
2. Which of the following sets are bounded below, which are bounded above, and which are bounded neither below nor above ?

(i) $\{-1, -2, -3, -4, -5, \ldots\}$.

(ii) $\{1, 2, 3, 4, 5, \ldots\}$.

(iii) $\{2, 2^2, 2^3, \ldots, 2^n, \ldots\}$.

(iv) $\left\{1, \frac{1}{4}, \left(\frac{1}{4}\right)^2, \left(\frac{1}{4}\right)^3, \ldots, \left(\frac{1}{4}\right)^n, \ldots\right\}$

(v) $\left\{x : x = (-1)^n \frac{1}{n}, n \in N\right\}$

(vi) $\{x : x = (-2)^n, n \in N\}$.

3. Find the supremum and infimum, if they exist, of the following sets.

(i) $\{x : x = 1 + (1/n), n \in N\}$.

(ii) $\{x : x = 1 - (1/n), n \in N\}$.

(iii) $\{x \in R : -5 < x < 3\}$.

(iv) $\{x \in R : x = 2^n, n \in N\}$.

(v) $\{\pi + 1, \pi + \frac{1}{2}, \pi + \frac{1}{3}, \ldots\}$.

(vi) $\{3, 4, 12, 20\}$.

(vii) $\left\{-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\right\}$.

(viii) $\left\{m + \frac{1}{n} : m, n \in N\right\}$.

(ix) $\left\{\frac{1}{5n} : n \in Z, n \neq 0\right\}$.

(x) $\left\{\frac{3n + 2}{2n + 1} : n \in N\right\}$.

4. Given an example of a non-empty bounded subset S of R, whose supremum and infimum both belong to $R - S$.
5. Prove that the set R^- of negative real numbers is not bounded below.
6. Prove that the set of all real numbers R is not bounded.
7. Find the l.u.b. for the following sets :

(i) $(1, 2] \cup [3, 8)$.

(ii) The empty set.

(iii) $\left\{\pi + \frac{1}{2}, \pi + \frac{1}{4}, \pi + \frac{1}{8}, \ldots\right\}$.

8. Find the l.u.b. and the g.l.b. of the set

$$S = \left\{\frac{2n + 1}{3n + 2} : n \in N\right\}.$$

9. Find g.l.b. and l.u.b. of the following sets S :

(i) $S = \{x \in Q : x = \frac{(-1)^n}{n}, n \in N\}$.

(ii) $S = \left\{ x \in Q : x = (-1)^n \left(\frac{1}{4} - \frac{4}{n} \right), n \in N \right\}$.

(iii) $S = \left\{ \left(1 - \frac{1}{n}\right) \sin \frac{n\pi}{2}, n \in N \right\}$.

(vi) $S = \left\{ x \in Q : x = (-1)^n \left(\frac{1}{n} - \frac{4}{n} \right), n \in N \right\}$.

10. Give an example of a set which is :
    (i) bounded above but not below
    (iii) bounded below but not above
    (iii) neither bounded above nor bounded below
    (iv) both bounded above and below.
11. If $A \neq \varnothing$ is bounded below and $-A$ denotes the set of all $-x$ for whic $x \in A$, then prove that $-A \neq \varnothing$, that $-A$ is bounded above, and that $-\sup(-A) = \inf A$.
12. If $A \neq \varnothing$, $B \neq \varnothing$ and $x \leq y$, $\forall x \in A$ and $\forall y \in B$, then prove that :
    (i) $\sup A \leq y, \forall y \in B$
    (ii) $\sup A \leq \inf B$.
13. If $B \supseteq A (\neq \varnothing)$ and B is bounded, then $\sup B \geq \sup A \geq \inf A \geq \inf B$

Answers 2

1. Inf $S = 4$.
2. (i) Bounded above but not bounded below.
    (ii) Bounded below but not bounded above.
    (iii) Bounded below but not bounded above.
    (iv) Bounded below as well as bounded above. The l.u.b. of the set = 1 and the g.l.b. = 0. Here the l.u.b. is a member of the set while the g.l.b. does not belong to the set.
    (v) Bounded below as well as bounded above. Here the l.u.b. of the set = $\frac{1}{2}$ and the g.l.b. = -1 and bot the l.u.b. and the g.l.b. are members of the set.
3. (i) Sup = 2, inf = 1.
    (ii) Sup = 1, inf = 0.
    (iii) Sup = 3, inf = -5.
    (iv) Inf = 2, sup does not exist.
    (v) Sup = $\pi + 1$, inf = π.
    (vi) Sup = 20, inf = 3.
    (vii) Sup = 0, inf = -1.
    (viii) Inf = 1, sup does not exist.
    (ix) Sup = 1/5, inf = $-1/5$.
    (x) Sup = 5/3. inf = 3/2
4. $S = \{ x : x \in \boldsymbol{R} \text{ and } 2 < x < 3 \}$ *i.e.*, S = the open interval $]2,3[$

7. (i) 8. (ii) No. (iii) $\pi + \frac{1}{2}$

8. l.u.b. $= \frac{2}{3}$ and g.l.b. $= \frac{3}{5}$

9. (i) g.l.b. $= -1$ and l.u.b. $= \frac{1}{2}$

(ii) g.l.b. $= -7/4$ and l.u.b. $= 15/4$

(iii) g.l.b. $= -1$ and l.u.b. $= 3$

(iv) g.l.b. $= -3/2$ and l.u.b. $= 3$

10. (i) The set of negative integers

(ii) The set of positive integers

(iii) The set of rational numbers

(iv) The set $S = \left\{\frac{1}{n} : n \in N\right\}$

1.15 Completeness

Completeness of a system S of numbers with respect to boundedness is defined as follows :

Definition: *A system S of numbers is said to be* **complete** *if every non-empty subset of S, which is bounded above has a member of S for its supremum.*

For example, the set Z of all integers is complete with respect to boundedness.

Let S be any non-empty subset of Z which is bounded above. Then S must possess a greatest integer. For, if S does not possess a greatest integer, then however great an integer u we may take, there exists an integer $x \in S$ such that $x > u$ and so S cannot be bounded above. But this is a contradiction. Hence S must contain a greatest integer, say s.

Then $s \in S$ and $x \leq s \forall x \in S$. So s is an upper bound for S and no integer less than s can be an upper bound for S.

$\therefore$ sup $S = s \in S$.

Hence for any non-empty subset S of integers which is bounded above, the sup S is an element of S.

Thus every non-empty subset of Z which is bounded above has a member of Z for its supremum. Therefore the set Z of all integers is complete with respect to boundedness.

1.16 Order-Completeness Axiom for Real Numbers

Every non-empty subset of real numbers which is bounded above has a supremum.

This property of real numbers is known as *order-completeness* or simply *completeness.*

Roughly speaking, this axiom means that R (regarded as a set of points on a line) has no holes in it. If S is any non-empty subset of R which is bounded above, then the set of all upper bounds of S must have a smallest member *i.e.* S must possess the supremum which is a member of R.

The set of rational numbers Q does not satisfy the order completeness axiom as we shall show later. Thus it is the completeness property which enables us to distinguish between the set Q of rational numbers and the set R of real numbers. Both the sets Q and R form ordered fields. The field R possesses the completeness property while the field Q does not possess it.

The set R thus satisfies (1) Field axioms (2) Order axioms and (3) Completeness axiom and *hence* R *is a complete ordered field.*

Complete ordered field

Definition: *An ordered field F is said to be a* **complete ordered field** *if every non-empty subset S of F which is bounded above has an element of F for its supremum.*

The field R of real numbers is a complete ordered field while the field Q of rational numbers is an ordered field but is not complete.

In fact, it is a characterisation of the set R *of real numbers that it is a* **complete ordered field.** An ordered field F is complete if and only if it is the field of real numbers. Thus we may define the real numbers as the elements which form a complete ordered field. This is a characteristic property of real numbers.

The following theorem asserts the existence of the infimum of any non-empty subset of R which is bounded below.

Theorem 1: *Any non-empty subset of real numbers which is bounded below has an infimum.*

Proof: Let $S \subset \mathrm{R}$ and $S \neq \emptyset$.

Let S be bounded below. Then to show that S has an infimum.

Let us denote by T the set of negatives of the members of S *i.e.*,

$$T = \{-x : x \in S\}$$

or $$T = \{y : y \in \mathrm{R} \text{ and } y = -x \text{ for some } x \in S\}.$$

First we shall show that T is bounded above.

Since S is bounded below, therefore let ν be a lower bound for S so that

$$x \geq \nu \quad \forall \ x \in S$$

$$\Rightarrow \quad -x \leq -\nu \ \forall \ x \in S$$

$$\Rightarrow \quad y \leq -\nu \ \forall \ y \in T. \qquad [\because \ y \in T \Rightarrow y = -x \text{ for some } x \in S]$$

Thus $-v$ is an upper bound for T and so T is bounded above.

$\therefore$ by the completeness axiom, T has the supremum, say t.

We shall show that $-t$ is the infimum of S *i.e.*, $-t$ is the greatest lower bound of S.

Since t is an upper bound of T, therefore $-t$ is a lower bound of S.

Let w be any lower bound of S. Then $-w$ is an upper bound of T.

Now $-w$ is an upper bound of T and t is the supremum of T.

$\therefore \quad t \leq -w$

$\Rightarrow \quad -t \geq w.$

Thus $-t$ is a lower bound of S and if w is any lower bound of S, then $w \leq -t$.

Hence $-t$ is the greatest lower bound of S *i.e.*, $-t = \inf S$.

Hence the theorem.

Theorem 2: *The set* N *of natural numbers is not bounded above.*

Proof. If possible, let N be bounded above.

Since $1 \in \mathrm{N}$, therefore $\mathrm{N} \neq \varnothing$. Thus N is a non-empty subset of R which is bounded above. Therefore by order-completeness property N must have supremum, say, s.

Then $\quad x \leq s \quad \forall\ x \in \mathrm{N}$

$\Rightarrow \quad (x+1) \leq s \quad \forall\ x \in \mathrm{N} \qquad [\because\ x \in \mathrm{N} \Rightarrow x+1 \in \mathrm{N}]$

$\Rightarrow \quad x \leq s-1 \ \forall x \in \mathrm{N}$

$\Rightarrow \quad s-1$ is an upper bound of N.

But $s-1 < s$ and we have assumed that s is the supremum of N. Thus we get an upper bound of N which is less than the supremum of N. But this contradicts the fact that s is the supremum of N.

Hence N is not bounded above.

Theorem 3: *The set* Q *of rational numbers is not order-complete i.e., the ordered field of rational numbers is not a complete ordered field.* **(Kanpur 2009)**

Proof: We shall show that there exists a non-empty subset of Q which is bounded above but which does not have any rational number for its supremum.

Let us consider the set S of all those positive rational numbers whose squares are less than 2 *i.e.*, let

$$S = \{x : x \in \mathrm{Q}^+ \text{ and } 0 < x^2 < 2\}.$$

Since $1 \in S$, therefore S is non-empty. Also S is bounded above, 2 being an upper bound for S. Note that $2^2 = 4 > 2$ and so $x \in \mathrm{Q}^+$ and $x^2 < 2 \Rightarrow x < 2$.

Thus S is a non-empty subset of Q which is bounded above. We shall show that S does not have any rational number a as its least upper bound. The following cases arise :

Case I: $a \leq 0$. Since every member of S is positive, therefore, a cannot be an upper bound of S and so it cannot be the supremum for S.

Case II: $a > 0$ and $0 < a^2 < 2$.

Let $$b = \frac{4 + 3a}{3 + 2a} \quad \ldots(1)$$

Then $$b^2 - 2 = \left(\frac{4 + 3a}{3 + 2a}\right)^2 - 2 = \frac{a^2 - 2}{(3 + 2a)^2} \quad \ldots(2)$$

and $$b - a = \frac{4 + 3a}{3 + 2a} - a = \frac{4 - 2a^2}{3 + 2a} = \frac{2(2 - a^2)}{3 + 2a} \quad \ldots(3)$$

Since a is a positive rational number, it follows from (1) that b is also a positive rational number. Also, since $a^2 < 2$, from (2) we find that $b^2 < 2$ and from (3) we find that $b > a$. Thus b is a positive rational number such that $0 < b^2 < 2$ and so by the definition of the set S, we have $b \in S$. Also, since $b > a$, therefore a cannot be an upper bound for S.

Case III: $a > 0$ and $a^2 = 2$. This is not possible because we know that there is no rational number whose square is 2.

Case IV: $a > 0$ and $a^2 > 2$.

Let us take $b = \frac{4 + 3a}{3 + 2a}$, as we have taken in case II. But now a^2 being > 2, from (1), (2) and (3) we see that b is a positive rational number such that $b^2 > 2$ and $b < a$ which means that $2 < b^2 < a^2$.

If y be any arbitrary member of S, then we have

$$0 < y^2 < 2 < b^2 < a^2$$

i.e., $$0 < y < b < a.$$

This shows that both a and b are upper bounds for S and that a cannot be the supremum for S because b is an upper bound for S which is less than a.

Since, by the law of trichotomy, the possibilities discussed above are mutually exclusive and exhaustive, therefore, it follows that if a is any rational number, then a cannot be the least upper bound of S.

Hence the system of rational numbers does not satisfy order completeness property.

1.17 Archimedean Property of Real Numbers

The order-completeness property of real numbers has important consequences. The most important of them is the following *Archimedean property of real numbers.*

Theorem 1. *Let a be any real number and b any positive real number. Then there exists a positive integer n such that*

$$nb > a.$$

Proof: If $a \leq 0$, the theorem is obvious because then for every positive integer n we have $nb > a$. [Note that $n > 0, b > 0 \Rightarrow nb > 0$ and so if $a \leq 0$, then we have $nb > a$]. So now let a be > 0.

In this case assume that there exists no positive integer n such that $nb > a$.

Then we have $nb \leq a \quad \forall\ n \in \mathbb{N}$.

It means that a is an upper bound of the non-empty subset S of R given by

$$S = \{b, 2b, 3b, 4b, \ldots\} = \{nb : n \in \mathbb{N}\}.$$

$\therefore$ by the completeness property of R, S must have the supremum, say s.

Then $nb \leq s$ for all $n \in \mathbb{N}$ and so

$$(n+1)\,b \leq s \text{ for all } n \in \mathbb{N} \qquad [\because\ n \in \mathbb{N} \Rightarrow n+1 \in \mathbb{N}]$$

i.e., $nb + b \leq s$ fo all $n \in \mathbb{N}$ *i.e.*, $nb \leq s - b$ for all $n \in \mathbb{N}$.

$\therefore$ $s - b$ is an upper bound of S.

Since $b > 0$, therefore $s - b < s$.

It means that we have an upper bound for S which is less than the supremum s of S. But this contradicts the definition of the supremum.

Hence our initial assumption is wrong and so there must exist some positive integer n such that

$$nb > a.$$

Archimedean ordered field

Definition: *An ordered field F is said to be an Archimedean ordered field if $\forall\ x, y \in F$, $y > 0$, there exists some $n \in \mathbb{N}$ such that $ny > x$.*

For example, the field R of real numbers is an Archimedean ordered field.

Corollary 1: *For every real number a there exists a positive integer n such that $n > a$.*

Proof: Since 1 is a positive real number, therefore by the Archimedean property of real numbers, for every real number a, there exists a positive integer n such that

$$n.1 > a \quad i.e., \quad n > a.$$

Corollary 2: *For any positive real number x, there exists a positive integer n such that $1/n < x$.*

Proof: By Archimedean property of real numbers, for every positive real number x, there exists a positive integer n such that

$$nx > 1. \qquad [\text{Note that we have taken } b = x \text{ and } a = 1 \text{ in the result of theorem 1}]$$

Since $n > 0$, therefore dividing both sides of the inequality $nx > 1$ by n, we have

$$x > 1/n \quad i.e., \quad 1/n < x.$$

Theorem 2: *For any real number x, there exist two integers m and n such that $m < x < n$.*

Proof: Since 1 is a positive real number, therefore by the Archimedean property of real numbers, for every real number x, there exists a positive integer n such that

$$n.1 > x \quad i.e., \quad n > x. \qquad ...(1)$$

Now $x \in \mathbf{R} \Rightarrow -x \in \mathbf{R}$. Therefore by the result (1), there exists a positive integer m_1 such that

$$m_1 > -x \quad i.e., \quad -m_1 < x. \qquad ...(2)$$

Since $-m_1$ is also an integer, therefore setting $-m_1 = m$, we see from (1) and (2) that there exist two integers m and n such that

$$m < x < n.$$

Hence the theorem is proved.

Theorem 3: *For any real number x, there exists a unique integer n such that*

$$n \leq x < n + 1.$$

Proof: Consider the set $S = \{y : y \in \mathbf{Z} \text{ and } y \leq x\}$.

Then S is a non-empty subset of integers and is bounded above by x. Therefore S has its supremum, say n, in Z. This n is the greatest integer belonging to the set S.

Thus there exists a greatest integer n such that $n \leq x$.

Then $n + 1$ being $> n$, we have $n + 1 > x$. Hence there exists an integer n such that

$$n \leq x < n + 1.$$

The uniqueness of the integer n satisfying the inequality

$$n \leq x < n + 1$$

follows from the fact that the supremum of a set which is bounded above is unique.

Note: If x is any real number, then we have just shown that there exists one and only one integer n such that $n \leq x < n + 1$. The integer n is called the integral part of x and is denoted by $[x]$. Also $x - [x]$ is called the fractional part of x and is always non-negative.

Corollary: *For any $x \in \mathbf{R}$, there exists a unique integer n such that*

$$x - 1 < n \leq x.$$

Theorem 4: *For any $x \in \mathbf{R}$ there exists a unique integer n such that*

$$x - 1 \leq n < x.$$

Proof: For any $x \in \mathbf{R}$, we know that there exist two integers n_1 and n_2 such that $n_2 < x < n_1$.

Now let n be the largest integer among $n_2, n_2 + 1, n_2 + 2, ..., n_1$ such that $n < x$. Then

$$n + 1 \geq x \text{ or } n \geq x - 1 \text{ or } x - 1 \leq n.$$

Thus we have $n < x$ and $x - 1 \leq n$.

Combining these two inequalities, we have

$$x - 1 \leq n < x.$$

1.18 Representation of Real Numbers as Points on a Straight Line

The geometric representation of real numbers is suggestive and profitably useful to understand more clearly certain properties of real numbers.

Consider any straight line. Take a point O on it. The line is divided into two parts. The portion of the line to the right of O is called the positive part and the portion of the line to the left of O is called the negative part. Let A be any point on the positive part. Let the points O and A represent the numbers 0 and 1 respectively.

Taking OA as unit, we can associate with each real number, exactly one point on the line. Represent the positive real numbers by points to the right of O and the negative real numbers by points to the left of O. Also, each point on the line corresponds to one and only one real number. The line representing the real numbers is called the **real line R**. Thus we talk of a real number as a point of the real line R.

Dedikind-Cantor axiom: *To every real number there corresponds a unique point on a directed line and conversely, to every point on a directed line there corresponds a unique real number.*

We can say that there is a one-to-one correspondence between the real numbers and the points of a directed line. That is why the directed line is called the **real line** or **real axis** and **a real number is called a point of the real line.**

Note: If $a, b \in R$, $a < b$, then the point a lies to the left of the point b.

If $a, b \in R$, $a > b$, then the point a lies to the right of the point b.

The negative numbers lie to the left of O and the positive numbers lie to the right of O.

If $a, b, c \in R$ and $a < c < b$, then the point c lies between the points a and b.

If $a, b \in R$, then $|a - b|$ is called the distance between the points a and b.

1.19 The Denseness Property of The Real Number System

Theorem 1: *Between any two distinct real numbers there always lies a rational number and therefore infinitely many rational numbers.* **(Kanpur 2009)**

Proof: Let a and b be any two distinct real numbers and let $a < b$ so that $b - a > 0$.

Since $b - a > 0$, therefore by the Archimedean property of real numbers, there exists a positive integer n such that

$$n(b - a) > 1 \quad i.e. \quad nb > na + 1. \qquad ...(1)$$

Also there exists a unique integer m such that

$$m - 1 \le na < m$$

so that $$na + 1 \ge m > na. \qquad ...(2)$$

From (1) and (2), we have

$$nb > na + 1 \ge m > na.$$

This gives $$na < m < nb$$

or $$a < \frac{m}{n} < b.$$

Since m and n are integers and $n \neq 0$, therefore m / n is a rational number. If we denote m / n by r, then we have

$$a < r < b$$

and thus there exists a rational number r lying between a and b.

Repeating the above argument for a and r and r and b, we get rational numbers say r_1 and r_2 such that

$$a < r_1 < r \quad \text{and} \quad r < r_2 < b.$$

Combining these, we get $a < r_1 < r < r_2 < b$.

Continuing to proceed in this way we get infinitely many rational numbers between any two distinct real numbers a and b.

Theorem 2: *Between any two distinct real numbers there always lies an irrational number and therefore infinitely many irrational numbers.*

Proof: Let a and b be any two distinct real numbers and let $a < b$ so that $b - a > 0$.

Let α be any positive irrational number. Since $b - a > 0$, therefore by the Archimedean property of real numbers, there exists a positive integer n such that

$$n(b - a) > \alpha$$

i.e., $$b - a > \frac{\alpha}{n} \quad \text{i.e.,} \quad b > a + \frac{\alpha}{n}.$$

But since α is positive, we have

$$a + \frac{\alpha}{n} > a + \frac{\alpha}{2n} > a.$$

Hence $$b > a + \frac{\alpha}{n} > a + \frac{\alpha}{2n} > a. \qquad ...(1)$$

Now $\left(a + \frac{\alpha}{n}\right) - \left(a + \frac{\alpha}{2n}\right) = \frac{\alpha}{2n}$ which is an irrational number, α being an irrational number. It means that both the numbers $a + \frac{\alpha}{n}$ and $a + \frac{\alpha}{2n}$ cannot be rational otherwise their difference will be a rational number. So at least one of these two numbers is irrational. Let us denote it by x. Then from (1), we have

$$a < x < b$$

and thus there exists an irrational number x lying between a and b.

Repeating the above argument for a and x and x and b, we get irrational numbers say x_1 and x_2 such that

$$a < x_1 < x \quad \text{and} \quad x < x_2 < b.$$

Combining these, we get

$$a < x_1 < x < x_2 < b.$$

Continuing to proceed in this way we get infinitely many irrational numbers between any two distinct real numbers a and b.

Theorem 3: *Between any two distinct real numbers, there lie an infinite number of real numbers.*

The proof follows from any of the two theorems 1 and 2 above.

Comprehensive Exercise 3

1. Assuming as order-completeness axiom for real numbers that every non-empty subset of real numbers which is bounded below has an infimum, prove that every non-empty subset of real numbers which is bounded above has a supremum.
2. If a and b are any real numbers and $a > 1$, then prove that there exists a positive integer n such that $a^n > b$.
3. Show that the set S of all those positive rational numbers whose square is less than 3 is a non-empty subset of the set Q of all rational numbers, which is bounded above. Show also that S has no rational number for its supremum.

   [Hint. Proceed as in theorem 3 of article 1.6. Take $b = \dfrac{3 + 2a}{2 + a}$.]
4. Prove that the system Q of rational numbers has the Archimedean property *i.e.*, is an Archimedean ordered field.

1.20 Neighbourhood of a Point

Definition: *A subset N of $\mathbf{R}$ is said to be a neighbourhood of a point $p \in \mathbf{R}$ if there exists a real number $\varepsilon > 0$ such that*

$$]\, p - \varepsilon, p + \varepsilon\, [\subset N.$$

Since $p \in]\, p - \varepsilon, p + \varepsilon\, [$ and $]\, p - \varepsilon, p + \varepsilon\, [\subset N$,

therefore if N is a neighbourhood of p, we must have $p \in N$.

Thus $N \subset \mathbf{R}$ is a neighbourhood of a point $p \in \mathbf{R}$ if there exists an open interval contained in N whose centre is the point p.

For example, if N be the closed interval $[3, 5]$, then N is a neighbourhood of the point $4 \in [3, 5]$ because $\varepsilon = \frac{1}{4}$ is a positive real number such that

$$\left]4 - \frac{1}{4}, 4 + \frac{1}{4}\right[\subset [3, 5].$$

We shall use the abbreviated form 'nbd' or 'nhd' for the word neighbourhood.

Theorem 1: A characterisation of neighbourhood *i.e.*, an equivalent definition of neighbourhood.

A subset N of $\mathbf{R}$ is a neighbourhood of a point $p \in \mathbf{R}$ if and only if there exists an open interval $]a, b[$ containing p and contained in N i.e., if and only if there exists an open interval $]a, b[$ such that

$$p \in]a, b[\subset N.$$

Proof: 'Only if' part: First suppose that N is a nbd of p. Then by our definition of nbd of a point, there exists a positive real number ε such that

$$]p - \varepsilon, p + \varepsilon[\subset N.$$

Now $]p - \varepsilon, p + \varepsilon[$ is an open interval containing p and contained in N. Thus if N is a nbd of p, then there exists an open interval $]p - \varepsilon, p + \varepsilon[$ such that

$$p \in]p - \varepsilon, p + \varepsilon[\subset N.$$

'If ' part. Conversely suppose there exists an open interval $]a, b[$ such that

$$p \in]a, b[\subset N.$$

Then to prove that N is a nbd of p.

Let us take a positive real number ε equal to (or even less than) the minimum of the two positive real numbers $p - a$ and $b - p$.

Then $\varepsilon > 0$ is such that

$$\varepsilon \leq p - a \text{ and } \varepsilon \leq b - p \quad \text{i.e.,} \quad a \leq p - \varepsilon \text{ and } b \geq p + \varepsilon$$

$$\text{i.e.,} \quad a \leq p - \varepsilon < p < p + \varepsilon \leq b$$

$$\text{i.e.,} \quad]p - \varepsilon, p + \varepsilon[\subset]a, b[\subset N.$$

Thus there exists $\varepsilon > 0$ such that

$$p \in]p - \varepsilon, p + \varepsilon[\subset N.$$

Hence N is a nbd of p.

ε-neighbourhood of p. From the above theorem we conclude that if p is any real number, then any open interval I containing p is a neighbourhood of p and consequently all supersets of I are also neighbourhoods of p.

However, for any positive real number ε, $]p - \varepsilon, p + \varepsilon[$ is an open interval containing p and consequently it is a neighbourhood of p. It is a *symmetric nbd* of p. We often use this form of nbd of p and call it an ε-neighbourhood of p and denote it

by $N(p, \varepsilon)$ or by $N_\varepsilon(p)$. We refer to ε as the **radius** of $N(p, \varepsilon)$ and the point p itself is the **mid-point** or the **centre** of $N(p, \varepsilon)$. It is evident that

$$x \in N(p, \varepsilon) \text{ iff } |x - p| < \varepsilon.$$

Thus $N(p, \varepsilon)$ *i.e.*, ε-nbd of $p = \{x : x \in \mathbf{R} \text{ and } |x - p| < \varepsilon\}$

$$= \{x : x \in \mathbf{R} \text{ and } p - \varepsilon < x < p + \varepsilon\} =]p - \varepsilon, p + \varepsilon[.$$

Geometrically speaking an ε-nbd of p is the set of all the points on the real line which are within ε distance of p on either side of it.

Deleted neighbourhood of p. If from a nbd of a point p, the point p itself is deleted or excluded, then we get a **deleted neighbourhood** of p. Thus if N is a nbd of a point p, then the set $N - \{p\}$ is a deleted nbd of p. **(Kanpur 2010)**

A *symmetric deleted nbd* of a point p will be of the form

$$]p - \varepsilon, p + \varepsilon[- \{p\},$$

where ε is any positive real number.

We shall denote it by $N^{(d)}(p, \varepsilon)$.

Thus an ε-nbd of p is a set of the form

$$\{x \in \mathbf{R} : |x - p| < \varepsilon\}$$

and a deleted symmetric nbd of p is a set of the form

$$\{x \in \mathbf{R} : 0 < |x - p| < \varepsilon\}.$$

Precisely speaking a symmetric deleted nbd of p is the union of two bounded intervals

$$]p - \varepsilon, p[\text{ and }]p, p + \varepsilon[.$$

Theorem 2: *A non-empty subset A of $\mathbf{R}$ is a nbd of $p \in \mathbf{R}$ if and only if there exists a positive integer n such that*

$$\left]p - \frac{1}{n}, p + \frac{1}{n}\right[\subset A.$$

Proof: Let a non-empty subset A of $\mathbf{R}$ be a nbd of a point $p \in \mathbf{R}$. Then there exists $\varepsilon > 0$ such that

$$p \in]p - \varepsilon, p + \varepsilon[\subset A.$$

Now given any positive real number ε, we can always choose a positive integer n so large that $1/n < \varepsilon$.

But $1/n < \varepsilon \Rightarrow p + (1/n) < p + \varepsilon$.

Also $1/n < \varepsilon \Rightarrow -1/n > -\varepsilon$

$\Rightarrow$ $p - (1/n) > p - \varepsilon$.

$\therefore$ $p - \varepsilon < p - (1/n) < p + (1/n) < p + \varepsilon$

$\Rightarrow$ $\left]p - \frac{1}{n}, p + \frac{1}{n}\right[\subset]p - \varepsilon, p + \varepsilon[.$

Hence if A is a nbd of p, there exists a positive integer n such that

$$\left]p-\frac{1}{n}, p+\frac{1}{n}\right[\subset A.$$

Conversely suppose there exists a positive integer n such that

$$\left]p-\frac{1}{n}, p+\frac{1}{n}\right[\subset A.$$

Then A is a nbd of p because $\left]p-\frac{1}{n}, p+\frac{1}{n}\right[$ is an open interval containing p and contained in A.

Hence the theorem.

Illustrative Examples

Example 18: *Any open interval is a nbd of each of its points.*

Solution: Let $]a, b[$ be any open interval and x be any arbitrary point of $]a, b[$. Then to show that $]a, b[$ is a nbd of x.

If we take ε as the minimum of the two positive numbers $x-a$ and $b-x$, then $\varepsilon > 0$ is such that

$$x \in]x-\varepsilon, x+\varepsilon[\subset]a, b[.$$

Hence $]a, b[$ is a nbd of x.

Since x is an arbitrary point of $]a, b[$, therefore we conclude that any open interval is a nbd of each of its points.

Example 19: *A closed interval $[a, b]$ is a nbd of each of its points except the two end points a and b.*

Solution: Let $x \in]a, b[$.

Take $\quad \varepsilon = \min\{x-a, b-x\}$.

Then $\quad \varepsilon \le x-a \Rightarrow a \le x-\varepsilon$

and $\quad \varepsilon \le b-x \Rightarrow x+\varepsilon \le b$.

$\therefore \quad a \le x-\varepsilon < x+\varepsilon \le b$.

Thus $\varepsilon > 0$ is such that

$$x \in]x-\varepsilon, x+\varepsilon[\subset]a, b[\subset [a, b].$$

$\therefore$ $[a, b]$ is a nbd of $x \in]a, b[$.

$\therefore$ $[a, b]$ is a nbd of each point of $]a, b[$.

Again $[a, b]$ is not a nbd of $a \in [a, b]$. For if we take any positive real number ε, then $]a-\varepsilon, a+\varepsilon[\not\subset [a, b]$. Note that if $x \in]a-\varepsilon, a+\varepsilon[$ and $a-\varepsilon < x < a$, then $x \notin [a, b]$ and therefore $]a-\varepsilon, a+\varepsilon[\not\subset [a, b]$. Thus there exists no $\varepsilon > 0$ such that

$$]a-\varepsilon, a+\varepsilon[\subset [a, b].$$

Hence $[a, b]$ is not a nbd of a.

Similarly $[a, b]$ is not a nbd of $b \in [a, b]$. For if we take any positive real number ε, then $]\, b - \varepsilon, b + \varepsilon\, [\not\subset [a, b]$. Note that if $x \in]\, b - \varepsilon, b + \varepsilon\, [$ and $b < x < b + \varepsilon$, then $x \notin [a, b]$ and so $]\, b - \varepsilon, b + \varepsilon\, [\not\subset [a, b]\ \forall \varepsilon > 0$. Hence $[a, b]$ is not a nbd of b.

Example 20: *The set of rational numbers* Q *is not a nbd of any of its points.*

Solution: Let $p \in \mathbf{Q}$. For any positive real number ε, $p - \varepsilon$ and $p + \varepsilon$ are two distinct real numbers and we know that between any two distinct real numbers there lie infinite irrational numbers which are not members of Q.

$\therefore$ $\quad]\, p - \varepsilon, p + \varepsilon\, [\not\subset \mathbf{Q} \quad \forall \varepsilon > 0.$

$\therefore$ Q is not a nbd of p.

Since p is an arbitrary point of Q, therefore Q is not a nbd of any of its points.

Example 21: *Is it true that the null set* $\varnothing$ *is a nbd of each of its points ?*

Solution: Yes. The null set $\varnothing$ is a nbd of each of its points because there is no point at all in $\varnothing$ and so there is no point in $\varnothing$ of which it is not a nbd.

Example 22: *Show that a non-empty finite set cannot be a nbd of any of its points.*

Solution: Let S be any non-empty finite set and let $x \in S$. Then for any positive real number ε, the open interval $]\, x - \varepsilon, x + \varepsilon\, [$ is an infinite set *i.e.*, it contains an infinite number of distinct elements and so it cannot be a subset of a finite set S. Thus there exists no $\varepsilon > 0$ such that

$$]\, x - \varepsilon, x + \varepsilon\, [\subset S.$$

$\therefore$ S is not a nbd of x and hence S is not a nbd of any of its points.

Example 23: *Show that the set* $\mathbf{Z}^+$ *of all positive integers is not a nbd of any of its points.*

Or

Show that the set N *of all natural numbers is not a neighbourhood of any of its points.*

Solution: Let $x \in \mathbf{Z}^+$. Then

$$]\, x - \varepsilon, x + \varepsilon\, [\not\subset \mathbf{Z}^+ \quad \forall \varepsilon > 0.$$ [Give argument as in Ex. 20]

Hence $\mathbf{Z}^+$ is not a nbd of any $x \in \mathbf{Z}^+$.

Example 24: *Show that the set* R *of all real numbers is a nbd of each of its points.*

Solution: Let $x \in \mathbf{R}$. Then for every positive real number ε, we have

$$x \in]\, x - \varepsilon, x + \varepsilon\, [\subset \mathbf{R}.$$

Hence R is a nbd of every $x \in \mathbf{R}$.

Example 25: *Show that any set* S *cannot be a nbd of any point of the set* $\mathbf{R} - S$.

Solution: Let $x \in \mathbf{R} - S$. Then $x \notin S$.

Since $\forall \varepsilon > 0$, $x \in]\, x - \varepsilon, x + \varepsilon\, [$, therefore

$$]\, x - \varepsilon, x + \varepsilon\, [\not\subset S \quad \forall \varepsilon > 0.$$

Hence S is not a nbd of any $x \in \mathbf{R} - S$.

Example 26: *Which of the following subsets of* R *are nbds of* 3 *? Give reasons.*

(*i*) $]2, 4[$; (*ii*) $[2, 4[$; (*iii*) $]2, 4]$; (*iv*) $[2, 4]$

(*v*) $]3, 7[$; (*vi*) $]3, 5]$, (*vii*) $[3, 6[$ (*viii*) $[2, 4] - \left\{3\frac{1}{4}\right\}$.

Solution: (i) Since $]2, 4[$ is an open interval and $3 \in]2, 4[$, therefore $]2, 4[$ is a nbd of 3.

(ii) Since $3 \in$ the open interval $]2, 4[$ which is a subset of $[2, 4[$, therefore $[2, 4[$ is a nbd of 3.

(iii) Since $3 \in]2, 4[\subset]2, 4]$, therefore $]2, 4]$ is a nbd of 3.

(iv) Since there exists an open interval $]2, 4[$ such that

$$3 \in]2, 4[\subset [2, 4],$$

therefore $[2, 4]$ is a nbd of 3.

(v) Since $3 \notin]3, 7[$, therefore $]3, 7[$ cannot be a nbd of 3.

(vi) Since $3 \notin]3, 5]$, therefore $]3, 5]$ is not a nbd of 3.

(vii) $[3, 6[$ is not a nbd of $3 \in [3, 6[$ because

$$]3 - \varepsilon, 3 + \varepsilon[\not\subset [3, 6[\ \forall \varepsilon > 0.$$

(viii) $[2, 4] - \left\{3\frac{1}{4}\right\}$ is a nbd of 3 since there exists an open interval $\left]3 - \frac{1}{5}, 3 + \frac{1}{5}\right[$ such that

$$3 \in \left]3 - \frac{1}{5}, 3 + \frac{1}{5}\right[\subset \left([2, 4] - \left\{3\frac{1}{4}\right\}\right).$$

Example 27: *Let* $I_n = \left]-\frac{1}{n}, 1 + \frac{1}{n}\right[$ *be an open interval for each* $n \in \mathbb{N}$, *the set of natural numbers. Find* $\bigcap_{n=1}^{\infty} I_n$ *and show that it is a nbd of each of its points with the exception of two points.*

Solution: First we shall show that

$$\bigcap_{n=1}^{\infty} I_n = [0, 1].$$

We have $x \in [0, 1] \Rightarrow 0 \le x \le 1$

$\Rightarrow$ $x > -\frac{1}{n}$ and $x < 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$

$\Rightarrow$ $x \in \left]-\frac{1}{n}, 1 + \frac{1}{n}\right[$ for all $n \in \mathbb{N}$

$\Rightarrow$ $x \in \bigcap_{n=1}^{\infty} I_n$.

$\therefore$ $[0, 1] \subset \bigcap_{n=1}^{\infty} I_n$. ...(1)

Again let $x \in \bigcap_{n=1}^{\infty} I_n$ *i.e.*, $x \in I_n \ \forall\ n \in \mathbf{N}$. Then to show that $x \in [0, 1]$. If possible let $x \notin [0, 1]$.

Then either $x > 1$ or $x < 0$.

If $x > 1$, let $x = 1 + b$ where $b > 0$. Then there exists $m \in \mathbf{N}$ such that $1/m < b$ so that $1 + (1/m) < 1 + b$ or $1 + b > 1 + (1/m)$. But then $x = 1 + b \notin \left]-\frac{1}{m}, 1+\frac{1}{m}\right[$ which contradicts our assumption that $x \in I_n \ \forall\ n \in \mathbf{N}$.

Similarly if $x < 0$, let $x = -b$ where $b > 0$. Then there exists $m \in \mathbf{N}$ such that $1/m < b$ or $-1/m > -b$. But then $x = -b \notin \left]-\frac{1}{m}, 1+\frac{1}{m}\right[$ which contradicts our assumption that $x \in I_n \ \forall n \in \mathbf{N}$.

$\therefore$ if $x \in \bigcap_{n=1}^{\infty} I_n$, then definitely $x \in [0, 1]$,

so that $\bigcap_{n=1}^{\infty} I_n \subset [0, 1]$. ...(2)

From (1) and (2), we conclude that

$$\bigcap_{n=1}^{\infty} I_n = [0, 1].$$

Now $[0, 1]$ is a closed interval. It is a nbd of each of its points except its end points 0 and 1.

1.21 Properties of Neighbourhoods

Theorem 1: *On the real line* $\mathbf{R}$ *, for each point* $p \in \mathbf{R}$ *, there exists at least one nbd of* p.

Proof: If $p \in \mathbf{R}$, then $\mathbf{R}$ is always a nbd of p because for each $\varepsilon > 0$, we have

$$p \in]p - \varepsilon, p + \varepsilon[\subset \mathbf{R}.$$

Theorem 2: *If* N *is a nbd of any point* $p \in \mathbf{R}$*, then* $p \in N$.

Proof: If N is a nbd of $p \in \mathbf{R}$, then by the definition of a nbd of a point, there exists $\varepsilon > 0$ such that

$$p \in]p - \varepsilon, p + \varepsilon[\subset N.$$

Hence if N is a nbd of p, then $p \in N$.

Theorem 3: *Any superset of a nbd of a point is also a nbd of that point.*

Proof: Let N be a nbd of a point $p \in \mathbf{R}$.

Then there exists $\varepsilon > 0$ such that

$$p \in]p - \varepsilon, p + \varepsilon[\subset N. \quad ...(1)$$

If $M \supset N$, then from (1) it follows that

$$p \in]p - \varepsilon, p + \varepsilon[\subset N \subset M,$$

so that M is also a nbd of p.

Theorem 4: *The intersection of two nbds of a point is also a nbd of that point.*

Proof: Let M, N be two nbds of a point p. Then there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$]\, p-\varepsilon_1, p+\varepsilon_1\, [\subset M,$$

and $$]\, p-\varepsilon_2, p+\varepsilon_2\, [\subset N.$$

Take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then

$$]\, p-\varepsilon, p+\varepsilon [\subset]\, p-\varepsilon_1, p+\varepsilon_1\, [\subset M,$$

and $$]\, p-\varepsilon, p+\varepsilon\, [\subset]\, p-\varepsilon_2, p+\varepsilon_2\, [\subset N.$$

It follows that $]\, p-\varepsilon, p+\varepsilon\, [\subset M \cap N$ and consequently $M \cap N$ is a nbd of p.

Theorem 5: *On the real line* $\mathbf{R}$ *for each point* $p \in \mathbf{R}$ *and each nbd* N *of* p*, there exists a nbd* M *of* p *such that* $M \subset N$ *and* M *is a nbd of each of its points.*

Proof: Since N is a nbd of p, therefore there exists $\varepsilon > 0$ such that

$$p \in]\, p-\varepsilon, p+\varepsilon\, [\subset N.$$

Let $M =]\, p-\varepsilon, p+\varepsilon\, [$. Then as $]\, p-\varepsilon, p+\varepsilon\, [$ is an open interval containing p, so it is a nbd of p and also a nbd of each of its points.

Hence there exists a nbd M of p such that $M \subset N$ and M is a nbd of each of its points.

1.22 Some More Theorems on Neighbourhoods

Theorem 1: *If* a *and* b *are any two distinct real numbers, then there exist neighbourhoods of* a *and* b *which are disjoint. (This is known as* **Haousdorff property**.*)*

Proof: Take $\varepsilon = \frac{1}{3}|b-a|$. Then the nbd $N(a, \varepsilon)$ of a and the nbd $N(b, \varepsilon)$ of b are such that no real number exists which is a member of both these nbds.

Theorem 2: *Let* a *be any point of the nbd* $N(p, \varepsilon)$*. Then there exists a nbd of* a *which is entirely contained in* $N(p, \varepsilon)$*.*

Proof: $a \in N(p, \varepsilon) \Rightarrow |p-a| < \varepsilon$ so that

$$\varepsilon - |p-a| > 0.$$

Choose a positive real number δ such that

$$\delta < \varepsilon - |p-a|. \qquad \ldots(1)$$

We shall show that the nbd $N(a, \delta)$ of a is entirely contained in $N(p, \varepsilon)$ *i.e.*, $N(a, \delta) \subset N(p, \varepsilon)$.

Let $x \in N(a, \delta)$. Then

$$|x-a| < \delta < \varepsilon - |p-a|, \text{ by (1).} \qquad \ldots(2)$$

Now $|x-p| = |(x-a)+(a-p)| \le |x-a|+|a-p|$

$< \varepsilon - |p-a| + |a-p|$, by (2)

$= \varepsilon$.

Thus $|x-p| < \varepsilon$ which implies that $x \in N(p, \varepsilon)$.

$\therefore \quad x \in N(a, \delta) \Rightarrow x \in N(p, \varepsilon)$.

Hence $N(a, \delta) \subset N(p, \varepsilon)$.

Theorem 3: *Let a denote any point of the intersection M of the nbds $N(a_1, \varepsilon_1)$ and $N(a_2, \varepsilon_2)$ of a_1 and a_2. Then there exists a nbd of a which is entirely contained in M.*

Proof: Since $M = N(a_1, \varepsilon_1) \cap N(a_2, \varepsilon_2)$, therefore

$a \in M \Rightarrow a \in N(a_1, \varepsilon_1)$ and $a \in N(a_2, \varepsilon_2)$

$\Rightarrow |a-a_1| < \varepsilon_1$ and $|a-a_2| < \varepsilon_2$

$\Rightarrow \varepsilon_1 - |a-a_1| > 0$

and $\varepsilon_2 - |a-a_2| > 0$.

Let ε denote the minimum of the two positive real numbers

$\varepsilon_1 - |a-a_1|$ and $\varepsilon_2 - |a-a_2|$.

We shall show that the ε-nbd of a *i.e.* $N(a, \varepsilon)$ is entirely contained in M.

Let $x \in N(a, \varepsilon)$. Then

$$|x-a| < \varepsilon \le \varepsilon_1 - |a-a_1|. \qquad \ldots(1)$$

Now $|x-a_1| = |(x-a)+(a-a_1)|$

$\le |x-a| + |a-a_1|$

$< \varepsilon_1 - |a-a_1| + |a-a_1|$, by (1)

$= \varepsilon_1$.

Thus $|x-a_1| < \varepsilon_1$ which implies that $x \in N(a_1, \varepsilon_1)$.

$\therefore \quad x \in N(a, \varepsilon) \Rightarrow x \in N(a_1, \varepsilon_1)$.

$\therefore \quad N(a, \varepsilon) \subset N(a_1, \varepsilon_1)$. $\ldots(2)$

Similarly we can show that

$$N(a, \varepsilon) \subset N(a_2, \varepsilon_2). \qquad \ldots(3)$$

From (2) and (3), we have

$N(a, \varepsilon) \subset N(a_1, \varepsilon_1) \cap N(a_2, \varepsilon_2)$

$\Rightarrow N(a, \varepsilon) \subset M$.

Hence the result.

Comprehensive Exercise 4

1. Show that the closed interval [1, 3] is a nbd of 2 but not of any of its end points 1 and 3.
2. Show that the set Z of all integers is not a nbd of any of its points.
3. Give an example of each of the following :
   (i) a set which is a nbd of each of its points.
   (ii) a set which is not a nbd of any of its points.
   (iii) a set which is a nbd of each of its points with the exception of two points.
   (iv) a set which is a nbd of each of its points with the exception of one point.
4. Show that a set $N \subset \mathbf{R}$ is a nbd of a point $p \in \mathbf{R}$ if and only if there exists a positive rational number r such that
   $$]\, p - r, p + r\, [\subset N.$$
5. Show that the intersection of the family of all neighbourhoods of an arbitrary point $x \in \mathbf{R}$ is the singleton $\{x\}$.
   [**Hint.** If y be a point different from x, and $\varepsilon = |x - y|$, then obviously $]\, x - \varepsilon, x + \varepsilon\, [$ is a nbd of x that does not contain y.
6. Show that the set $S =]\, 2, 3\, [\cup]\, 5, 6\, [$ is a nbd of each of its points.
7. If $I = \bigcup_{n=1}^{\infty} I_n$, where each I_n is an open interval, then show that I is a nbd of each of its points.
8. Is the set $A = \{1, 2, 3, 4\}$ a nbd of 2 ?
9. Is Q the set of rational numbers a nbd of the point 4 ?
10. Show that the right half open interval $[\, 2, 3\, [$ is a nbd of each of its points except that of 2.
11. Show that the set of all irrational numbers is not a nbd of any of its points.

1.23 Adherent Points and Limit Points of a Set

Adherent point: **Definition:** *A point $p \in \mathbf{R}$ is said to be an* **adherent point** *of a set $A \subset \mathbf{R}$ if every neighbourhood of p contains a point of A. The set of all adherent points of A is called the adherence of A and is denoted by* **Adh** A.

From the above definition it is obvious that every point of A is an adherent point of A *i.e.*, $A \subset \text{Adh } A$.

Limit point or Limiting point.

Definition: *A point $p \in \mathbf{R}$ is said to be a* **limit point** *(or* **an accumulation point** *or* **a cluster point** *or* **a condensation point***) of a set $A \subset \mathbf{R}$ if every neighbourhood of p contains a point of A* **distinct from** *p.* **(Kanpur 2012)**

Thus p will be a limit point of A if and only if for each $\varepsilon > 0$, the open interval $]\, p - \varepsilon, p + \varepsilon\, [$ contains a point of A other than p.

Symbolically, a point $p \in \mathbf{R}$ will be a limit point of a subset A of $\mathbf{R}$ iff

$(N \cap A) - \{p\} \neq \varnothing$ for every nbd N of p

or $(N - \{p\}) \cap A \neq \varnothing$ for every nbd N of p

or $N \cap (A - \{p\}) \neq \varnothing$ for every nbd N of p

or $]\, p - \varepsilon, p + \varepsilon\, [\cap A) - \{p\} \neq \varnothing$ for every $\varepsilon > 0$.

From the definitions of a limit point and an adherent point of a set $A \subset \mathbf{R}$, it is obvious that every limit point of A is also an adherent point of A but the converse is not always true. For example 1 is an adherent point of the set $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ but it is not a limit point of this set.

A limit point of a set A may or may not belong to A.

Closed set: Definition: *A set $A \subset \mathbf{R}$ is said to be a* **closed set** *if it contains all its limit points.*

In order to show that a point p is not a **limit point** of a set A, it is enough to show that there exists a neighbourhood N of p, such that either $N \cap A = \{p\}$ or $N \cap A = \varnothing$. In other words a point p will not be a limit point of a set A if there exists some open interval I containing p which contains no points of A other than p or if there exists a real number $\varepsilon > 0$ such that the open interval $]\, p - \varepsilon, p + \varepsilon\, [$ contains no points of A other than p.

Isolated point: Definition: *A point $a \in A$ is said to be an* **isolated point** *of A if it is not a limit point of A i.e., if there exists a nbd of a which contains no points of A other than a itself. A set A is called a* **discrete set** *if all its points are isolated points.*

For example all the points of the set $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ are its isolated points and so it is a discrete set.

It should be noted that each point of a set A is either an isolated point of A or a limit point of A.

A dense-in-itself set and a perfect set: Definition: *A subset A of $\mathbf{R}$ is said to be* **dense-in-itself** *if it possesses no isolated points i.e., every point of A is a limit point of A. Further a set A is called a* **perfect set** *if it is dense-in-itself and if it contains all its limit points.* **(Kanpur 2012)**

Some important characterisations of limit points or equivalent definitions of a limit point.

Theorem 1: *A point $p \in \mathbf{R}$ is a limit point of a set $A \subset \mathbf{R}$ if and only if every neighbourhood of p contains* **infinitely many** *points of A.*

Proof: First suppose that every nbd of p contains infinitely many points of A. Then obviously every nbd of p contains a point of A which is different from p, and consequently, p is a limit point of A.

Conversely suppose that p is a limit point of A. Then to prove that every nbd of p contains infinitely many points of A.

Suppose there exists a nbd N of p which contains only finitely many points of A. Then there exists $\varepsilon > 0$ such that the open interval $]\, p - \varepsilon, p + \varepsilon\,[$ contains only finitely many points of A. If p is the only point of A which is contained in $]\, p - \varepsilon, p + \varepsilon\,[$, then p is not a limit point of A. If $]\, p - \varepsilon, p + \varepsilon\,[$ contains points of A other than p also, then since their number has been assumed to be finite, let they be $p_1, p_2, \ldots, p_n$. Out of these n points of A let p_r be the point which is nearest to p and let $|\, p_r - p\,| = \varepsilon_1$, so that,

$$\varepsilon_1 = \min \{|\, p_1 - p\,|, |\, p_2 - p\,|, \ldots, |\, p_n - p\,|\}.$$

Then the real number $\varepsilon_1 > 0$ is such that the open interval $]\, p - \varepsilon_1, p + \varepsilon_1\,[$ contains no points of A other than p. Thus there exists a nbd of p which contains no points of A other than p and so p is not a limit point of A.

Thus if there exists a nbd N of p which contains only finitely many points of A, then p is not a limit point of A. But this contradicts the hypothesis that p is a limit point of A. Hence if p is a limit point of A, then every nbd of p must contain infinitely many points of A.

Note: The above characterisation of a limit point will be very useful to solve some problems.

Theorem 2: *A point $p \in \mathbf{R}$ is a limit point of a set $A \subset \mathbf{R}$ iff for each neighbourhood N of p,*

$$(A \cap N) - \{p\} \neq \varnothing.$$

Proof: First suppose that p is a limit point of A. Let N be any nbd of p. Then there exists $\varepsilon > 0$ such that

$$p \in\,]\, p - \varepsilon, p + \varepsilon\,[\subset N. \qquad \ldots(1)$$

Since p is a limit point of A, therefore the open interval $]\, p - \varepsilon, p + \varepsilon\,[$ must contain a point of A other than p.

$\therefore$ $(A \cap\,]\, p - \varepsilon, p + \varepsilon\,[\,) - \{p\} \neq \varnothing$

$\Rightarrow$ $(A \cap N) - \{p\} \neq \varnothing$, from (1).

$\therefore$ if p is a limit point of A, then for each nbd N of p,

$$(A \cap N) - \{p\} \neq \varnothing.$$

Conversely suppose that for every nbd N of p, we have

$$(A \cap N) - \{p\} \neq \varnothing.$$

Then to prove that p is a limit point of A.

Take any positive real number ε. Then $]\, p - \varepsilon, p + \varepsilon\,[$ is a nbd of p and so by hypothesis

$$(\,]\, p - \varepsilon, p + \varepsilon\,[\cap A) - \{p\} \neq \varnothing$$

i.e. $]\, p - \varepsilon, p + \varepsilon\,[$ contains a point of A other than p.

Thus for every $\varepsilon > 0$, the open interval $]\, p - \varepsilon, p + \varepsilon\,[$ contains a point of A other than p. Hence p must be a limit point of A .

Some more theorems on limit points

Theorem 3: *If a non-empty subset S of* R *which is bounded above has no maximum member, then its supremum is a limit point of the set S.*

Proof: Let S be a non-empty subset of R which is bounded above. Then by the order-completeness property of real numbers, S has a supremum in R.

Let $\sup S = u$.

Since S has no maximum member, therefore $u \notin S$.

Take any positive real number ε. Since $\sup S = u$, therefore $u - \varepsilon$ cannot be an upper bound for S and so there exists $x \in S$ such that $x > u - \varepsilon$.

Since $u + \varepsilon > u$ and u is sup S, therefore $u + \varepsilon$ is also an upper bound for S and so $x \in S \Rightarrow x < u + \varepsilon$.

Thus for every $\varepsilon > 0$, there exists $x \in S$ such that

$$u - \varepsilon < x < u + \varepsilon.$$

Since $u \notin S$, therefore $x \neq u$.

$\therefore$ every ε-nbd $]\, u - \varepsilon, u + \varepsilon\,[$ of u contains a point x of S which is different from u. Hence u is a limit point of S.

Theorem 4: *If a non-empty subset S of* R *which is bounded below has no minimum member, then its infimum is a limit point of the set S.*

Proof: Let S be a non-empty subset of R which is bounded below. Then by the order-completeness property of real numbers, S has an infimum in R.

Let $\inf S = v$.

Since S has no minimum member, therefore $v \notin S$.

Take $\varepsilon > 0$. Since $\inf S = v$, therefore $v + \varepsilon$ cannot be a lower bound for S and so there exists $x \in S$ such that $x < v + \varepsilon$.

Since $v - \varepsilon < v$ and v is inf S, therefore $v - \varepsilon$ is also a lower bound for S and so $x \in S \Rightarrow x > v - \varepsilon$.

Thus for every $\varepsilon > 0$, there exists $x \in S$ such that

$$v - \varepsilon < x < v + \varepsilon.$$

Since $p \notin S$, therefore $x \neq p$.

$\therefore$ every ε-nbd $]p - \varepsilon, p + \varepsilon[$ of p contains a point x of S which is different from p. Hence p is a limit point of S.

1.24 Derived Sets

Definition: *The set of all limit points of a set $A \subset \mathbf{R}$ is called the derived set of A and is denoted by $D(A)$ or by A'.*

Thus $D(A) = \{x : x \text{ is a limit point of } A\}$.

Again the derived set of $D(A)$ is called the second derived set of A and is denoted by $D^2(A)$ or by A''. In general, the nth derived set of A is denoted by $D^{(n)}(A)$ or by $A^{(n)}$.

It can be easily seen that Adh $A = A \cup D(A)$.

Also it can be easily seen that if a set A is finite, then A has no limit point and consequently $D(A) = \varnothing$.

Definition. *A set is said to be of* **first species** *if it has only a finite number of derived sets. It is said to be of* **second species** *if the number of its derived sets is infinite.*

Note that if a set is of first species, then its last derived set must be empty.

A set whose nth derived set is a finite set so that its $(n+1)$th derived set is empty is called a set of ***n*th order**.

Illustrative Examples

Example 28: *Find the derived set of the set Q of all rational numbers.*

Solution: We shall show that every real number is a limit point of the set Q.

Let p be any real number and let $\varepsilon > 0$ be given. Then $p - \varepsilon$ and $p + \varepsilon$ are two distinct real numbers and we know that between two distinct real numbers there lie infinitely many rational numbers. Therefore for every $\varepsilon > 0$, the open interval $]p - \varepsilon, p + \varepsilon[$ contains at least one point of Q other than p. Hence p is a limit point of Q.

Since p is an arbitrary real number, therefore every real number is a limit point of Q. Hence the set of the limit points of Q is the set of all real numbers R.

$\therefore$ $D(Q) = R$.

Example 29: *Find the limiting points of the set of irrational numbers.*

Solution: We shall show that every real number is a limit point of the set R – Q of irrational numbers.

Let $p \in R$ and $\varepsilon > 0$ be given. Then between two distinct real numbers $p - \varepsilon$ and $p + \varepsilon$ there lie infinitely many irrational numbers. Therefore for every $\varepsilon > 0$, the open interval $]p - \varepsilon, p + \varepsilon[$ contains at least one point of the set of

irrational numbers which is distinct from p. Hence every real number p is a limit point of the set of irrational numbers *i.e.*, $D(\mathrm{R} - \mathrm{Q}) = \mathrm{R}$.

Example 30: *Show that every point of the set* R *of real numbers is a limit point of* R.

Solution: Let $p \in \mathrm{R}$. Then for every $\varepsilon > 0$, the open interval $]\, p - \varepsilon, p + \varepsilon\,[$ contains infinitely many real numbers. So for every $\varepsilon > 0$, the open interval $]\, p - \varepsilon, p + \varepsilon\,[$ contains at least one point of the set R which is distinct from p. Hence every real number p is a limit point of R *i.e.*, $D(\mathrm{R}) = \mathrm{R}$.

Note: We have $D(\mathrm{Q}) = \mathrm{R}$, $D^2(\mathrm{Q}) = D(\mathrm{R}) = \mathrm{R}$, $D^3(\mathrm{Q}) = \mathrm{R}$, $D^4(\mathrm{Q}) = \mathrm{R}$ and so on. Thus for every positive integer n, $D^{(n)}(\mathrm{Q}) = \mathrm{R}$. Therefore the number of derived sets of the set Q is infinite and so the set Q is of the second species.

Since each point of the set Q is its limit point, therefore the set Q has no isolated point. Hence the set Q is dense-in-itself. But the set Q is not perfect because it does not contain all its limit points. We have $D(\mathrm{Q}) = \mathrm{R} \not\subset \mathrm{Q}$.

Example 31: *Show that the set* N *of natural numbers has no limit points.*

Solution: Let p be any real number. The open interval $\left]\, p - \frac{1}{4}, p + \frac{1}{4}\,\right[$ whose length is $\frac{1}{2}$ contains at the most one natural number. Thus $\left]\, p - \frac{1}{4}, p + \frac{1}{4}\,\right[$ is a nbd of p which does not contain infinitely many points of the set N. Hence p is not a limit point of N.

Thus no real number p is a limit point of N *i.e.*, $D(\mathrm{N}) = \varnothing$.

Note. Since no point of the set N is a limit point of N, therefore all the points of N are isolated points. Hence N is a discrete set.

Example 32: *Show that the set* Z *of all integers has no limit points.*

Solution: Proceed as in Ex. 30 and show that no real number p is a limit point of the set Z. Hence $D(\mathrm{Z}) = \varnothing$.

Alternative solution: We shall show that every real number p is not a limit point of the set Z.

(i) If $p \in \mathrm{Z}$, then $\left]\, p - \frac{1}{4}, p + \frac{1}{4}\,\right[\cap \mathrm{Z} = \{p\}$.

(ii) If $p \notin \mathrm{Z}$ *i.e.*, if p is a real number which is not an integer then for some integer j, we have $j < p < j + 1$. Thus $]\, j, j + 1\,[$ is a nbd of p which does not contain any point of Z.

From (i) and (ii), we see that whatever real number p we may take, there exists a nbd of p which does not contain any point of Z other than p. Hence p is not a limit point of Z. Thus every real number p is not a limit point of Z and so Z has no limit points *i.e.*, $D(\mathrm{Z}) = \varnothing$.

Note: Since $D(\mathrm{Z}) = \varnothing \subset \mathrm{Z}$, therefore the set Z is a closed set.

Example 33: *Show that a finite set has no limit points.* **(Kanpur 2009)**

Solution: Let S be any finite set. We shall show that every real number p is not a limit point of the set S.

Since the set S is finite, therefore if we take any real number $\varepsilon > 0$, the open interval $]\, p - \varepsilon, p + \varepsilon\,[$ contains only finitely many points of the set S. Thus $]\, p - \varepsilon, p + \varepsilon\,[$ is a nbd of p which does not contain infinitely many points of S and so p is not a limit point of S.

Thus every real number p is not a limit point of S and so a finite set S has no limit points *i.e.*, the derived set of a finite set is empty.

Note 1: If S is a finite set, then $D(S) = \varnothing \subset S$. Hence every finite set is a closed set.

Note 2: The null set $\varnothing$ has no limit points *i.e.*, the derived set of the null set is the null set.

Thus $D(\varnothing) = \varnothing \subset \varnothing$ and so the null set $\varnothing$ is a closed set.

Example 34: *Find the limit points of the open interval* $]\, 0, 1\,[$.

Solution: Let $S =]\, 0, 1\,[$.

First we shall show that every point of the closed interval $[0, 1]$ is a limit point of S.

Let p be any point of $[0, 1]$ and ε be any positive real number. Then the open interval $]\, p - \varepsilon, p + \varepsilon\,[$ contains infinitely many points of $S =]\, 0, 1\,[$ and so it contains at least one point of $[0, 1]$ other than p. Hence p is a limit point of $]\, 0, 1\,[$.

Now we shall show that if $p \notin [0, 1]$ *i.e.*, if $p < 0$ or if $p > 1$, then p is not a limit point of $]\, 0, 1\,[$.

Suppose $p \notin [0, 1]$. If we take a positive real number ε such that ε is less than the distance of the point p from each of the end points 0 and 1 of the closed interval $[0, 1]$ *i.e.*, ε is less than each of the positive real numbers $|p - 0|$ and $|p - 1|$, then the open interval $]\, p - \varepsilon, p + \varepsilon\,[$ does not contain any point of the set $S =]\, 0, 1\,[$ and consequently p is not a limit point of S.

Thus if $p \notin [0, 1]$, then p is not a limit point of $]\, 0, 1\,[$ and if $p \in [0, 1]$, then p is a limit point of $]\, 0, 1\,[$. Hence p in a limit point of $]\, 0, 1\,[$ iff $p \in [0, 1]$.

$\therefore \qquad D(]\, 0, 1\,[) = [0, 1]$.

Note: Here we see that the points 0 and 1 do not belong to $[0, 1]$, but they are limit points of $]\, 0, 1\,[$. The set $]\, 0, 1\,[$ is dense-in-itself but is not perfect.

Example 35: *Find the limit points of the closed interval* $[0, 1]$.

Solution: Let $S = [0, 1]$.

Proceeding exactly as in Ex. 33, we can show that a real number p is a limit point of $[0, 1]$ iff $p \in [0, 1]$.

$\therefore \qquad D([0, 1]) = [0, 1]$.

Note 1: Since each point of $]\, 0, 1\,[$ is its limit point, therefore the set $[0, 1]$ is dense-in-itself. Also no point outside $]\, 0, 1\,[$ is a limit point of $[0, 1]$ *i.e.*, $[0, 1]$

contains all its limit points and therefore $[0, 1]$ is also a closed set. Since the set $[0, 1]$ is dense-in-itself and is also closed, therefore it is a perfect set.

Also if $S = [0, 1]$, then $D(S) = S$, $D^2(S) = S$, and so on. Thus $D^{(n)}(S) = S$ for every positive integer n. Therefore the closed interval $S = [0, 1]$ is a set of second species.

Note 2: Proceeding exactly as in Ex. 33, we can show that

$$D([0, 1[) = [0, 1] \quad \text{and} \quad D(]0, 1]) = [0, 1].$$

Example 36: *Find the set of the limit points of the set*

$$S = \{1/n : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}.$$

Solution: We shall show that the set $S = \{1/n : n \in \mathbb{N}\}$ has only one limit point, namely 0.

First, we show that 0 is a limit point of S.

Let $\varepsilon > 0$ be given. Then by Archimedean property of real numbers there exists a positive integer m such that $1/m < \varepsilon$, so that

$$-\varepsilon < 0 < 1/m < \varepsilon.$$

$\therefore$ for every $\varepsilon > 0$, the open interval $]-\varepsilon, \varepsilon[$ contains a point of S other than 0, namely $1/m$. Therefore 0 is a limit point of S.

Now we shall show that no real number p other than 0 can be a limit point of S. The following cases arise :

(i) $p < 0$. The open interval $]p - 1, 0[$ is a nbd of p which contains no point of S and consequently p is not a limit point of S.

(ii) $p > 1$. The open interval $]1, p + 1[$ is a nbd of p which contains no point of S and so p is not a limit point of S.

(iii) $0 < p < 1$ but $p \notin S$. In this case $1/p > 1$ and $1/p$ is not an integer. Therefore there exists a positive integer m such that

$$m < \frac{1}{p} < m + 1 \quad i.e., \quad \frac{1}{m+1} < p < \frac{1}{m}.$$

Thus $\left]\frac{1}{m+1}, \frac{1}{m}\right[$ is a nbd of p which contains no point of S and so p is not a limit point of S.

(iv) $p = 1$. The open interval $]\frac{1}{2}, 2[$ is a nbd of 1 which contains no point of S other than 1 and so 1 is not a limit point of S.

(v) $p \neq 1$ and $p \in S$. If $p = 1/m$, where $m \in \mathbb{N}$ and $m \neq 1$, then $\left]\frac{1}{m+1}, \frac{1}{m-1}\right[$ is a nbd of p which contains no point of S other than p and so p is not a limit point of S.

Hence no real number other than 0 is a limit point of S. Therefore 0 is the only limit point of S.

We have $D(S) = \{0\}$. Also $D^2(S) = D\{0\} = \varnothing$.

$\therefore$ the set S is of the first species and of first order.

Here we observe that the limit point 0 of S is not a point of the set S and so the set S is not closed.

Example 37: *Find the limit points of the set*

$$S = \left\{\frac{n}{n+1} : n \in \mathrm{N}\right\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots\right\}.$$

Solution: We shall show that the given set S has only one limit point, namely 1.

First we show that 1 is a limit point of S.

We observe that $\dfrac{n}{n+1} = \dfrac{(n+1)-1}{n+1} = 1 - \dfrac{1}{n+1}$.

Let $\varepsilon > 0$ be given. By Archimedean property of real numbers there exists a positive integer m such that

$$m > 1/\varepsilon \qquad \text{or} \qquad m + 1 > \frac{1}{\varepsilon}$$

$$\text{or} \quad \frac{1}{m+1} < \varepsilon \qquad \text{or} \qquad -\frac{1}{m+1} > -\varepsilon$$

$$\text{or} \quad 1 - \frac{1}{m+1} > 1 - \varepsilon \qquad \text{or} \qquad \frac{m}{m+1} > 1 - \varepsilon$$

$$\text{or} \quad 1 - \varepsilon < \frac{m}{m+1} < 1 + \varepsilon.$$

$\therefore$ for every $\varepsilon > 0$, the open interval $]1 - \varepsilon, 1 + \varepsilon[$ contains a point of S other than 1, namely $m/(m+1)$. Therefore 1 is a limit point of S.

Now we shall show that no real number p other than 1 can be a limit point of S. The following cases arise :

(i) $p > 1$. The open interval $]1, p + 1[$ is a nbd of p which contains no point of S and so p is not a limit point of S.

(ii) $p < 1$ and $p \notin S$. In this case let x_0 be the point of S which is nearest to p. If we choose a positive real number ε such that $\varepsilon < |p - x_0|$, then the open interval $]p - \varepsilon, p + \varepsilon[$ contains no point of S and so p is not a limit point of S.

(iii) $p < 1$ and $p \in S$.

Let $p = \dfrac{n}{n+1}$, where $n \in \mathrm{N}$.

Since $\dfrac{1}{2} < \dfrac{2}{3} < \dfrac{3}{4} < \ldots$, *i.e.*, $\dfrac{n}{n+1} < \dfrac{n+1}{(n+1)+1}$ for all $n \in \mathrm{N}$,

therefore the open interval

$$\left]\frac{n-1}{(n-1)+1}, \frac{n+1}{(n+1)+1}\right[$$

is a nbd of $n/(n+1)$ which contains no point of S other than $n/(n+1)$ and so p is not a limit point of S.

Hence no real number other than 1 is a limit point of S. Therefore 1 is the only limit point of S.

We have $D(S) = \{1\}$.

Note. Since the only limit point of S is 1 which is not a point of the set S, therefore each point of S is an isolated point and so the set S is a discrete set.

Example 38: *Find the derived set of the set*

$$S = \left\{\frac{1}{m} + \frac{1}{n} : m \in \mathrm{N}, n \in \mathrm{N}\right\}.$$

Solution: To find the derived set of S *i.e.*, to find the set of all limit points of S, we proceed as follows :

Keep m fixed and vary n. Then as n gets bigger, $1/n$ gets nearer to 0 so that $1/m + 1/n$ gets nearer to $1/m$ and consequently $1/m$ is a limit point of S. Since m is any member of N, therefore all the points of the set $\{1/m : m \in \mathrm{N}\}$ are limit points of the given set S.

Again vary both m and n so that $1/m + 1/n$ gets nearer and nearer to 0 and consequently 0 is also a limit point of S. Thus

$D(S) = \left\{\frac{1}{m} : m \in \mathrm{N}\right\} \cup \{0\}$. Note that $\left\{\frac{1}{m} : m \in \mathrm{N}\right\}$ and $\left\{\frac{1}{n} : n \in \mathrm{N}\right\}$ are the same sets so that it is immaterial if we keep n fixed and vary m.

Here $D^2(S) = \{0\}$, so that $D^3(S) = \varnothing$.

The set S is therefore of the first species and of second order.

1.25 Existence of Limit Points of a Set Bolzano-Weierstrass Theorem

We have seen that a finite set has no limit points. Also we have observed that an infinite set may or may not have a limit point. For example the infinite set N of natural numbers has no limit point whereas the infinite set $S = \{1/n : n \in \mathrm{N}\}$ has 0 as its limit point. We observe that the set S is bounded. Now we shall give a theorem which gives us a set of sufficient conditions for a set to have a limit point. This theorem is known as Bolzano-Weierstrass theorem and it is one of the most important theorems of Real Analysis.

Bolzano-Weierstrass theorem: *Every infinite bounded set of real numbers has a limit point.* **(Kanpur 2009)**

Proof. Let S be an infinite bounded set of real numbers and let h and k be its infimum and supremum respectively. Note that both h and k exist by order-completeness property of real numbers.

Let us define a set H as follows :

A real number x belongs to H iff it exceeds at most a finite number of members of S *i.e.*,

$H = \{x : x \in \mathrm{R}$ and the number of elements of S which are less than x is finite$\}$.

Since $h = \inf S$, therefore $x \geq h, \forall\ x \in S$.

$\therefore$ h exceeds no member of S and so by our definition of H, $h \in H$. Thus $H \neq \varnothing$.

Also no number greater than k is in H. For let $y > k = \sup S$. Then $x < y, \forall\ x \in S$. Therefore y exceeds all the members of S which are infinite because S is an infinite set. Therefore if $y > k$, then by our definition of H, $y \notin H$.

$\therefore$ the set H is bounded above with k as an upper bound.

Thus H is a non-empty subset of R bounded above by k. Hence by order-completeness property, H must have a supremum in R.

Let $\sup H = p$. We shall show that p is a limit point of S.

Let $\varepsilon > 0$ be given.

Since $\sup H = p$, therefore $p - \varepsilon$ cannot be an upper bound for H. Therefore there exists $x \in H$ such that $x > p - \varepsilon$ *i.e.*, $p - \varepsilon < x$.

Since $x \in H$, therefore, it exceeds at most a finite number of members of S and consequently $p - \varepsilon$ also exceeds at most a finite number of members of S.

Again since $p = \sup H$, $p + \varepsilon$ cannot belong to H and so $p + \varepsilon$ exceeds an infinite number of members of S.

Now $p - \varepsilon$ exceeds only a finite number of members of S and $p + \varepsilon$ exceeds an infinite number of members of S. It means that the nbd $]\ p - \varepsilon, p + \varepsilon\ [$ of p contains infinitely many points of S.

Since ε is arbitrary, therefore every ε-nbd of p contains infinitely many points of S. Hence p is a limit point of S. This proves the theorem.

Remark: The conditions in the above theorem are only sufficient conditions for a set S to have a limit point. These conditions are not necessary for a set S to have a limit point. Even an infinite unbounded set may have a limit point. For example the set Q of rational numbers is unbounded and still it has limit points. We know that every real number is a limit point of the set Q.

1.26 Some Theorems On Derived Sets

Theorem 1: *If $\varnothing$ be the empty set, then its derived set $D(\varnothing) = \varnothing$.*

Proof: If p be any point of R, then R is a nbd of p and $\mathrm{R} \cap \varnothing = \varnothing$ *i.e.*, R contains no point of $\varnothing$ because the empty set $\varnothing$ has no points at all. Therefore p is not a limit point of $\varnothing$. Thus no point of R is a limit point of $\varnothing$ and hence

$$D(\varnothing) = \varnothing.$$

Theorem 2: *If A and B are any subsets of* R, *then*

$$A \subset B \Rightarrow D(A) \subset D(B).$$

Proof: It is given that $A \subset B$.

Let $x \in D(A)$ *i.e.*, let x be any limit point of A. Then

$x \in D(A) \Rightarrow$ every nbd N of x contains a point of A other than x

$\Rightarrow$ every nbd N of x contains a point of B other than x $[\because B \supset A]$

$\Rightarrow$ x is a limit point of B

$\Rightarrow$ $x \in D(B)$.

$\therefore A \subset B \Rightarrow D(A) \subset D(B)$.

Theorem 3: *If A and B are any subsets of* R, *then*

$$D(A \cap B) \subset D(A) \cap D(B).$$

Proof. We know that $A \subset B \Rightarrow D(A) \subset D(B)$.

[See theorem 2 above.]

Now $A \cap B \subset A$ and $A \cap B \subset B$.

$\therefore$ $D(A \cap B) \subset D(A)$ and $D(A \cap B) \subset D(B)$.

Hence $D(A \cap B) \subset D(A) \cap D(B)$.

Theorem 4. *If A and B are any subsets of* R, *then*

$$D(A \cup B) = D(A) \cup D(B).$$

Proof: We know that $A \subset B \Rightarrow D(A) \subset D(B)$.

Now $A \subset A \cup B$ and $B \subset A \cup B$.

$\therefore$ $D(A) \subset D(A \cup B)$ and $D(B) \subset D(A \cup B)$.

$\therefore$ $D(A) \cup D(B) \subset D(A \cup B)$. ...(1)

Now we shall show that $D(A \cup B) \subset D(A) \cup D(B)$.

Let $x \in D(A \cup B)$. Then to show that $x \in D(A) \cup D(B)$.

For this we shall show that $x \notin D(A) \cup D(B) \Rightarrow x \notin D(A \cup B)$.

Suppose $x \notin D(A) \cup D(B)$. Then $x \notin D(A)$ and $x \notin D(B)$ *i.e.*, x is neither a limit point of A nor a limit point of B.

$\therefore$ there exists a nbd N of x which contains no point of A other than x and there exists a nbd M of x which contains no point of B other than x.

Since the intersection of two nbds of x is also a nbd of x, therefore $N \cap M$ is a nbd of x which contains no point of A other than x and no point of B other than x *i.e.*, which contains no point of $A \cup B$ other than x. Consequently x is not a limit point of $A \cup B$ *i.e.*, $x \notin D(A \cup B)$.

Thus $x \notin D(A) \cup D(B) \Rightarrow x \notin D(A \cup B)$.

$\therefore$ $x \in D(A \cup B) \Rightarrow x \in D(A) \cup D(B)$.

$\therefore$ $D(A \cup B) \subset D(A) \cup D(B)$. ...(2)

From (1) and (2), we conclude that

$$D(A \cup B) = D(A) \cup D(B).$$

Theorem 5: *If A be any subset of* R, *then*

$$x \in D(A) \Rightarrow x \in D(A - \{x\}).$$

Proof: We have $x \in D(A) \Rightarrow x$ is a limit of point A

$\Rightarrow$ every nbd N of x contains a point of A other than x

$\Rightarrow$ every nbd N of x contains a point of $A - \{x\}$ other than x

$\Rightarrow x$ is a limit point of $A - \{x\}$

$\Rightarrow \quad x \in D(A - \{x\}). \quad \therefore \; D(A) \subset D(A - \{x\}).$

Theorem 6: *The derived set of any bounded set is again a bounded set.*

Proof. Let S be a bounded set.

Then there exist $h, k \in$ R such that $S \subset [h, k]$.

We shall show that no member of the derived set $D(S)$ can be less than h or greater than k.

Let $p < h$. Then we shall show that p cannot be a limit point of S.

Let $\varepsilon = h - p$, so that $\varepsilon > 0$. Also then $h = p + \varepsilon$.

Since h is a lower bound for S, therefore $x \in S \Rightarrow x \geq h$

i.e., $\quad x \in S \Rightarrow x \nless p + \varepsilon. \qquad [\because \; h = p + \varepsilon]$

$\therefore \varepsilon > 0$ is such that the open interval $]\, p - \varepsilon, p + \varepsilon\, [$ contains no point of S.

$\therefore$ if $p < h$, then p is not a limit point of S.

Similarly we can show that if $p > k$, then p is not a limit point of S.

$\therefore$ all the limit points of S lie in the closed interval $[h, k]$.

Hence $D(S) \subset [h, k]$ *i.e.*, $D(S)$ is bounded.

Theorem 7: *Every bounded infinite set has the greatest and the smallest limit points i.e., the derived set of any infinite bounded set attains its bounds.*

Proof: Let S be an infinite bounded set. Since S is bounded, therefore there exist $h, k \in$ R such that $S \subset [h, k]$.

Now $S \subset [h, k] \Rightarrow D(S) \subset D([h, k])$.

But the derived set of the closed interval $[h, k]$ is $[h, k]$ *i.e.*, $D([h, k]) = [h, k]$.

$\therefore D(S) \subset [h, k]$, so that $D(S)$ is bounded.

Since S is an infinite bounded set, therefore by Bolzano-Weierstrass theorem, S has at least one limit point and so $D(S) \neq \varnothing$.

Thus $D(S)$ is a non-empty bounded subset of R.

$\therefore$ by order-completeness property of R, $D(S)$ has infimum as well as supremum in R.

Let $\quad \inf D(S) = p$ and $\sup D(S) = q$.

We shall show that both p and q belong to $D(S)$ *i.e.*, both p and q are limit points of S.

Let $\varepsilon > 0$ be given.

Now $p = \inf \text{ of } D(S) \Rightarrow$ there exists some $x \in D(S)$

such that $p \leq x < p + \varepsilon$

$\Rightarrow \quad p - \varepsilon < x < p + \varepsilon$

$\Rightarrow \quad x \in]\, p - \varepsilon, p + \varepsilon\, [$

$\Rightarrow \quad]\, p - \varepsilon, p + \varepsilon\, [$ is a nbd of some $x \in D(S)$

$\Rightarrow \quad]\, p - \varepsilon, p + \varepsilon\, [$ is a nbd of x which is a limit point of S

$\Rightarrow \quad]\, p - \varepsilon, p + \varepsilon\, [$ contains infinitely many points of S.

Thus for every $\varepsilon > 0$, the open interval $]\, p - \varepsilon, p + \varepsilon\, [$ contains infinitely many points of S and consequently p is a limit point of S.

Similarly we can show that q is also a limit point of S.

Thus both p and q *i.e.*, the inf and sup of $D(S)$ belong to $D(S)$. So p is the smallest and q the greatest member of $D(S)$.

$\therefore$ the set S has smallest and greatest limit points.

Definition: *The smallest and the greatest limit points of an infinite bounded set S, as they do exist by the above theorem, are called the* **lower limit (limit inferior)** and the **upper limit (limit superior)** *of S respectively.*

We usually denote these limits by $p = \underline{\lim}\, S$ and $q = \overline{\lim}\, S$.

Not that $\underline{\lim}\, S \leq \overline{\lim}\, S$.

1.27 Interior of a Set

Interior point of a set: Definition: *Let S be a subset of* R. *A point $p \in S$ is said to be an interior point of S if there exists a nbd $]\, p - \varepsilon, p + \varepsilon\, [$ of p which is entirely contained in S i.e., if there exists $\varepsilon > 0$ such that*

$$]\, p - \varepsilon, p + \varepsilon\, [\subset S.$$

(Kanpur 2012)

Interior of a set: Definition

If S is a subset of R, *then the set of all interior points of S is called the interior of S and is denoted by S^o or int (S). Obviously $S^o \subset S$.*

Illustrations

1. *Every point of an open interval is an interior point of the interval.*
   Let S denote the open interval $]\, a, b\, [$ and let $p \in S$ so that $a < p < b$. If we take $\varepsilon = \min \{p - a, b - p\}$ then $\varepsilon > 0$ is such that $]\, p - \varepsilon, p + \varepsilon\, [\subset S$. Therefore p is an interior point of S.
2. *Every point of a closed interval except the end points of the interval is an interior point of the interval.*
   Let S denote the closed interval $[a, b]$.
   We have $a \in S$. There exists no $\varepsilon > 0$ such that
   $]\, a - \varepsilon, a + \varepsilon\, [\subset S$.

Hence a is not an interior point of S.

Similarly b is also not an interior point of S.

But every point $p \in]\, a, b\,[$ is an interior point of $[a, b]$ as can be easily shown.

Hence if $S = [a, b]$, then $S^\circ =]\, a, b\,[$.

3. Let S denote the set of rational numbers Q. Then $S^\circ = \varnothing$.

If $p \in$ Q, then every nbd of p contains irrationals also. So there exists no $\varepsilon > 0$ such that $]\, p - \varepsilon, p + \varepsilon\,[\subset$ Q. Therefore p is not an interior point of Q. Hence Q has no interior points so that $Q^\circ = \varnothing$.

In a similar manner we can show that if $S = Z^+$, then $S^\circ = \varnothing$. Also $Z^\circ = \varnothing$.

4. If S denotes the set of real numbers R, then $S^\circ =$ R because every real number is an interior point of R.

1.28 Open Sets

Definition: *Let S be a subset of* R. *Then S is said to be open if every point of S is an interior point of S.*

The following two characterizations of an open set immediately follow from its definition.

(*i*) *A set S is open iff $S^\circ = S$*

(*ii*) *A set S is open iff it contains a neighbourhood of each of its points.*

Illustrations

1. *Every open interval is an open set, but the converse is not true.*

Let S denote the open interval $]\, a, b\,[$. Let $p \in S$ so that $a < p < b$.

Take $\varepsilon = \min\{p - a, b - p\}$. Then $\varepsilon > 0$ is such that

$$]\, p - \varepsilon, p + \varepsilon\,[\subset S.$$

Thus to each $p \in S$, there exists $\varepsilon > 0$ such that

$$]\, p - \varepsilon, p + \varepsilon\,[\subset S$$

i.e., S contains a nbd of each of its points.

Hence every open interval is an open set.

The converse need not be true *i.e., an open set need not be an open interval.* For example, if $S =]\, 2, 3\,[\cup]\, 5, 7\,[$, then S is an open set but S is not an open interval.

2. *A closed or semi-closed interval is not an open set.*

Let S denote the closed interval $[a, b]$. We have $a \in S$, but there exists no $\varepsilon > 0$ such that $]\, a - \varepsilon, a + \varepsilon\,[\subset S$. Thus S does not contain a nbd of the point $a \in S$. Hence S is not an open set.

Similarly, if $S = [\, a, b\,[$, then S is not open because $a \in S$ and there exists no $\varepsilon > 0$ such that $]\, a - \varepsilon, a + \varepsilon\,[\subset S$.

Also if $S =]\, a, b\,]$, then S is not open because $b \in S$ and there exists no $\varepsilon > 0$ such that $]\, b - \varepsilon, b + \varepsilon\,[\subset S$.

3. *Every non-empty finite set is not an open set.*

Let S denote any non-empty finite subset of R. If $p \in S$, then for every $\varepsilon > 0$, the open interval $]\, p - \varepsilon, p + \varepsilon\,[$ contains infinitely many points of R and so $]\, p - \varepsilon, p + \varepsilon\,[$ cannot be a subset of a finite set S. Thus there exists no nbd of p which is contained in S. Hence S is not an open set.

4. *The set of rational numbers* Q *is not an open set.*

We have $Q^o = \varnothing$, so that $Q^o \neq Q$. Hence Q is not an open set.

5. *The set of all real numbers* R *and the empty set* $\varnothing$ *are open.*

The set R of all real numbers is an open set. If $p \in R$, then for every $\varepsilon > 0$, we have $]\, p - \varepsilon, p + \varepsilon\,[\subset R$. Thus R contains a neighbourhood of each of its points. Hence R is an open set.

Now we show that the null set $\varnothing$ is an open set.

Since $\varnothing$ has no points the condition that $\varnothing$ contains a neighbourhood of each of its points is vaccuously satisfied. Hence $\varnothing$ is open.

Some important properties of open sets

Theorem 1: *The union of an arbitrary family of open sets is open.*

Proof: Let $\{G_\lambda : \lambda \in \Lambda\}$ be an arbitrary family of open sets. Here Λ is an index set and is such that for every $\lambda \in \Lambda$, G_λ is an open set.

Let $G = \cup \{G_\lambda : \lambda \in \Lambda\}$. Then in order to show that G is an open set, we shall show that every point of G is an interior point of G.

Let $p \in G$. Since G is the union of the family $\{G_\lambda\}$, therefore $p \in G_\lambda$ for some $\lambda \in \Lambda$.

Since G_λ is an open set and $p \in G_\lambda$, therefore there must exist some $\varepsilon > 0$ such that $]\, p - \varepsilon, p + \varepsilon\,[\subset G_\lambda \subset G$. Note that G is the union of the family $\{G_\lambda\}$.

Thus $]\, p - \varepsilon, p + \varepsilon\,[\subset G$ and so p is an interior point of G.

Since every point of G is an interior point of G, therefore G is an open set.

Hence the union of an arbitrary family of open sets is an open set.

Note: In particular the union of two open sets is an open set.

Theorem 2: *The intersection of a finite collection of open sets is an open set.*

(Kanpur 2009)

Proof: Let $G = \bigcap_{i=1}^{n} G_i$, where each G_i is an open set.

If $G = \varnothing$, then G is an open set and the proof is complete. So let $G \neq \varnothing$.

Let $p \in G$. Then $p \in G_i$ for each $i = 1, 2, \ldots, n$.

Since each G_i is an open set, therefore for every $i = 1, 2, \ldots, n$, there exists $\varepsilon_i > 0$ such that

$$]\, p - \varepsilon_i, p + \varepsilon_i\,[\subset G_i.$$

Let $\varepsilon = \min [\varepsilon_1, \varepsilon_2, ..., \varepsilon_n]$.

Then $]p - \varepsilon, p + \varepsilon[\subset]p - \varepsilon_i, p + \varepsilon_i[\subset G_i$ for each i

$\Rightarrow$ $]p - \varepsilon, p + \varepsilon[\subset \bigcap_{i=1}^{n} G_i$

$\Rightarrow$ $]p - \varepsilon, p + \varepsilon[\subset G$

$\Rightarrow$ p is an interior point of G.

Thus every point p of G is an interior point of G and so G is an open set.

Hence a finite intersection of open sets is an open set.

Remark. The intersection of an infinite collection of open sets is not necessarily an open set. The following example supports this statement.

Let $G_n =]-1/n, 1/n[, n \in \mathbb{N}$.

Then each G_n is an open set because every open interval is an open set.

Now $\bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty}]-1/n, 1/n[= \{0\}$ which is not open because there exists no $\varepsilon > 0$ such that $]-\varepsilon, \varepsilon[\subset \{0\}$.

Hence the intersection of an infinite collection of open sets is not necessarily an open set.

1.29 Closed Sets

Definition: *Let F be a subset of* R. *Then F is said to be closed if its complement F' is open.*

Note that $F' = \mathbb{R} - F = \{x : x \in \mathbb{R} \text{ and } x \notin F\}$. **(Kanpur 2010)**

Illustrations :

1. *Every closed interval is a closed set.*

Let $F = [a, b]$. Then $F' =]-\infty, a[\cup]b, \infty[$.

Here $]-\infty, a[$ and $]b, \infty[$ are both open sets and we know that the union of two open sets is also an open set.

$\therefore$ $F' =]-\infty, a[\cup]b, \infty[$ is an open set. Hence by definition F is closed.

2. *The right half open interval $[a, b[$ is not closed*

Let $F = [a, b[$. Then $F' =]-\infty, a[\cup [b, \infty[$.

Obviously F' is not an open set because $b \in F'$ and b is not an interior point of F'.

Note that there exists no $\varepsilon > 0$ such that $]b - \varepsilon, b + \varepsilon[\subset F'$.

Since F' is not open, therefore F is not closed.

Remark: Similarly we can show that every open interval $]a, b[$ is not a closed set and the left half open interval $]a, b]$ is also not a closed set.

3. *Every singleton set in* R *is closed.*

Let $F = \{a\}$. Then $F' =]-\infty, a[\cup]a, \infty[$ which is open being the union of two open sets. Hence by definition F is closed.

4. *The set* Q *of all rationals is not closed.*

We have Q′ = R − Q = the set of all irrational numbers.

The set Q′ is not open because if $p \in Q'$, then for every $\varepsilon > 0$ the open interval $]p - \varepsilon, p + \varepsilon[$ contains rational numbers also and so is not a subset of Q′. Thus if $p \in Q'$, then p is not an interior point of Q′ and so Q′ is not open. Hence Q is not closed.

5. *The set of all real numbers* R *and the null set* ∅ *are closed.*

We have R′ = ∅. Since ∅ is open, therefore R′ is open. Hence R is closed.

Again ∅′ = R. Since R is open, therefore ∅′ is open. Hence ∅ is closed.

Some important properties of closed sets.

Theorem 1: *The union of a finite collection of closed sets is closed.*

Proof: Let $F_1, F_2, \ldots, F_n$ be n closed sets. Then to prove that $\bigcup_{i=1}^{n} F_i$ is closed.

Since each F_i $(i = 1, 2, \ldots, n)$ is a closed set, therefore each F_i' is an open set.

By theorem 2 of article 1.28, we know that the intersection of a finite collection of open sets is open.

$\therefore \quad \bigcap_{i=1}^{n} F_i'$ is open.

But by De Morgan's law $\left(\bigcup_{i=1}^{n} F_i \right) = \bigcap_{i=1}^{n} F_i'$.

$\therefore \quad \left(\bigcup_{i=1}^{n} F_i \right)'$ is an open set. Hence by definition of a closed set, $\bigcup_{i=1}^{n} F_i$ is a closed set.

Theorem 2: *The intersection of an arbitrary collection of closed sets is closed.*

Proof: Let $\{F_\lambda : \lambda \in \Lambda\}$ be an arbitrary family of closed sets. Here Λ is an index set and is such that for every $\lambda \in \Lambda$, F_λ is a closed set.

Now to prove that $\cap \{F_\lambda\}$ is a closed set. Since each F_λ is a closed set, therefore each F_λ' is an open set.

By theorem 1 of article 1.28, we know that the union of an arbitrary family of open sets is open.

$\therefore \quad \cup \{F_\lambda' : \lambda \in \Lambda\}$ is open.

But by De Morgan's law, $|\cap \{F_\lambda : \lambda \in \Lambda\}|' = \cup \{F_\lambda' : \lambda \in \Lambda\}$.

$\therefore \quad |\cap \{F_\lambda : \lambda \in \Lambda\}|'$ is an open set.

Hence $\cap \{F_\lambda : \lambda \in \Lambda\}$ is a closed set.

Example 39: *Every finite subset of* R *is a closed set.*

Solution: Let $F = \{a_1, a_2, \ldots, a_n\}$ be a finite subset of R. Then to prove that F is closed.

We know that every singleton in R is a closed set. Also the union of a finite collection of closed sets is closed. The set F can be expressed as the union of n singletons *i.e.*, we can write

$$F = \{a_1\} \cup \{a_2\} \cup \ldots \cup \{a_n\}.$$

Hence F is a closed set.

Example 40: *Give examples to show that the union of an infinite collection of closed sets is not necessarily closed.*

Solution: Let $F_n = [1/n, 1], n \in \mathbb{N}$.

Then each F_n is a closed set in R because each closed interval is a closed set.

Now $\cup \{F_n : n \in \mathbb{N}\}$

$$= \{1\} \cup [\frac{1}{2}, 1] \cup [\frac{1}{3}, 1] \cup [\frac{1}{4}, 1] \cup \ldots =]0, 1].$$

Since $]0, 1[$ is not a closed set in R, therefore it follows that the union of an infinite collection of closed sets is not necessarily closed.

As an other illustration consider the set Q of all rational numbers.

We can write $\mathbb{Q} = \cup \{r\}. r \in \mathbb{Q}$

Since each singleton set in R is closed, therefore Q is expressed as an infinite union of closed sets. But we know that Q is not closed since its complement Q′ is not open. Hence it follows that an infinite union of closed sets is not necessarily closed.

1.30 Closure of a Set

Definition: *Let A be any subset of* R. *Then the closure of A is defined as the smallest closed set containing A and is denoted by $\bar{A}$ or cl (A).*

The following two results are quite obvious :

(*i*) *$\bar{A}$ is the intersection of the family of all closed sets containing A.*

(*ii*) *A is closed iff $\bar{A} = A$.*

Some theorems connecting closed sets and limit points.

Theorem 1: A characterization of closed sets. *Let A be a subset of* R. *Then A is closed iff $D(A) \subset A$ i.e., A is closed iff A contains all its limit points.*

Proof: Let A be closed so that A' is open. If $D(A) = \varnothing$, then obviously $D(A) \subset A$.

So let $D(A) \neq \varnothing$. Let $p \in D(A)$ *i.e.*, let p be a limit point of A. We wish to show that $p \in A$.

Suppose, if possible, $p \notin A$. Then $p \in A'$ and, since A' is open, there exists $\varepsilon > 0$ such that $]p - \varepsilon, p + \varepsilon[\subset A'$. Consequently $]p - \varepsilon, p + \varepsilon[\cap A = \varnothing$ *i.e.*, $]p - \varepsilon, p + \varepsilon[$ contains no point of A, contradicting the fact that p is a limit point of A. Hence we must have $p \in A$.

Thus if A is closed, then $D(A) \subset A$.

Conversely, we assume that $D(A) \subset A$ *i.e.*, A contains all its limit points. Then we wish to show that A is closed. By definition A will be closed if A' is open.

Let $p \in A'$. Then $p \notin A$, and so $p \notin D(A)$ because by hypothesis $D(A) \subset A$.

Now $p \notin D(A) \Rightarrow p$ is not a limit point of A

$\Rightarrow$ there exists $\varepsilon > 0$ such that $]\, p - \varepsilon, p + \varepsilon \,[$ contains no point of A other than p

$\Rightarrow$ there exists $\varepsilon > 0$ such that $]\, p - \varepsilon, p + \varepsilon \,[$ contains no point of A because $p \notin A$

$\Rightarrow$ there exists $\varepsilon > 0$ such that $]\, p - \varepsilon, p + \varepsilon \,[\subset A'$

Thus A' contains a neighbourhood of each of its points and so A' is open. Hence A is closed.

Some Illustrations

1. Every finite set is closed. Note that a finite set has no limit points and so the condition that a finite set contains all its limit points is vaccuously satisfied.

2. The set of all rational numbers Q is not closed because $D(\mathrm{Q}) = \mathrm{R}$ is not contained in Q.

Theorem 2: *Let A be any subset of* R. *Then $D(A)$ is a closed set.*

Proof: In order to prove that $D(A)$ is closed we shall show that $D(A)$ contains all its limit points. [Refer theorem 1 above.]

Let p be any limit point of $D(A)$. Then every ε-nbd of p contains infinitely many points of $D(A)$ and since each point of $D(A)$ is a limit point of A, every ε-nbd of p must contain infinitely many points of A. Thus, p is also a limit point of A and so $p \in D(A)$. Therefore $D(A)$ contains all its limit points and so $D(A)$ is closed.

Theorem 3: *Let A be any subset of* R. *Then $\overline{A} = A \cup D(A)$ i.e., $\overline{A}$ is the set of all adherent points of A.*

Proof: We shall first prove that $A \cup D(A)$ is a closed set.

Let p be any limit point of $A \cup D(A)$. Then either p is a limit point of A or a limit point of $D(A)$. If p is a limit point of A, then $p \in D(A)$. If p is a limit point of $D(A)$, then since $D(A)$ is closed, $p \in D(A)$. So in either case $p \in D(A)$ and surely then $p \in A \cup D(A)$. Thus $A \cup D(A)$ contains all its limit points and hence $A \cup D(A)$ is closed.

Now $A \cup D(A)$ is a closed set containing A, and since $\overline{A}$ is the smallest closed set containing A, we have

$$\overline{A} \subset A \cup D(A). \qquad \ldots(1)$$

Also $A \subset \overline{A} \Rightarrow D(A) \subset D(\overline{A})$. $\qquad \ldots(2)$

[Refer theorem 2 of article 1.26]

Since $\overline{A}$ is closed, we have $D(\overline{A}) \subset \overline{A}$. $\qquad \ldots(3)$

[Refer theorem 1 of article 1.30]

$\therefore$ from (2) and (3), we have $D(A) \subset \overline{A}$.

Moreover, $A \subset \overline{A}$, and $D(A) \subset \overline{A} \Rightarrow A \cup D(A) \subset \overline{A}$. ...(4)

$\therefore$ from (1) and (4), we have $\overline{A} = A \cup D(A)$.

Some Illustrations

1. If $A =]1, 2[$, then $D(A) = [1, 2]$.

   $\therefore \quad \overline{A} = A \cup D(A) =]1, 2[\cup [1, 2] = [1, 2]$.

2. If Z is the set of all integers, then $\overline{Z} = Z \cup D(Z) = Z \cup \varnothing = Z$.

3. If Q is the set of all rational numbers, then

   $$\overline{Q} = Q \cup D(Q) = Q \cup R = R.$$

4. Consider the finite set $A = \{1, 3, 7\}$. We have

   $$\overline{A} = A \cup D(A) = A \cup \varnothing = A.$$

Illustrative Examples

Example 41: *Show that the union of two closed sets is also a closed set.*

Solution: Let F_1, F_2 be two closed sets. Then to prove that $F_1 \cup F_2$ is also a closed set *i.e.*, $(F_1 \cup F_2)'$ is an open set.

Since both F_1 and F_2 are closed sets, therefore both F_1' and F_2' are open sets. But the intersection of two open sets is an open set.

$\therefore \quad F_1' \cap F_2'$ is an open set

$\Rightarrow \quad (F_1 \cup F_2)'$ is an open set

[$\because$ by De Morgan's law $(F_1 \cup F_2)' = F_1' \cap F_2'$]

$\Rightarrow \quad F_1 \cup F_2$ is a closed set.

Example 42: *Show that the union of two open sets is an open set.*

Solution: Let G_1, G_2 be two open sets. Then to prove that $G = G_1 \cup G_2$ is also an open set.

In order to show that G is an open set, we shall show that every point of G is an interior point of G.

Let $p \in G = G_1 \cup G_2$. Then $p \in G_n$ for some $n \in \{1, 2\}$.

Since each G_n is an open set and $p \in G_n$, therefore there must exist some $\varepsilon > 0$ such that $]p - \varepsilon, p + \varepsilon[\subset G_n \subset G$. Note that $G_1 \subset G$ and $G_2 \subset G$.

Thus $]p - \varepsilon, p + \varepsilon[\subset G$ and so p is an interior point of G.

Since every point of G is an interior point of G, therefore G is an open set. Hence the union of two open sets is an open set.

1.31 Countability of Sets

Denumerable Sets. Definition

A set A is said to be **denumerable (or countably infinite)** *if there exists a one-to-one correspondence between the set A and the set* N *of natural numbers i.e., if there exists a one-one mapping from the set* N *of natural numbers onto the set A.*

Example: Consider the set $A = \{1/2, 2/3, 3/4, \ldots.\}$. Then A is denumerable because the mapping $f : \mathrm{N} \Rightarrow A$ defined by

$$f(n) = \frac{n}{n+1} \quad \forall\, n \in \mathrm{N}$$

is bijective *i.e.*, one-one and onto.

Countable and uncountable sets:

Definition: *A set which is either finite or denumerable is called a* **countable set.**

Thus a set A is countable $\Leftrightarrow$ A is finite or there exists a one-one mapping f of N onto A.

A set which is neither finite nor denumerable is said to be **uncountable** or **non-denumerable.**

Thus there are many sizes of infinite sets. The smallest size is called countable.

Example 43: *The set* Z *of all integers is countable i.e., there exists a one-one onto map between the set of integers and the set of natural numbers.*

Solution: Let N denote the set of natural numbers and Z denote the set of integers. Consider the mapping $f : \mathrm{N} \to \mathrm{Z}$ defined by

$f(r) = (r-1)/2$ when r is odd and

$f(r) = -r/2$ when r is even.

f is one-one. We have $f(1) = 0, f(2) = -1, f(3) = 1, f(4) = -2, f(5) = 2,$ and so on.

Thus we easily observe that under the mapping f distinct elements of N have distinct f-images in Z. Therefore f is one-one.

f is onto. Let $y \in \mathrm{Z}$. If $y \geq 0$, then

$2y + 1 \in \mathrm{N}$ and $f(2y+1) = \{(2y+1) - 1\}/2 = y.$

Again if $y < 0$, then $-2y \in \mathrm{N}$ and $f(-2y) = \{-(-2y)\}/2 = y.$

Thus $y \in \mathrm{Z} \Rightarrow$ there exists some element $x \in \mathrm{N}$ such that $f(x) = y$. Therefore f is onto.

Thus the mapping f gives us a one-to-one correspondence between N and Z. Hence Z is denumerable.

Theorem 1: *Every subset of a countable set is countable.*

Proof: Let A be any countable set and let $B \subset A$. If B is finite, there is nothing to prove.

Therefore we suppose that A is an infinite countable set and B is an infinite subset of A.

Since A is countably infinite *i.e.*, denumerable, therefore we can write A as an infinite sequence

$$< a_1, a_2, a_3, \ldots, a_n, \ldots >$$

Let n_1 be the smallest positive integer such that $a_{n_1} \in B$. Again let n_2 be the next smallest positive integer such that $n_2 > n_1$ and $a_{n_2} \in B$ and so on.

Then $B = \{a_{n_1}, a_{n_2}, \ldots\}$.

Obviously the mapping $f : N \to B$ such that

$$f(k) = a_{n_k}$$

is one-one and onto.

Hence B is countable.

Theorem 2: *Every superset of an uncountable set is uncountable.*

Proof: Let A be an uncountable set and $B \supset A$. Then to prove that B is also uncountable.

Suppose B is countable. Since $A \subset B$, therefore A must be countable which is against our hypothesis.

Hence B must be uncountable.

Theorem 3: *Union of a finite number of countable sets is countable.*

Proof: Let A_1 and A_2 be two countable sets. The members of A_1 and A_2 can be arranged in a definite order. Thus we can write

$$A_1 = \{x_1, x_2, x_3, \ldots\}$$

and $$A_2 = \{y_1, y_2, y_3, \ldots\}.$$

Since subsets of countable sets are countable, therefore we can assume that A_1 and A_2 have no common members. For if A_1 and A_2 have common members, we can take subsets of A_1 and A_2 so that they have no common members.

$\therefore$ The set $A_1 \cup A_2$ is the same as if the union of A_1 and A_2 is taken without having any common member.

Let us write $A_1 \cup A_2 = \{z_1, z_2, z_3, \ldots\}$,

where $z_{2n-1} = x_n, z_{2n} = y_n, n = 1, 2, 3, \ldots$.

Then we find that every member of $A_1 \cup A_2$ occupies a definite place in this arrangement.

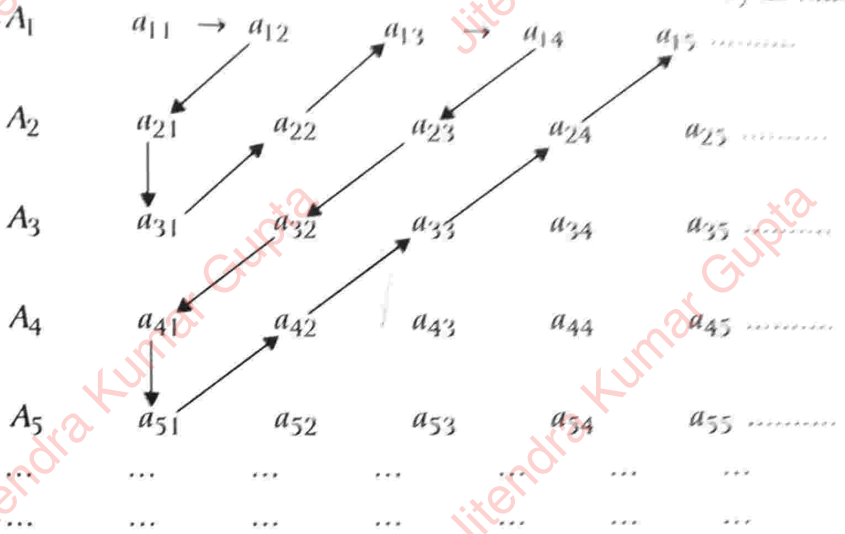
$\therefore$ $A_1 \cup A_2$ is countable.

Repeating the above argument with a finite number of countable sets, we conclude that the union of a finite number of countable sets is countable.

Theorem 4: *The union of a countable family of countable sets is countable.*

(Kanpur 2010)

Proof: Let $\{A_n\}$ be a countable family of sets such that each A_n is countable. We may enumerate, the elements of each $A_n, n = 1, 2, 3, \ldots$ in an array as follows :



If we run out of elements in any set, *i.e.*, if any of the sets is finite, we just put down x's in the spot where an element should go. In the above array a_{ij} stands for the jth element of the ith set. Let us define the height of the element a_{ij} to be $i + j$. With this definition, the height of a_{11} is 2. Similarly a_{12}, a_{21} are the only elements of height 3. There are exactly n elements namely $a_{n1}, a_{(n-1)2}, \ldots, a_{1n}$ of height $n + 1$. Therefore we arrange the elements of $\cup A_n$ according to their height as follows :

a_{11} ;

a_{21}, a_{12} ;

a_{31}, a_{22}, a_{13} ;

$a_{41}, a_{32}, a_{23}, a_{14}$;

... ...

... ...

$a_{n1}, a_{(n-1)2}, \ldots, a_{1n}$;

... ...

... ...

From the above scheme it is evident that a_{pq} is the qth element of $(p + q - 1)$th row. Thus all the elements of $\cup A_n$ have been arranged in an infinite sequence as

$$\{a_{11}, a_{21}, a_{12}, a_{31}, a_{22}, a_{13}, a_{41}, a_{32}, a_{23}, a_{14}, \ldots\}.$$

In fact, the map

$$f : \cup A_n \to \mathbf{N}$$

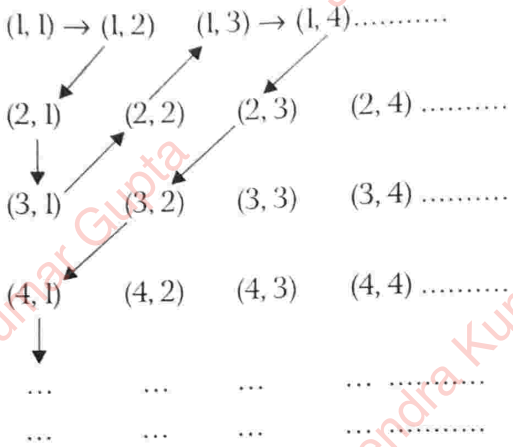
defined by $f(a_{pq}) = \dfrac{(p + q - 2)(p + q - 1)}{2} + q$

gives an enumeration of $\cup A_n$.

Hence the union of a countable family of countable sets is countable.

Theorem 5: *The set* $N \times N$ *is countable.* **(Kanpur 2011)**

Proof: We have $N \times N = \{(m, n) : m, n \in N\}$. To prove that $N \times N$ is countable. We may arrange the set $N \times N$ as shown below :



We observe that in the above arrangement every member of $N \times N$ is placed in a definite position such that no member escapes. For example the member (i, j) of $N \times N$ occurs in the ith row and in the jth column.

Hence the set $N \times N$ is countable.

1.32 Dedikind's Theorem or Dedikind's Axiom or Dedikind's Property of Real Numbers

Let L *and* U *be two subsets of* R *such that*

(i) $L \neq \emptyset, U \neq \emptyset$ *(i.e., each set has at least one element)*

(ii) $L \cup U = R$ *(i.e., each real number is either in* L *or in* U*)*

and (iii) $x \in L, y \in U \Rightarrow x < y$ *(i.e., each member of* L *is smaller than each member of* U*);*

then the subset L *has the greatest member or the subset* U *has the smallest member i.e., there exists* $\alpha \in R$ *such that*

$$x < \alpha \Rightarrow x \in L, y > \alpha \Rightarrow y \in U.$$

In the following theorem we shall establish the equivalence between the Order-completeness property and Dedikind's theorem.

Theorem: *To prove that Order-completeness axiom* $\Leftrightarrow$ *Dedikind's theorem.*

Proof: First we shall prove that

Completeness axiom $\Rightarrow$ Dedikind's axiom.

Let L and U be two subsets of R such that

(i) $L \neq \emptyset, U \neq \emptyset$ (ii) $L \cup U = R$

and (iii) $x \in L, y \in U \Rightarrow x < y$.

Then we wish to prove that there exists $\alpha \in R$ such that

$$x < \alpha \Rightarrow x \in L,\ y > \alpha \Rightarrow y \in U.$$

From (i), $L \neq \emptyset$.

Also $U \neq \emptyset$, so let $y \in U$.

Then from (iii), we have

$$x < y,\ \forall\ x \in L.$$

$\therefore$ y is an upper bound for L.

Thus L is a non-empty subset of R which is bounded above. So by order-completeness property of real numbers, L must have its supremum in R.

Let $\sup L = \alpha$.

We shall show that $\alpha \in R$ is such that

$$x < \alpha \Rightarrow x \in L,\ y > \alpha \Rightarrow y \in U.$$

Let $y > \alpha$.

Then $y \notin L$ because $\alpha = \sup L$.

But from (ii), every real number is either in L or in U.

So $y \notin L \Rightarrow y \in U$.

Thus $y > \alpha \Rightarrow y \in U$.

Again let $x < \alpha$. Since $\alpha = \sup L$, therefore x cannot be an upper bound for L. So there exists $z \in L$ such that $z > x$ *i.e.*, $x < z$. Now from (iii), $z \in L$ and $x < z \Rightarrow x \notin U \Rightarrow x \in L$.

Thus $x < \alpha \Rightarrow x \in L$.

$\therefore$ there exists $\alpha \in R$ such that

$$x < \alpha \Rightarrow x \in L \text{ and } y > \alpha \Rightarrow y \in U.$$

Hence completeness axiom $\Rightarrow$ Dedikind's axiom.

Now we shall prove that

Dedikind's axiom $\Rightarrow$ Completeness axiom.

Let S be a non-empty subset of R which is bounded above.

Then we have to prove that S has its supremum in R.

Since S is bounded above, therefore let u be an upper bound for S *i.e.*,

$$x \leq u, \forall\ x \in S.$$

Let $U = \{s : s \in R \text{ and } s \text{ is an upper bound for } S\}$.

Since u is an upper bound for S, therefore $u \in U$ and so $U \neq \emptyset$. Let $L = R - U$.

Then $S \subset L$ and $L \neq \emptyset$. Also $L \cup U = (R - U) \cup U = R$.

Now let $x \in L,\ y \in U$.

Then $x \neq y$ because by definition of L, L and U are disjoint.

Also we cannot have $x > y$. Because if $x > y$, then

$y \in U \Rightarrow y$ is an upper bound for S

$\Rightarrow x$ is also an upper bound for S [$\because x > y$]

$\Rightarrow x \in U$, by definition of U

$\Rightarrow x \in L \cap U$ [$\because x \in L$]

$\Rightarrow L \cap U \neq \varnothing$, which is a contradiction.

$\therefore$ $x \in L, y \in U \Rightarrow x < y$.

$\therefore$ by Dedikind's axiom, there exists $\alpha \in R$ such that

$x < \alpha \Rightarrow x \in L, y > \alpha \Rightarrow y \in U$.

Now we shall show that $\alpha = \sup S$.

If $y > \alpha$, then $y \in U$ and so $y \notin L$ because L and U are disjoint.

Thus if $y > \alpha$, then y cannot be in L and hence can also not be in S because $S \subset L$.

Thus no real number $y > \alpha$ is in S.

α is an upper bound for S *i.e.*,

$x \leq \alpha, \forall\ x \in S$.

Now let $\beta < \alpha$. Then

$\beta < \alpha \Rightarrow \beta \in L \Rightarrow \beta \notin U$

$\Rightarrow \beta$ is not an upper bound for S.

Thus α is an upper bound for S and no real number $\beta < \alpha$ can be an upper bound for S.

$\therefore$ $\alpha = \sup S$.

Hence Dedikind's axiom $\Rightarrow$ Completeness axiom.

Comprehensive Exercise 5

1. Give an example of a set which has
   (i) no limit point (iii) exactly one limit point
   (iii) exactly two limit points (iv) an infinite number of limit points
   (v) every point of the set as its limit point.
2. Give an example of a set having the numbers 1 and -1 as the only limit points.
3. Find the derived set of the set of points in the open interval $]a, b[$.
4. If a and b are real numbers such that $b > a$, then show that the derived set of each of the intervals $]a, b[$, $]a, b]$, $[a, b[$ and $[a, b]$ is the interval $[a, b]$.
5. Show that the derived set of the open interval $]2, 3[$ is the closed interval $[2, 3]$.

6. Show that the derived set of the closed interval $[3, 5]$ is the closed interval $[3, 5]$ itself. Hence show that the closed interval $[3, 5]$ is a set of second species.

7. Let $S = R - \{1/n : n \in N\}$. Show that each point of R is a limit point of S.

8. Let $S = \left\{3 + \frac{1}{n} : n \in N\right\}$. Show that the only limit point of S is 3.

9. If A' denotes the derived set of A, then find a set A such that
   (i) $A \cap A' = \varnothing$. (ii) $A = A'$.
   (iii) $A' \subset A$. (iv) $A \subset A'$.

10. Find the derived set of the set $S = \{1, 3, 7, 11\}$.

11. Give an example of two subsets A and B of R such that
$$D(A \cap B) \neq D(A) \cap D(B).$$

12. Show that a point $p \in R$ is a limit point of a set $S \subset R$ iff for each positive integer n, the open interval $]p - 1/n, p + 1/n[$ contains a point of S other than p.

13. Show that a point $p \in R$ is a limit point of a set $S \subset R$ iff for each positive rational number r,
$$]p - r, p + r[\cap S - \{p\} \neq \varnothing.$$

14. Find the derived set of each of the following sets :
   (i) The open ray $]a, \infty[$. (ii) The open ray $]-\infty, a[$.
   (iii) $\{1 + 3^{-n} : n \in N\}$. (iv) $\left\{1 - \frac{3}{n} : n \in N\right\}$.

15. Find the limit points of the set $\left\{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, \ldots\right\}$.

16. Find the limit points of the set $S = \left\{\frac{3n + 2}{2n + 1} : n \in N\right\}$.

17. If $S = \left\{1 + \frac{(-1)^n}{n} : n \text{ is a natural number}\right\}$, find the limit points of the set S.

18. Show that the set $S = \left\{1 + \frac{(-1)^n}{2^n} : n \text{ is a positive integer}\right\}$ is bounded. Show that 1 is a limit point of S. Are there any other limit points of S ?

19. (a) Prove that every infinite bounded set of real numbers has at least one limit point. **(Kanpur 2011)**
   (b) Verify the converse of the above theorem in case of the set
$$S = \left\{1, \frac{1}{2}, -3, \frac{1}{4}, 5, \frac{1}{6}, -7, \frac{1}{8}, 9, \ldots\right\}.$$

20. Choose the correct answer :
If S is closed, then the complement (relative to any open set containing S) of S is
(A) open (B) closed (C) neither open nor closed.

21. Give examples for the sets (a) which are neither open nor closed
(b) which are both open and closed.

22. Prove that a set is closed iff its complement is open.
It is given that a set S is not closed, does it imply that it is an open set ? Justify this by giving examples. **(Kanpur 2010)**

Answers 5

1. (i) The set of positive integers $\mathbf{Z}^+$

(ii) The set $\{1/n : n \in \mathbf{Z}^+\}$

(iii) The set $\left\{\frac{1}{2}, -\frac{1}{2}, \frac{2}{3}, -\frac{2}{3}, \frac{3}{4}, -\frac{3}{4}, \ldots, \frac{n}{n+1}, \frac{-n}{n+1}, \ldots\right\}$

Note that 1 and – 1 are the only limit points of this set

(iv) The open interval $]1, 2[$

(v) The closed interval $[2, 5]$

2. The set given in part (iii) of Ex. 27

3. The closed interval $[a, b]$

9. (i) The set $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}\right\}$

(i) The closed interval $[3, 4]$;

(iii) The set $A = \left\{0, \frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \left(\frac{1}{2}\right)^4, \ldots\right\}$

(iv) The open interval $]0, 1[$

10. The empty set $\varnothing$

11. Take $A = [0, 1[$, $B =]1, 2]$

14. (i) The closed ray $[a, \infty[$

(ii) The closed ray $]-\infty, a]$

(iii) The singleton $\{1\}$

(iv) The singleton $\{1\}$

15. 1 and –1

16. $\frac{3}{2}$

17. 1

20. (B)

21. (a) (i) The set of rational numbers Q

(ii) The right half open interval $[2, 3[$

(iii) The left half open interval $]2, 3]$

(b) Set $\varnothing$ and R are both and closed

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. For all $x, y \in \mathbf{R}$, the triangle inequality is
   (a) $|x + y| \leq |x| + |y|$ (b) $|xy| = |x||y|$
   (c) $|x - y| \leq |x| + |y|$ (d) $|x + y| = |x| + |y|$.
2. "To every real number there corresponds a unique point on a directed line and conversely, to every point on a directed line there corresponds a unique real number". This is known as:
   (a) Cauchy's axiom (b) Weierstrass axiom
   (c) Dedikind-Cantor axiom (d) Cantor's axiom
3. If a and b are any two real numbers such that $a \leq b$, then the set $\{x \in \mathbf{R} : a < x < b\}$ is known as :
   (a) right-half open interval (b) open interval
   (c) left-half open interval (d) closed interval.
4. The set $\{x : x \in \mathbf{R}, a \leq x \leq b\}$ is known as :
   (a) open interval (b) left-half open interval
   (c) closed interval (d) none of these.
5. Which of the following sets is bounded below but not bounded above ?
   (a) $\mathbf{N}$ (b) $\mathbf{Z}$
   (c) $\mathbf{Q}$ (d) $\mathbf{R}$
6. Let a be any real number and b any positive real number. Then there exists a positive integer n such that
   (a) $nb \geq a$ (b) $nb > a$
   (c) $nb \leq a$ (d) $nb < a$.
7. The least upper bound of the set $\left\{\frac{1}{n} : n \in \mathbf{N}\right\}$ is
   (a) 1 (b) -1
   (c) 0 (d) none of these.
8. The greatest lower bound of the set of positive even integers is
   (a) 0 (b) 2
   (c) 1 (d) none of these.
9. The least upper bound of the set $\{x : x = 1 - (1/n), n \in \mathbf{N}\}$ is
   (a) 0 (b) -1
   (c) 1 (d) none of these.

10. The set $S = \{ x : 2 \leq x < 4\}$ is such that it contains
(a) g.l.b. but not l.u.b. (b) g.l.b. and l.u.b.
(c) l.u.b. but not g.l.b. (d) none of these.

11. Which of the following set is *nbd* of each of its points ?
(a) Set **N** of natural numbers
(b) Set **Q** of rational numbers
(c) Set **Z** of integers
(d) open interval $]a, b[$.

12. Neighbourhood of $\frac{1}{2}$ is the set
(a) $]0, \frac{1}{2}[$ (b) $[-\frac{1}{2}, \frac{1}{2}]$
(c) **R** (d) none of these.

13. Derived set of the set **Q** of all rational numbers is
(a) **Q** (b) **R**
(c) **Z** (d) none of these.

14. The set of all rational numbers is
(a) closed (b) open
(c) neither open nor closed (d) none of these.

15. The set of all integers is
(a) countable (b) uncountable
(c) finite (d) none of these.

16. The set of all real numbers is
(a) closed (b) open
(c) both open and closed (d) none of these.

17. The union of infinite number of open sets is
(a) an open set (b) a closed set
(c) need not be an open set (d) none of these.

18. If A and B are any subsets of **R**, then $A \subset B \Rightarrow$
(a) $D(A) \subset D(B)$ (b) $D(A) \supset D(B)$
(c) $D(A) = D(B)$ (d) none of these.

19. Every singleton set in **R** is
(a) open (b) closed
(c) neither open nor closed (d) none of these.

20. If S denotes the set of real numbers **R**, then $S^\circ =$
(a) **Q** (b) **Z**
(c) **R** (d) none of these.

Fill in the Blank(s)

Fill in the blanks "....." so that the following statements are complete and correct.

1. If S be any subset of the set $\mathbf{R}$ of real numbers and if there exists a real number u, such that $x \leq u \; \forall \; x \in S$, then u is called an ..... of S.
2. Every non-empty set of real numbers which is bounded above has its ..... in $\mathbf{R}$.
3. If a and b are any real numbers and $a > 1$, then there exists a positive integer n such that a^n ..... b.
4. Any ..... interval is a *nbd* of each of its points.
5. The set of all limit points of a set $A \subset \mathbf{R}$ is called the ..... of A.
6. If S is a finite set, then $D(S)$ = ...
7. Let F be a subset of $\mathbf{R}$. Then F is said to be closed if its complement F' is ..... .
8. $\overline{A}$ is the intersection of the family of all closed sets containing .....
9. Every ..... of a countable set is countable.
10. For any two real numbers a, b one and only one of the following is true

    $a > b, a = b, a < b$.

    It is known as the law of .....

True or False

Write 'T' for true and 'F' for false statement.

1. The field of real numbers is an ordered field.
2. There is no rational number whose square is 3.
3. For all $x, y \in \mathbf{R}, |x + y| \geq ||x| - |y||$.
4. The set $\mathbf{R}^+$ of positive real numbers is bounded above.
5. Between any two distinct real numbers, there lie an infinite number of real numbers.
6. A closed interval $[a, b]$ is a *nbd* of each of its points.
7. The intersection of two *nbds* of a point need not be a *nbd* of that point.
8. Every point of the set $\mathbf{R}$ of real numbers is a limit point of $\mathbf{R}$.
9. Every open interval is not an open set.
10. Union of a finite number of countable sets is not countable.
11. The intersection of an infinite collection of open sets is an open set.
12. If A and B are any subsets of $\mathbf{R}$, then

    $D(A \cup B) = D(A) \cup D(B)$.

Answers

Multiple Choice Questions

1.	(a)	2.	(c)	3.	(b)	4.	(c)	5.	(a)
6.	(b)	7.	(a)	8.	(b)	9.	(c)	10.	(a)
11.	(d)	12.	(c)	13.	(b)	14.	(c)	15.	(a)
16.	(c)	17.	(a)	18.	(a)	19.	(b)	20.	(c)

Fill in the Blank(s)

1.	upper bound	2.	supremum
3.	$>$	4.	open
5.	derived set	6.	ϕ
7.	open	8.	A
9.	subset	10.	trichotomy

True or False

1.	*T*	2.	*T*	3.	*T*	4.	*F*	5.	*T*
6.	*F*	7.	*F*	8.	*T*	9.	*F*	10.	*F*
11.	*F*	12.	*T*						

□

Jitendra Kumar Gupta

Chapter

Sequences

2.1 Introduction

In the present chapter we shall study a special class of functions, namely sequences. The study of sequences plays an important role in Analysis.

2.2 Sequence

Definition: *Let S be any non-empty set. A function whose domain is the set* N *of natural numbers and whose range is a subset of S, is called a sequence in the set S.*

In other words a sequence in a set S is a rule which assigns to each natural number a unique element of S.

A sequence whose range is a subset of R is called a *real sequence* or a sequence of real numbers.

In this chapter we shall study only real sequences. Therefore the term sequence will be used to denote a real sequence.

If s is a sequence, then the image $s(n)$ of $n \in \mathbb{N}$ is usually denoted by s_n. It is customary to denote the sequence s by the symbol $< s_n >$ or by $\{s_n\}$. The image s_n of n is called the nth term of the sequence.

A sequence can be described in several different ways.

1. Listing in order, the first few elements of a sequence, till the rule for writing down different elements becomes clear. For example, $< 1, 8, 27, 64, \ldots >$ is the sequence whose nth term is n^3.
2. Defining a sequence by a formula for its nth term. For example, the sequence $< 1, 8, 27, 64, \ldots >$ can also be written as $< 1, 8, \ldots, n^3, \ldots >$ or as $< n^3 : n \in \mathbb{N} >$ or simply as $< n^3 >$.
3. Defining a sequence by a Recursion formula *i.e.* by a rule which expresses the nth term in terms of the $(n-1)$th term. For example, let

$$a_1 = 1, a_{n+1} = 3a_n \text{, for all } n \geq 1.$$

These relations define a sequence whose nth term is 3^{n-1}.

Illustrations

1. $< \frac{1}{n} >$ is the sequence $< 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots >$.
2. $< \frac{n}{n+1} >$ is the sequence $< \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots >$.
3. $< (-1)^n / n >$ is the sequence $< -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots >$.
4. Let $s_1 = 1, s_2 = 1$ and $s_{n+2} = s_{n+1} + s_n$ for all $n \geq 1$.

   From the above formula, $s_3 = s_2 + s_1 = 2$,

   $s_4 = s_3 + s_2 = 3, s_5 = s_4 + s_3 = 5$ and so on.

$\therefore$ $< s_n > = < 1, 1, 2, 3, 5, \ldots >$.

Range of a sequence: *The set of all distinct terms of a sequence is called its range.*

$\therefore$ The range of a sequence $< s_n >$ = the set $\{s_1, s_2, \ldots\}$.

Example: The range of the sequence $< (-1)^n > = \{-1, 1\}$, a finite set.

Example: The range of the sequence

$$\frac{1}{n} = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\},$$

is an infinite set.

Constant sequence: *A sequence $< s_n >$ defined by $s_n = a$ for all $n \in \mathbb{N}$ is called a constant sequence.*

The sequence $< s_n > = < a, a, a, \ldots >$ is a constant sequence.

Its range = the singleton $\{a\}$ is a finite set.

Equality of two sequences: *Two sequences $< s_n >$ and $< t_n >$ are said to be equal if $s_n = t_n \; \forall n \in \mathbb{N}$.*

2.3 Operations on Sequences

Since sequences of real numbers are real valued functions, we define the sum, difference, product etc. of two sequences as follows :

Let $< s_n >, < t_n >$ be two sequences. Then the sequences having nth terms $s_n + t_n$, $s_n - t_n$ and $s_n\, t_n$ are respectively called the *sum*, *difference* and *product* of the sequences $< s_n >$ and $< t_n >$. These sequences are denoted by $< s_n + t_n >, < s_n - t_n >$ and $< s_n\, t_n >$ respectively.

If $t_n \neq 0$ for all $n \in \mathbb{N}$, then the sequence whose nth term is $1 / t_n$ is known as the *reciprocal of the sequence* $< t_n >$ and is denoted by $< 1 / t_n >$. Also the sequence whose nth term is s_n / t_n is called the *quotient* of the sequence $< s_n >$ by the sequence $< t_n >$ and it is denoted by $< s_n / t_n >$.

If $c \in \mathbb{R}$, then the sequence having nth term cs_n is called the *scalar multiple* of $< s_n >$ by c. This sequence is denoted by $< cs_n >$.

2.4 Subsequences

Let $< s_n >$ be any sequence. If $< n_1, n_2, \ldots, n_k, \ldots >$ be a strictly increasing sequence of positive integers i.e., $i > j \Rightarrow n_i > n_j$, then the sequence

$$< s_{n_1}, s_{n_2}, \ldots, s_{n_k}, \ldots >$$

is called a subsequence of $< s_n >$.

From the condition $i > j \Rightarrow n_i > n_j$, we conclude that the order of the various terms in the subsequence is the same as it is in the sequence.

Illustrations :

1. Let $< s_n > = < 1, 0, 1, 0, 1, 0, \ldots >$ *i.e.*, $s_1 = 1, s_2 = 0, s_3 = 1, s_4 = 0, \ldots\ldots$.

Take $n_1 = 1, n_2 = 3, n_3 = 5, \ldots\ldots$ Then $< n_r >$ is a sequence of positive integers such that $n_1 < n_2 < n_3 < \ldots\ldots$.

Hence $< 1, 1, 1, \ldots >$ is a subsequence of $< s_n >$.

Similarly if we take $n_1 = 2, n_2 = 4, n_3 = 6, \ldots$, then the sequence $< 0, 0, 0, \ldots >$ is also a subsequence of $< s_n >$.

2. The sequence of primes $< 2, 3, 5, 7, 11, \ldots >$ is a subsequence of the sequence of natural numbers $< 1, 2, 3, 4, \ldots\ldots >$.

3. The sequence $< 7^2, 3^2, 15^2, 11^2, 19^2, \ldots >$ is not a subsequence of the sequence $< 1^2, 2^2, 3^2, 4^2, \ldots >$.

Here $n_1 = 7, n_2 = 3, n_3 = 15, n_4 = 11, n_5 = 19$. But $< 7, 3, 15, 11, 19, \ldots >$ is not a strictly increasing sequence of positive integers.

2.5 Bounded Sequences

(Kanpur 2008)

Definition 1: *A sequence* $< s_n >$ *is said to be bounded above if the range set of* $< s_n >$ *is bounded above i.e., if there exists a real number* k_1 *such that*

$$s_n \leq k_1 \text{ for all } n \in \mathbb{N}.$$

The number k_1 is called an **upper bound** of the sequence $< s_n >$.

Definition 2: *A sequence* $< s_n >$ *is said to be bounded below if the range set of* $< s_n >$ *is bounded below i.e., if there exists a real number* k_2 *such that*

$$s_n \geq k_2 \text{ for all } n \in \mathbb{N}.$$

The number k_2 is called a **lower bound** of the sequence $< s_n >$.

Definition 3: *A sequence* $< s_n >$ *is said to be bounded if the range set of* $< s_n >$ *is both bounded above and bounded below i.e., if there exist two real numbers* k_1 and k_2 *such that*

$$k_2 \leq s_n \leq k_1 \text{ for all } n \in \mathbb{N}.$$

Equivalently, a sequence $< s_n >$ is bounded if and only if there exists a real number $K > 0$ such that

$$|s_n| \leq K \text{ for all } n \in \mathbb{N}.$$

It is not necessary that a sequence be bounded above or bounded below.

A sequence $< s_n >$ is said to be **unbounded** if it is either unbounded below or unbounded above.

Definition 4: *The least number say, M, if it exists, of the set of the upper bounds of* $< s_n >$ *is called the* **least upper bound** (**l.u.b.**) *or the* **supremum** (**sup**) *of the sequence* $< s_n >$.

The greatest number say, m, if it exists, of the set of the lower bounds of $< s_n >$ *is called the* **greatest lower bound** (**g.l.b.**) *or the* **infimum** (**inf.**) *of the sequence* $< s_n >$.

Note 1: If the range of a sequence is a finite set , then the sequence is bounded because a finite set is always bounded.

Note 2: Every subsequence of a bounded sequence is bounded.

Illustrations:

1. The sequence $< \frac{1}{n} >$ is bounded since $\left|\frac{1}{n}\right| \leq 1$ for all $n \in \mathbb{N}$.
2. The sequence $< \frac{n}{n+1} >$ is bounded since $\frac{1}{2} \leq \frac{n}{n+1} < 1$ for all $n \in \mathbb{N}$.
3. The sequence $< (-1)^n >$ is bounded since $|(-1)^n| \leq 1$ for all $n \in \mathbb{N}$. In fact $|(-1)^n| = 1$ for all $n \in \mathbb{N}$.
4. The sequence $< (-1)^n / n >$ is bounded since $|(-1)^n / n| \leq 1$ for all $n \in \mathbb{N}$.
5. The sequence $< s_n >$ defined by $s_n = 1 + (-1)^n$ for all $n \in \mathbb{N}$, is bounded since the range set of the sequence is $\{0, 2\}$, which is a finite set.

6. The sequence $< n^2 >$ is bounded below by 1 but not bounded above.
7. The sequence $< -n^2 >$ is bounded above by -1 but not bounded below.
8. The sequence $< s_n > = < (-1)^n n >$ is neither bounded below nor bounded above.

For any positive real number K, there exists a positive integer $2m$ such that $2m > K$.
It gives that $s_{2m} > K$. Hence $< s_n >$ is not bounded above.
Similarly it can be shown that $< s_n >$ is not bounded below.

Theorem: *A sequence $< s_n >$ is bounded iff there exist $m \in \mathbb{N}$, $l \in \mathbb{R}$ and $a > 0$ such that*

$$| s_n - l | < a \text{ for all } n \geq m.$$

Proof: Let $< s_n >$ be a bounded sequence. Then there exist two real numbers k_1, k_2 such that $k_1 < s_n < k_2$ for all $n \in \mathbb{N}$

$$\Rightarrow \quad k_1 - \frac{k_1 + k_2}{2} < s_n - \frac{k_1 + k_2}{2} < k_2 - \frac{k_1 + k_2}{2} \quad \forall\, n \in \mathbb{N}$$

$$\Rightarrow \quad \frac{k_1 - k_2}{2} < s_n - \frac{k_1 + k_2}{2} < \frac{k_2 - k_1}{2} \quad \forall\, n \in \mathbb{N}$$

$$\Rightarrow \quad -a < s_n - l < a \quad \forall\, n \in \mathbb{N} \quad \text{where} \quad l = \frac{k_1 + k_2}{2} \quad \text{and} \quad a = \frac{k_2 - k_1}{2}$$

$$\Rightarrow \quad | s_n - l | < a \quad \forall\, n \in \mathbb{N}$$

$\Rightarrow \quad | s_n - l | < a \quad \forall\, n \geq m$, where $m = 1 \in \mathbb{N}$, $l \in \mathbb{R}$ and $a > 0$.

Conversely, let there exist $l \in \mathbb{R}$, $a > 0$ and $m \in \mathbb{N}$ such that $| s_n - l | < a$ for all $n \geq m$.
This gives $l - a < s_n < l + a$ for all $n \geq m$.
Choose $k_1 = \min \{ s_1, s_2, \ldots, s_{m-1}, l - a \}$.
Then $k_1 \leq s_n$, for all $n \in \mathbb{N}$.
Again choose $k_2 = \max \{ s_1, s_2, \ldots, s_{m-1}, l + a \}$.
Then $s_n \leq k_2$ for all $n \in \mathbb{N}$.
$\therefore \quad k_1 \leq s_n \leq k_2$ for all $n \in \mathbb{N}$.
Hence $< s_n >$ is a bounded sequence.

Note: In view of the above theorem we conclude that a sequence is bounded even if $k_1 \leq s_n \leq k_2$ for $n \geq m$. However in such a case it is not necessary that k_1 is a lower bound and k_2 is an upper bound of the sequence $< s_n >$.

2.6 Convergent Sequences

Definition: *A sequence $< s_n >$ is said to converge to a number l, if for any given $\varepsilon > 0$ there exists a positive integer m such that*

$$| s_n - l | < \varepsilon \text{ for all } n \geq m.$$

The number l is called the **limit** *of the sequence $< s_n >$ and we write $s_n \to l$ as $n \to \infty$ or* $\lim_{n \to \infty} s_n = l$ *or simply* $\lim s_n = l$.

The positive integer m depends on the value of ε.

The phrase '$|s_n - l| < \varepsilon$ for all $n \geq m$' expresses the fact that the absolute value of the difference between s_n and l can be made less than ε from some stage onwards *i.e.*, $l - \varepsilon < s_n < l + \varepsilon$ from some stage onwards.

The truth of the statement that the sequence $< s_n >$ converges to l depends upon showing that all except a finite number of terms of the sequence must lie in the open interval $]\, l - \varepsilon, l + \varepsilon\, [$, whatever $\varepsilon > 0$ we take. If we can find even one ε for which infinitely many terms of the sequence lie outside $]\, l - \varepsilon, l + \varepsilon\, [$, the sequence will not converge to l. The number of terms lying outside $]\, l - \varepsilon, l + \varepsilon\, [$ depends upon ε. The smaller the ε, the larger the number of terms which lie outside $]\, l - \varepsilon, l + \varepsilon\, [$.

From the above discussion, we conclude that *'a sequence converges to l iff it lies ultimately in each open interval around l'.*

Theorem 1: *If $< s_n >$ is a sequence of non-negative numbers such that $\lim s_n = l$, then $l \geq 0$.*

Proof: Suppose, if possible, $l < 0$. Then $-l > 0$.

Since $\lim s_n = l$, for $\varepsilon = -(l/2) > 0$, there exists $m \in \mathbb{N}$ such that $|s_n - l| < -l/2$ for all $n \geq m$.

In particular, $|s_m - l| < -l/2$

i.e., $$l + \frac{l}{2} < s_m < l - \frac{l}{2}$$

i.e., $s_m < l/2$ or $s_m < 0$, because by assumption $l < 0$.

But by hypothesis we have $s_m \geq 0$ because it is given that $s_n \geq 0$ for all n.

Hence our assumption is wrong. So we must have $l \geq 0$.

Remark: In the above proof ε can be taken any positive real number such that $0 < \varepsilon \leq -l$, say $\varepsilon = -l$ or $\varepsilon = -l/2$ or $\varepsilon = -l/3$ etc.

Note 1: A negative number cannot be the limit of a sequence of non-negative numbers.

2. If $\lim s_n = l$ and $l < 0$ then there exists a positive integer m such that $s_n < 0$ for all $n \geq m$.

3. A sequence $< s_n >$ is called a null sequence if $\lim s_n = 0$.

Theorem 2: *A sequence cannot converge to more than one limit i.e., the limit of a sequence is unique.*

Proof: If possible, suppose a sequence $< s_n >$ converges to two distinct numbers l and l'. Since $l \neq l'$, therefore $|l - l'| > 0$.

Let $\varepsilon = \frac{1}{2}|l - l'|$, then $\varepsilon > 0$.

Now $< s_n >$ converges to $l \Rightarrow$ there exists $m_1 \in \mathbb{N}$ such that

$$|s_n - l| < \varepsilon \text{ for all } n \geq m_1. \quad \ldots(1)$$

Similarly $< s_n >$ converges to $l' \rightarrow$ there exists $m_2 \in \mathbb{N}$ such that

$$|s_n - l'| < \varepsilon \text{ for all } n \geq m_2. \quad \ldots(2)$$

Let $m = \max\{m_1, m_2\}$.

Then (1) and (2) hold for all $n \geq m$.

We have for all $n \geq m$

$$|l - l'| = |(s_n - l) - (s_n - l')| \leq |s_n - l| + |s_n - l'|$$
$$< \varepsilon + \varepsilon = 2\varepsilon = |l - l'|.$$

Thus $|l - l'| < |l - l'|$, which is absurd. Hence our initial assumption that $l \neq l'$ is wrong and we must have $l = l'$ *i.e.*, the limit of a sequence is unique.

Note: After taking $\varepsilon = \frac{1}{2}|l - l'|$, we can also give the following argument :

Since $< s_n >$ converges to both l and l', therefore it lies ultimately in both the intervals $]l - \varepsilon, l + \varepsilon[$ and $]l' - \varepsilon, l' + \varepsilon[$. This is impossible since these two intervals have no real number in common. Hence our assumption is wrong and thus the sequence cannot converge to more than one limit.

Theorem 3: *If $< s_n >$ converges to l, then any subsequence of $< s_n >$ also converges to l.*

Proof: Let $< s_{n_k} >$ be any subsequence of $< s_n >$. Then by definition of subequence, $n_1, n_2, \ldots, n_k, \ldots$ are positive integers such that

$$n_1 < n_2 < \ldots < n_k < \ldots$$

Now $n_1 \geq 1 \Rightarrow n_k \geq k$ (by induction).

Since $< s_n >$ converges to l, so given $\varepsilon > 0$, there exists a positive integer m such that

$$|s_k - l| < \varepsilon \text{ for all } k \geq m.$$

For $k \geq m$ we have $n_k \geq k \geq m$.

$\therefore$ $|s_{n_k} - l| < \varepsilon$ for all $n_k \geq m$.

$\therefore$ $< s_{n_k} >$ converges to l.

Corollary: *All subsequences of a convergent sequence converge to the same limit.*

Proof: By theorem 3, any subsequence of a sequence converges to the same limit as the limit of the sequence and by theorem 2, the limit of a sequence is unique. This shows that all subsequences of a convergent sequence have the same limit.

Note: To show that a given sequence is not convergent it is enough to show that two of its subsequences converge to different limits.

Illustration: The sequence $< (-1)^n >$ is not convergent.

The two subsequences $< 1, 1, 1, \ldots. >$ and $< -1, -1, -1, \ldots >$ of the given sequence converge respectively to 1 and -1 which are different.

Theorem 4: *If the subsequences* $< s_{2n-1} >$ *and* $< s_{2n} >$ *of the sequence* $< s_n >$ *converge to the same limit* l, *then the sequence* $< s_n >$ *converges to* l.

Proof: Let $\varepsilon > 0$ be given. Then, since $\lim s_{2n-1} = l$, there exists $m_1 \in \mathbb{N}$ such that $|s_{2n-1} - l| < \varepsilon \quad \forall n \geq m_1$.

Similarly $\lim s_{2n} = l \Rightarrow$ for $\varepsilon > 0$, there exists $m_2 \in \mathbb{N}$ such that

$$|s_{2n} - l| < \varepsilon \quad \forall n \geq m_2 .$$

Let $m = \max. \{m_1, m_2\}$.

Then $|s_{2n-1} - l| < \varepsilon$ and $|s_{2n} - l| < \varepsilon \; \forall n \geq m$.

$\therefore$ $|s_n - l| < \varepsilon \quad \forall n \geq 2m - 1$.

Hence $< s_n >$ converges to l.

Theorem 5: *Every convergent sequence is bounded.* **(Kanpur 2008)**

Proof: Let $< s_n >$ be a sequence which converges to l. Take $\varepsilon = 1$. Then there exists a positive integer m such that

$$|s_n - l| < 1, \text{ for all } n \geq m,$$

i.e., $l - 1 < s_n < l + 1$, for all $n \geq m$.

Let $k = \min. \{s_1, s_2, \ldots, s_{m-1}, l - 1\}$,

and $K = \max. \{s_1, s_2, \ldots, s_{m-1}, l + 1\}$.

$\therefore$ $k \leq s_n \leq K$ for all $n \in \mathbb{N}$.

Hence the sequence $< s_n >$ is bounded.

Note: The converse of the above theorem need not be true. That is, a bounded sequence need not be convergent. For example, the sequence $< (-1)^n >$ is bounded but is not convergent.

Illustrative Examples

Example 1: *Show that the sequence* $< 1/n >$ *has the limit* 0.

Solution: For any given $\varepsilon > 0$, we have $\left|\frac{1}{n} - 0\right| < \varepsilon$ when $\frac{1}{n} < \varepsilon$ *i.e.*, when $n > \frac{1}{\varepsilon}$.

Let us choose a positive integer $m > 1/\varepsilon$. Then for all $n \geq m$, we have

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{m} < \varepsilon.$$

$\therefore$ $< \frac{1}{n} >$ converges to 0 *i.e.*, $< \frac{1}{n} >$ has the limit 0.

Example 2: *If* $s_n = k \; (\in \mathbb{R})$ *is a constant sequence, then* $\lim s_n = k$.

Solution: We have $|s_n - k| = |k - k| = 0$ for all $n \in \mathbb{N}$.

Given any $\varepsilon > 0, |s_n - k| = 0 < \varepsilon$ for all n

i.e., $|s_n - k| < \varepsilon$ for all $n \geq m = 1$.

Hence $\lim s_n = k$.

Example 3: *The sequence* $< s_n >$ *where* $s_n = 1/2^n$ *converges to '0'.*

Solution: We have $|s_n - 0| = \dfrac{1}{2^n}$.

$\therefore$ For any $\varepsilon > 0, |s_n - 0| < \varepsilon$ if $\dfrac{1}{2^n} < \varepsilon$

i.e., if $2^n > \dfrac{1}{\varepsilon}$ *i.e.*, if $n > \dfrac{\log (1/\varepsilon)}{\log 2}$.

Let us choose a positive integer $m > \left(\log \dfrac{1}{\varepsilon}\right) / \log 2$.

Then for all $n \geq m, |s_n - 0| < \varepsilon$.

$\therefore$ $< s_n >$ converges to 0.

Example 4: *Show that the sequence* $< s_n >$ *defined by* $s_n = r^n$ *converges to zero if* $|r| < 1$.

Solution: If $|r| < 1$, then we can write $|r| = \dfrac{1}{1+h}$, where $h > 0$. Since $h > 0$, therefore

$$(1+h)^n = 1 + nh + \frac{n(n-1)}{2} h^2 + \ldots + h^n \geq 1 + nh \text{ for all } n.$$

Now $|s_n - 0| = |r^n| = |r|^n = \dfrac{1}{(1+h)^n} \leq \dfrac{1}{1+nh}$, $\forall\ n$.

Let $\varepsilon > 0$ be given. Then $|s_n - 0| < \varepsilon$ if $\dfrac{1}{1+nh} < \varepsilon$ *i.e.*, if $n > \left(\dfrac{1}{\varepsilon} - 1\right) / h$.

If we take a positive integer $m > \left(\dfrac{1}{\varepsilon} - 1\right) / h$, then for all $n \geq m$, $|s_n - 0| < \varepsilon$.

Hence $< s_n >$ converges to zero.

Example 5: *Let* $< s_n >$ *be a sequence such that* $s_n \neq 0$ *for any n, and* $\dfrac{s_{n+1}}{s_n} \to l$. *Prove that if* $|l| < 1$, *then* $s_n \to 0$.

Solution: Since $|l| < 1$, hence there exists $\varepsilon_0 > 0$ such that

$$|l| + \varepsilon_0 = h < 1.$$

Now $\dfrac{s_{n+1}}{s_n} \to l \Rightarrow$ there exists a positive integer m such that

$$\left| \frac{s_{n+1}}{s_n} - l \right| < \varepsilon_0 \text{ for all } n \geq m.$$

We have $\left|\frac{s_{n+1}}{s_n}\right| = \left|\left(\frac{s_{n+1}}{s_n} - l\right) + l\right| \le \left|\frac{s_{n+1}}{s_n} - l\right| + |l|,$

$< \varepsilon_0 + |l|$, for all $n \ge m$

i.e., $\left|\frac{s_{n+1}}{s_n}\right| < h$, for all $n \ge m$. ...(1)

Replacing n by $m, m+1, \ldots, n-1$ successively in (1) and multiplying the corresponding sides of the resulting $n - m$ inequalities, we get

$$\left|\frac{s_{m+1}}{s_m}\right| \left|\frac{s_{m+2}}{s_{m+1}}\right| \ldots\ldots \left|\frac{s_n}{s_{n-1}}\right| < h^{n-m},$$

or $$\left|\frac{s_{m+1}}{s_m} \cdot \frac{s_{m+2}}{s_{m+1}} \cdot \ldots\ldots\ldots \frac{s_n}{s_{n-1}}\right| < h^{n-m},$$

or $$|s_n| < h^n \left(\frac{|s_m|}{h^m}\right), \text{ for all } n > m. \quad \ldots(2)$$

Again, since $0 < h < 1$, therefore, $h^n \to 0$ and hence, given $\varepsilon > 0$, there exists a positive integer m_1 such that

$$|h^n| < h^m \varepsilon / |s_m|, \text{ for all } n \ge m_1. \quad \ldots(3)$$

Let us choose a positive integer p such that $p >$ max. $\{m, m_1\}$.

From (2) and (3), we get

$|s_n| < \varepsilon$, for all $n \ge p$.

Hence $s_n \to 0$.

Example 6: *Find an* $m \in \mathbb{N}$ *such that* $\left|\frac{2n}{n+3} - 2\right| < \frac{1}{5}$ *for all* $n \ge m$.

Solution: We have $\left|\frac{2n}{n+3} - 2\right| < \frac{1}{5} \Rightarrow \left|\frac{2n - 2n - 6}{n+3}\right| < \frac{1}{5}$

$\Rightarrow \quad \frac{6}{n+3} < \frac{1}{5} \Rightarrow \frac{n+3}{6} > 5 \Rightarrow n > 27.$

If we take a positive integer $m > 27$, we have

$$\left|\frac{2n}{n+3} - 2\right| < \frac{1}{5}, \text{ for all } n \ge m.$$

Hence for $\varepsilon = 1/5$, the required least value of $m = 28$.

In fact, we can find $m \in \mathbb{N}$ for each $\varepsilon > 0$ such that

$$\left|\frac{2n}{n+3} - 2\right| < \varepsilon \text{ for all } n \ge m.$$

$\therefore \quad \lim \frac{2n}{n+3} = 2.$

Example 7: *Show that the sequence* $< s_n >$ *where* $s_n = \dfrac{2n^2 + 1}{2n^2 - 1}$, $\forall n \in N$ *converges to* 1.

Solution: Let $\varepsilon > 0$ be given.

We have $|s_n - 1| = \left|\dfrac{2n^2 + 1}{2n^2 - 1} - 1\right| = \left|\dfrac{2}{2n^2 - 1}\right|$

$$= \frac{2}{2n^2 - 1} < \varepsilon \text{ if } n > \sqrt{\left(\frac{2 + \varepsilon}{2\varepsilon}\right)}.$$

If we choose a positive integer $m > \sqrt{\left(\dfrac{2 + \varepsilon}{2\varepsilon}\right)}$, then for all $n \geq m$, $|s_n - 1| < \varepsilon$.

$\therefore \quad \lim s_n = 1.$

Example 8: *Show that the sequence* $< s_n >$ *where* $s_n = \dfrac{3n}{n + 5n^{1/2}}$ *has the limit 3.*

Solution: Let $\varepsilon > 0$ be given.

We have $\left|\dfrac{3n}{n + 5n^{1/2}} - 3\right| = \left|\dfrac{3n - 3n - 15n^{1/2}}{n + 5n^{1/2}}\right| = \dfrac{15n^{1/2}}{n + 5n^{1/2}}$

which is less than $\dfrac{15n^{1/2}}{n} = \dfrac{15}{n^{1/2}}$.

Hence $\dfrac{15n^{1/2}}{n + 5n^{1/2}}$ will be $< \varepsilon$ if $\dfrac{15}{n^{1/2}} < \varepsilon$ *i.e.* if $n > \dfrac{225}{\varepsilon^2}$.

If we choose a positive integer $m > \dfrac{225}{\varepsilon^2}$, then $|s_n - 3| < \varepsilon$ for all $n \geq m$.

Hence $\lim\limits_{n \to \infty} s_n = 3.$

Example 9: *Prove that* $\lim (1 / n^p) = 0$, $p > 0$.

Solution: Let $\varepsilon > 0$ be given.

We have $\left|\dfrac{1}{n^p} - 0\right| < \varepsilon \Rightarrow \dfrac{1}{n^p} < \varepsilon \Rightarrow n^p > \dfrac{1}{\varepsilon}$

$$\Rightarrow n > \left(\frac{1}{\varepsilon}\right)^{1/p}.$$

By Archimedean property, for $(1/\varepsilon)^{1/p} \in R$ there exists a positive integer $m > (1/\varepsilon)^{1/p}$. If we choose $m > (1/\varepsilon)^{1/p}$, then we have

$$\left|\frac{1}{n^p} - 0\right| < \varepsilon \text{ for all } n \geq m.$$

Hence $\lim \dfrac{1}{n^p} = 0$, when $p > 0$.

Example 10: *Show that the sequence* $< s_n >$, *where* $s_n = (-1)^{n-1} / n$, *converges to* 0.

Solution: We have $s_n = \frac{1}{n}$ if n is odd, $s_n = -\frac{1}{n}$ if n is even. The sequences $< s_{2n-1} >$ and $< s_{2n} >$ are subsequences of $< s_n >$.

$$< s_{2n-1} > = < 1, \frac{1}{3}, \frac{1}{5}, \ldots. > , < s_{2n} > = < -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, \ldots. >.$$

But $< 1, \frac{1}{3}, \frac{1}{5}, \ldots. >$ and $< \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots. >$ are subsequences of the sequence $< 1/n >$ which converges to 0.

$\therefore < s_{2n-1} >$ and $< s_{2n} >$ both converge to the same limit 0. Hence $< s_n >$ converges to 0. [By theorem 4, article 2.6]

Alternative Solution: Take any given $\varepsilon > 0$.

We have $| s_n - 0 | = | s_n | = \left| \frac{(-1)^{n-1}}{n} \right| = \frac{1}{n} < \varepsilon$ if $n > \frac{1}{\varepsilon}$.

Now by Archimedean property of real numbers, for a given real number $1/\varepsilon$ there exists a positive integer m such that $m > 1/\varepsilon$ or $1/m < \varepsilon$.

Then for all $n \geq m$, we have

$$| s_n - 0 | = \frac{1}{n} \leq \frac{1}{m} < \varepsilon.$$

Hence $< s_n >$ converges to zero.

Example 11: *Show that* $\lim \sqrt[n]{n} = 1$.

Solution: Let $\sqrt[n]{n} = 1 + h_n$, where $h_n \geq 0$.

$$\therefore \quad n = (1 + h_n)^n = 1 + nh_n + \frac{n(n-1)}{1.2} h_n^2 + \ldots + h_n^n$$

$$> \frac{n(n-1)}{2} h_n^2 \text{ for all } n, \text{ since } h_n \geq 0.$$

$$\therefore \quad h_n^2 < \frac{2}{n-1} \text{ for } n \geq 2 \quad i.e. \ |h_n| < \sqrt{\left(\frac{2}{n-1}\right)} \text{ for } n \geq 2.$$

Let $\varepsilon > 0$ be given. Then $|h_n| < \sqrt{\left(\frac{2}{n-1}\right)} < \varepsilon$, provided $\frac{2}{n-1} < \varepsilon^2$ *i.e.* $n > \frac{2}{\varepsilon^2} + 1$.

If we choose $m \in \mathbb{N}$ such that $m > \frac{2}{\varepsilon^2} + 1$, then we have

$$|h_n| < \varepsilon \ \forall n \geq m$$

i.e. $$|\sqrt[n]{n} - 1| < \varepsilon \ \forall n \geq m.$$

$\therefore \lim \sqrt[n]{n} = 1$.

Example 12: *Show that the sequence* $< s_n >$ *defined by*

$$s_n = \{\sqrt{(n+1)} - \sqrt{n}\}, \forall \ n \in \mathbb{N} \text{ is convergent.}$$

Solution: We have $s_n = \sqrt{(n+1)} - \sqrt{n}$

$$= \{\sqrt{(n+1)} - \sqrt{n}\} \frac{\{\sqrt{(n+1)} + \sqrt{n}\}}{\{\sqrt{(n+1)} + \sqrt{n}\}}$$

$$= \frac{1}{\sqrt{(n+1)} + \sqrt{n}} < \frac{1}{\sqrt{n} + \sqrt{n}} = \frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{n}} \quad i.e., \; s_n < \frac{1}{\sqrt{n}}.$$

Let $\varepsilon > 0$ be given. Then $|s_n - 0| < \frac{1}{\sqrt{n}} < \varepsilon$, provided $\sqrt{n} > \frac{1}{\varepsilon}$ *i.e.*, $n > \frac{1}{\varepsilon^2}$.

If m is a positive integer greater than $1/\varepsilon^2$, then

$|s_n - 0| < \varepsilon$ for all $n \geq m$. Hence $\lim s_n = 0$.

Example 13: *Show that the sequence* $< s_n >$ *where* $s_n = \sin n\pi\theta$ *and* θ *is a rational number such that* $0 < \theta < 1$, *is not convergent.*

Solution: Let $\theta = \frac{p}{q}$, where p and q are integers. Since $0 < \theta < 1$, we must have $q \geq 2$.

For $n = q, 2q, 3q, \ldots$, the terms of $< s_n >$ are $\sin \pi p, \sin 2\pi p, \sin 3\pi p, \ldots$, *i.e.*, $0, 0, 0, \ldots$

Thus $< s_n >$ contains a subsequence $< 0, 0, 0, \ldots >$ which converges to 0. Now for $n = q+1, 2q+1, 3q+1, \ldots$ the terms of $< s_n >$ are

$$\sin\left(\pi p + \frac{\pi p}{q}\right), \sin\left(2\pi p + \frac{\pi p}{q}\right), \sin\left(3\pi p + \frac{\pi p}{q}\right), \ldots.$$

i.e., $(-1)^p \sin(\pi p/q), (-1)^{2p} \sin(\pi p/q), (-1)^{3p} \sin(\pi p/q), \ldots.$

All these terms have absolute value $\sin(\pi p/q)$ and do not tend to zero since $0 < \pi p/q < \pi$. $[\because \; 0 < p/q < 1]$.

Thus the sequence $< s_n >$ contains a subsequence whose limit is 0 and a subsequence which (may or may not converge but certainly) does not have the limit zero. Hence we conclude that the given sequence is not convergent. [See corollary to theorem 3]

Remark: The sequence $< \sin n\pi\theta >$ obviously converges to 0 for $\theta = 0$ or $\theta = 1$.

2.7 Divergent Sequences

Definition 1: *A sequence* $< s_n >$ *is said to diverge to* $+\infty$ *if for any given* $k > 0$ *(however large), there exists* $m \in \mathbb{N}$ *such that* $s_n > k$ for all $n \geq m$.

If $< s_n >$ diverges to infinity, we write $s_n \to \infty$ as $n \to \infty$ or $\lim s_n = +\infty$.

Definition 2: *A sequence* $< s_n >$ *is said to diverge to* $-\infty$ *if for any given* $k < 0$ *(however small), there exists* $m \in \mathbb{N}$ *such that* $s_n < k$ *for all* $n \geq m$.

If $< s_n >$ diverges to minus infinity, we write

$s_n \to -\infty$ as $n \to \infty$ or $\lim s_n = -\infty$.

A sequence is said to be a divergent sequence if it diverges to either $+\infty$ *or* $-\infty$.

Illustrations:

1. $< 2, 4, 6, \ldots, 2n, \ldots >$ diverges to $+\infty$.
2. $< 3, 3^2, 3^3, \ldots, 3^n, \ldots >$ diverges to $+\infty$.
3. $< x, x^2, x^3, \ldots, x^n, \ldots >, x > 1$, diverges to $+\infty$.
4. $< -2, -4, -6, \ldots, -2n, \ldots >$ diverges to $-\infty$.
5. $< -3, -3^2, -3^3, \ldots, -3^n, \ldots >$ diverges to $-\infty$.
6. $< -x, -x^2, -x^3, \ldots, -x^n, \ldots >, x > 1$, diverges to $-\infty$.

2.7 Oscillatory Sequences

Definition: *A sequence $< s_n >$ is said to be an oscillatory sequence if it is neither convergent nor divergent.*

An oscillatory sequence is said to oscillate finitely or infinitely according as it is bounded or unbounded.

Illustrations

1. The sequence $< (-1)^n >$ oscillates finitely.
2. The sequence $< (-1)^n n >$ oscillates infinitely.

Theorem 1: *If a sequence $< s_n >$ diverges to infinity then any subsequence of $< s_n >$ also diverges to infinity.*

Proof: Let $< s_{n_k} >$ be any subsequence of the sequence $< s_n >$. Then by the definition of a subsequence $< n_1, n_2, \ldots, n_k, \ldots >$ is a strictly increasing sequence of positive integers. This implies $n_1 \geq 1 \Rightarrow n_k \geq k$ (by induction).

Take any given positive real number k_1.

Now $< s_n >$ diverges to $\infty \Rightarrow$ for $k_1 > 0$ there exists $m \in \mathbb{N}$ such that $s_n > k_1$ for all $n \geq m$ *i.e.*, $s_k > k_1 \; \forall k \geq m$.

For $k \geq m$, we have $n_k \geq k \geq m$ *i.e.*, $n_k \geq m$.

$\therefore \quad s_{n_k} > k_1$ for all $n_k \geq m$.

$\therefore \quad < s_{n_k} >$ diverges to infinity.

Note: If $s_{2n-1} \to \infty$ as $n \to \infty$ and $s_{2n} \to \infty$ as $n \to \infty$, then $s_n \to \infty$ as $n \to \infty$.

Theorem 2: *If $s_n > 0$ for all $n \in \mathbb{N}$, then*

$$s_n \to \infty \text{ as } n \to \infty \Leftrightarrow \frac{1}{s_n} \to 0 \text{ as } n \to \infty.$$

Proof: Let $s_n \to \infty$ as $n \to \infty$.

Let $\varepsilon > 0$ be given. Since $s_n \to \infty$ as $n \to \infty$, hence for $1/\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that

$s_n > 1/\varepsilon$ for all $n \geq m$

$\Rightarrow \quad \frac{1}{s_n} < \varepsilon \; \forall n \geq m \Rightarrow \left|\frac{1}{s_n}\right| < \varepsilon \; \forall n \geq m \qquad [\because s_n > 0]$

$\Rightarrow \quad \left|\frac{1}{s_n} - 0\right| < \varepsilon \; \forall n \geq m \Rightarrow \frac{1}{s_n} \to 0$ as $n \to \infty$.

Conversely, let $\frac{1}{s_n} \to 0$ as $n \to \infty$.

Take any given $k > 0$.

Now $\frac{1}{s_n} \to 0$ as $n \to \infty \Rightarrow$ for $1/k > 0$ there exists $m \in N$ such that

$$\left|\frac{1}{s_n} - 0\right| < \frac{1}{k} \text{ for all } n \geq m$$

$\Rightarrow \quad \frac{1}{s_n} < \frac{1}{k} \; \forall n \geq m \Rightarrow s_n > k \quad \forall \; n \geq m$

$\Rightarrow \quad s_n \to \infty$ as $n \to \infty$.

Theorem 3: *If the sequences $< s_n >$, $< t_n >$ diverge to infinity then $< s_n + t_n >$ and $< s_n \, t_n >$ diverge to infinity.*

Proof: Take any given $k > 0$. The sequence $< s_n >$ diverges to infinity $\Rightarrow$ for $k > 0$, there exists $m_1 \in N$ such that $s_n > k \; \forall n \geq m_1$. Again, the sequence $< t_n >$ diverges to infinity $\Rightarrow$ for $1 > 0$, there exists $m_2 \in N$ such that $t_n > 1 \; \forall n \geq m_2$.

Take $m = \max. \{m_1, m_2\}$.

$\therefore \quad s_n + t_n > k + 1 > k \quad \forall n \geq m \quad$ and $\quad s_n \, t_n > k.1 = k \quad \forall n \geq m.$

$\therefore$ Both $< s_n + t_n >$ and $< s_n \, t_n >$ diverge to infinity.

Theorem 4: *If $< s_n >$ diverges to infinity and $< t_n >$ is bounded then $< s_n + t_n >$ diverges to infinity.*

Proof: The sequence $< t_n >$ is bounded $\Rightarrow$ there exists $k_1 > 0$ such that

$$|t_n| < k_1 \; \forall n \in N.$$

The sequence $< s_n >$ diverges to infinity $\Rightarrow$ for $k > 0$ there exists $m \in N$ such that $s_n > k + k_1 \; \forall n \geq m$.

$\therefore$ For all $n \geq m$, we have

$$s_n + t_n \geq s_n - |t_n| \qquad [\because \; x \geq -|x|]$$
$$> k + k_1 - k_1 = k.$$

Thus for $k > 0$, there exists $m \in N$ such that $s_n + t_n > k$ for all $n \geq m$.

Hence $< s_n + t_n >$ diverges to infinity.

Corollary: *If $< s_n >$ diverges to infinity and $< t_n >$ converges then $< s_n + t_n >$ diverges to infinity.*

Illustrative Examples

Example 14: *Prove that the sequence* $< n^p >$ *where* $p > 0$ *diverges to infinity.*

Solution: Let $s_n = n^p$. Then $s_n > 0$ for all n as $n \in \mathbb{N}$ and $p > 0$.

$\therefore$ The sequence $< \frac{1}{s_n} > = < \frac{1}{n^p} >$ exists.

Since we know that $\frac{1}{n^p} \to 0$ as $n \to \infty$,

$\therefore$ $n^p \to \infty$ as $n \to \infty$.

Hence $< n^p >$ diverges to ∞.

Example 15: *Show that the sequence* $< \log \frac{1}{n} >$ *diverges to* $-\infty$.

Solution: Let $s_n = \log \frac{1}{n}$. Take any given $k < 0$.

Then $s_n < k$ if $\log \frac{1}{n} < k$ *i.e.,* if $-\log n < k$

i.e., if $\log n > -k$ *i.e.,* if $n > e^{-k}$.

If we take $m \in \mathbb{N}$ such that $m > e^{-k}$, then $s_n < k$ for all $n \geq m$.

Hence $s_n \to -\infty$ as $n \to \infty$.

Example 16: *If* $< t_n >$ *diverges to* ∞ *and* $s_n > t_n \ \forall \ n$, *then* $< s_n >$ *diverges to* ∞.

Solution: Take any given $k > 0$.

Since $< t_n >$ diverges to ∞, therefore, for $k > 0$ there exists $m \in \mathbb{N}$ such that

$t_n > k$ for all $n \geq m$

$\Rightarrow$ $s_n > k$ for all $n \geq m$. $[\because s_n > t_n \ \forall \ n \in \mathbb{N}]$

Hence $< s_n >$ diverges to ∞.

2.9 Algebra of Convergent Sequences

Theorem 1: *If* $\lim s_n = l$ *and* $\lim t_n = l'$ *then* $\lim (s_n + t_n) = l + l'$. *In other words the limit of the sum of two convergent sequences is the sum of their limits.*

Proof: Take any given $\varepsilon > 0$.

Since $\lim s_n = l$, therefore for a given positive real number $\varepsilon / 2$ there exists $m_1 \in \mathbb{N}$ such that

$|s_n - l| < \varepsilon / 2$ for all $n \geq m_1$.

Similarly, since $\lim t_n = l'$, there exists $m_2 \in \mathbb{N}$ such that $|t_n - l'| < \varepsilon / 2$ for all $n \geq m_2$.

Let $m = \max. \{m_1, m_2\}$. Then

$$|s_n - l| < \varepsilon / 2 \text{ and } |t_n - l'| < \varepsilon / 2 \text{ for all } n \geq m.$$

$\therefore$ for all $n \geq m$, we have

$$|(s_n + t_n) - (l + l')| = |(s_n - l) + (t_n - l')|$$

$$\leq |s_n - l| + |t_n - l'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus for any given $\varepsilon > 0$, there exists a positive integer m such that

$$|(s_n + t_n) - (l + l')| < \varepsilon \text{ for all } n \geq m.$$

$\therefore$ The sequence $< s_n + t_n >$ is convergent and

$$\lim (s_n + t_n) = l + l' = \lim s_n + \lim t_n.$$

Note: The converse of the above theorem need not be true.

Let $s_n = (-1)^n$ and $t_n = (-1)^{n+1}$.

Then $s_n + t_n = (-1)^n + (-1)^{n+1} = (-1)^n [1 + (-1)] = 0$.

Hence the sequence $< s_n + t_n >$ converges to 0, while $< s_n >$ and $< t_n >$ oscillate finitely.

Theorem 2: *If* $\lim s_n = l$ and $c \in \mathbb{R}$, *then* $\lim (cs_n) = cl$.

Proof: If $c = 0$, the theorem is obvious because then $\lim (cs_n) = 0 = 0 \cdot l$.

Let $c \neq 0$. Take any given $\varepsilon > 0$.

Since $\lim s_n = l$, hence for a given positive real number $\varepsilon / |c|$, there exists $m \in \mathbb{N}$ such that

$$|s_n - l| < \frac{\varepsilon}{|c|} \text{ for all } n \geq m.$$

Now for all $n \geq m$, we have

$$|cs_n - cl| = |c(s_n - l)| = |c||s_n - l| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon.$$

$\therefore$ $\lim (cs_n) = cl$.

Theorem 3: *If* $\lim s_n = l$ *and* $\lim t_n = l'$, *then* $\lim (s_n - t_n) = l - l'$.

Proof: By theorem 2,

$$\lim (-t_n) = \lim [(-1) t_n] = (-1) l' = -l'.$$

We have $\lim (s_n - t_n) = \lim [s_n + (-t_n)] = \lim s_n + \lim (-t_n)$, [by theorem 1]

$$= l - l'.$$

Corollary: *If* $< s_n >$ *and* $< t_n >$ *are convergent sequences such that* $s_n \leq t_n$ *for all* $n \in \mathbb{N}$ *and* $\lim s_n = l$, $\lim t_n = l'$, *then* $l \leq l'$.

Proof: By theorem 3, we have $\lim (t_n - s_n) = l' - l$. By hypothesis $t_n - s_n \geq 0$ for all $n \in \mathbb{N}$. Hence $l' - l \geq 0$, by theorem 1 of article 2.6. Thus $l' \geq l$ i.e., $l \leq l'$.

Theorem 4: *If* $\lim s_n = 0$ *and the sequence* $< t_n >$ *is bounded then*

$$\lim (s_n\, t_n) = 0.$$

Proof: The sequence $< t_n >$ is bounded $\Rightarrow$ there exists $k \in R^+$ such that

$$| t_n | < k \ \forall\ n \in N.$$

Take any given $\varepsilon > 0$.

Since $\lim s_n = 0$, therefore for a given positive real number ε / k there exists $m \in N$ such that $| s_n - 0 | = | s_n | < \frac{\varepsilon}{k}\ \forall n \geq m$.

Now for all $n \geq m$, we have

$$| s_n\, t_n - 0 | = | s_n\, t_n | = | s_n |\, | t_n | < \frac{\varepsilon}{k} \cdot k = \varepsilon.$$

Thus for any given $\varepsilon > 0$, there exists $m \in N$ such that $| s_n\, t_n - 0 | < \varepsilon$ for all $n \geq m$.

Hence $\lim (s_n\, t_n) = 0$.

Theorem 5: *If* $\lim s_n = l$ *and* $\lim t_n = l'$, *then* $\lim (s_n t_n) = ll'$.

Proof: Let $\varepsilon > 0$ be given. We have

$$\begin{aligned} |s_n\, t_n - ll'| &= |s_n t_n - lt_n + lt_n - ll'| = |t_n\, (s_n - l) + l\, (t_n - l')| \\ &\leq | t_n\, (s_n - l) | + | l\, (t_n - l') | \\ &= | t_n |\, | s_n - l | + | l |\, | t_n - l' |. \end{aligned} \quad \ldots(1)$$

Since the sequence $< t_n >$ is convergent, therefore it is bounded *i.e.* there exists a positive real number k such that

$$| t_n | \leq k \text{ for all } n \in N.$$

Since the sequences $< s_n >$ and $< t_n >$ are convergent, therefore there exist positive integers m_1 and m_2, such that

$$| s_n - l | < \frac{\varepsilon}{2k} \text{ for all } n \geq m_1 \quad \ldots(2)$$

and

$$| t_n - l' | < \frac{\varepsilon}{2\,(|\, l\,| + 1)} \text{ for all } n \geq m_2\ . \quad \ldots(3)$$

Let $m = \max. \{m_1, m_2\}$. From (1), (2) and (3), we have for all $n \geq m$

$$| s_n\, t_n - ll' | < k \cdot \frac{\varepsilon}{2k} + | l | \frac{\varepsilon}{2\,(|\, l\,| + 1)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\lim (s_n\, t_n) = l\, l'$.

Note 1: In the inequality (3) we have taken $(| l | + 1)$. Had we not done so, this inequality would have failed in case $l = 0$ Hence to include this case also we used this device.

2. The converse of the above theorem need not be true.

Let $s_n = (-1)^n$ and $t_n = (-1)^{n+1}$.

Then $s_n\, t_n = (-1)^n\, (-1)^{n+1} = (-1)^{2n+1} = -1$.

Thus $< s_n\ t_n >$ is a constant sequence and converges to -1 while $< s_n >$ and $< t_n >$ oscillate finitely.

Theorem 6: *If* $\lim s_n = l$ and $l \neq 0$, *then there exists a positive number* k *and a positive integer* m, *such that* $|s_n| > k$ for all $n \geq m$.

Proof: Let us choose $\varepsilon = \frac{1}{2}|l|$. Then $\varepsilon > 0$, since $l \neq 0$. Since $\lim s_n = l$, therefore there must exist a positive integer m, such that $|s_n - l| < \varepsilon$ for all $n \geq m$.

We can write $l = l - s_n + s_n$.

$$\therefore \quad |l| = |(l - s_n) + s_n|$$
$$\leq |l - s_n| + |s_n| < \varepsilon + |s_n| \text{ for all } n \geq m.$$
$$\therefore \quad |s_n| > |l| - \varepsilon \text{ for all } n \geq m$$
$$\text{or} \quad |s_n| > |l| - \frac{1}{2}|l| = \frac{1}{2}|l| \text{ for all } n \geq m.$$

Thus we have found a positive number $k = \frac{1}{2}|l|$ and a positive integer m, such that $|s_n| > k$ for all $n \geq m$.

Theorem 7: *If* $\lim t_n = l'$, $l' \neq 0$ and $t_n \neq 0\ \forall n$, *then* $\lim (1/t_n) = 1/l'$.

Proof: We have $\left|\frac{1}{t_n} - \frac{1}{l'}\right| = \frac{|l' - t_n|}{|t_n| \cdot |l'|}$. ...(1)

Since $l' \neq 0$, therefore by theorem 6, there exists a positive number k and a positive integer m_1, such that

$$|t_n| > k \text{ or } \frac{1}{|t_n|} < \frac{1}{k} \text{ for all } n \geq m_1. \quad \ldots(2)$$

Take any given $\varepsilon > 0$.

Since $\lim t_n = l'$, therefore for a given positive real number $k|l'|\varepsilon$, there exists $m_2 \in \mathbb{N}$ such that

$$|t_n - l'| < k|l'|\varepsilon \text{ for all } n \geq m_2. \quad \ldots(3)$$

Let $m = \max.\{m_1, m_2\}$. From (1), (2) and (3), we have

$$\left|\frac{1}{t_n} - \frac{1}{l'}\right| < \frac{1}{|l'|} \cdot \frac{1}{k} \cdot k|l'|\varepsilon \text{ for all } n \geq m$$

i.e., $$\left|\frac{1}{t_n} - \frac{1}{l'}\right| < \varepsilon \text{ for all } n \geq m.$$

Hence $\lim (1/t_n) = 1/l'$.

Theorem 8: *If* $\lim s_n = l$ *and* $\lim t_n = l'\ (\neq 0)$, $t_n \neq 0$ *for all* n, *then* $\lim (s_n / t_n) = l / l'$.

Proof: Since $t_n \neq 0\ \forall\ n$ and $l' \neq 0$, therefore, by theorem 7, $\lim (1/t_n) = 1/l'$.

Now $\lim\left(\frac{s_n}{t_n}\right) = \lim\left(s_n \cdot \frac{1}{t_n}\right) = (\lim s_n)\lim\left(\frac{1}{t_n}\right)$ [by theorem 5, article 2.9]

$$= l \cdot \frac{1}{l'} = \frac{l}{l'}.$$

Hence $\lim \frac{s_n}{t_n} = \frac{l}{l'} = \frac{\lim s_n}{\lim t_n}$.

Theorem 9: (Sandwich theorem). *If $< s_n >, < t_n >$ and $< u_n >$ are three sequences such that*

(i) for some positive integer k, $s_n \le u_n \le t_n$ for $n \ge k$,

(ii) $\lim s_n = \lim t_n = l$, then $\lim u_n = l$.

Proof: Let $\varepsilon > 0$ be given.

Since $\lim s_n = l$, therefore, there exists $m_1 \in \mathbb{N}$ such that

$$|s_n - l| < \varepsilon \text{ for all } n \ge m_1$$

i.e., $l - \varepsilon < s_n < l + \varepsilon$ for all $n \ge m_1$.

Similarly $\lim t_n = l$, therefore, there exists $m_2 \in \mathbb{N}$ such that

$$|t_n - l| < \varepsilon \text{ for all } n \ge m_2$$

i.e., $l - \varepsilon < t_n < l + \varepsilon$ for all $n \ge m_2$.

Let $m = \max.\{m_1, m_2, k\}$. Then, for $n \ge m$, we have

$$l - \varepsilon < s_n \le u_n \le t_n < l + \varepsilon$$

or $l - \varepsilon < u_n < l + \varepsilon$.

Thus $|u_n - l| < \varepsilon$ for all $n \ge m$.

Hence $\lim u_n = l$.

Corollary: *If $< s_n >$ and $< t_n >$ are two sequences such that $|s_n| \le |t_n| \forall n \ge k$ where $k \in \mathbb{N}$ and $\lim t_n = 0$, then $\lim s_n = 0$.*

Proof: Lim $t_n = 0 \Rightarrow \lim |t_n| = 0$ and $\lim(-|t_n|) = 0$.

We have $|s_n| \le |t_n| \ \forall n \ge k$

$\Rightarrow$ $-|t_n| \le s_n \le |t_n| \ \forall n \ge k$.

Hence by Sandwich theorem, $\lim s_n = 0$.

Note: If $|s_n| \le \alpha |t_n| \forall n \ge k$ where $k \in \mathbb{N}$ and α is a positive real number, then

$$\lim t_n = 0 \Rightarrow \lim s_n = 0.$$

For example, let $s_n = \frac{\cos n\pi}{n}$.

Then $|s_n| = \left|\frac{\cos n\pi}{n}\right| \le \frac{1}{n}$. $\quad [\because -1 \le \cos n\pi \le 1]$

$\therefore$ $|s_n| \le \left|\frac{1}{n}\right|$ and $\lim \frac{1}{n} = 0$.

Hence $\lim s_n = \lim \frac{\cos n\pi}{n} = 0.$

Theorem 10: (Cauchy's first theorem on limits)

If $\lim_{n \to \infty} s_n = l$, *then* $\lim_{n \to \infty} \frac{s_1 + s_2 + \ldots + s_n}{n} = l.$

Proof: Define a sequence $< t_n >$ such that

$$s_n = l + t_n \ \forall \ n \in \mathrm{N}.$$

$\therefore$ $\lim t_n = 0$

and $$\frac{s_1 + s_2 + \ldots + s_n}{n} = l + \frac{t_1 + t_2 + \ldots + t_n}{n}. \qquad \ldots(1)$$

In order to prove the theorem we wish to show that

$$\lim_{n \to \infty} \frac{t_1 + t_2 + \ldots + t_n}{n} = 0.$$

Let $\varepsilon > 0$ be given. Since $\lim t_n = 0$, therefore, there exists a positive integer m, such that

$$|t_n - 0| = |t_n| < \varepsilon / 2 \quad \forall \ n \geq m. \qquad \ldots(2)$$

Also, since every convergent sequence is bounded, hence there exists a real number $k > 0$ such that

$$|t_n| \leq k \ \forall n \in \mathrm{N}. \qquad \ldots(3)$$

Now for all $n \geq m$, we have

$$\left|\frac{t_1 + t_2 + \ldots + t_n}{n}\right| = \left|\frac{t_1 + t_2 + \ldots + t_m}{n} + \frac{t_{m+1} + t_{m+2} + \ldots + t_n}{n}\right|$$

$$\leq \frac{|t_1| + |t_2| + \ldots + |t_m|}{n} + \frac{|t_{m+1}| + |t_{m+2}| + \ldots + |t_n|}{n}$$

$$\leq \frac{mk}{n} + \frac{n-m}{n} \cdot \frac{\varepsilon}{2} \qquad \text{[From (2) and (3)]}$$

$$< \frac{mk}{n} + \frac{\varepsilon}{2}. \qquad \ldots(4)$$

$$\left[\because \ 0 \leq \frac{n-m}{n} < 1\right]$$

If m is fixed, then $\frac{mk}{n} < \frac{1}{2}\varepsilon$ if $n > \frac{2mk}{\varepsilon}$.

Let us choose a positive integer $p > \frac{2mk}{\varepsilon}$. Then

$$\frac{mk}{n} < \frac{1}{2}\varepsilon \text{ for } n \geq p. \qquad \ldots(5)$$

Let $M = \max. \{m, p\}$. From (4) and (5), we have

$$\left|\frac{t_1 + t_2 + \ldots + t_n}{n}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ \forall n \geq M.$$

Thus $\lim \frac{t_1 + t_2 + \ldots + t_n}{n} = 0$ and consequently (1) gives

$$\lim \frac{s_1 + s_2 + \ldots + s_n}{n} = l.$$

Note: The converse of the above theorem need not be true.

Consider the sequence $< s_n >$ where $s_n = (-1)^n$.

For this sequence,

$$\frac{s_1 + s_2 + \ldots + s_n}{n} = 0, \text{ if } n \text{ is even, and } = -\frac{1}{n}, \text{ if } n \text{ is odd.}$$

$\therefore$ $\lim \frac{s_1 + s_2 + \ldots + s_n}{n} = 0$, but $< s_n >$ is not convergent.

Theorem 11: (Cauchy's second theorem on limits). *If $< s_n >$ is a sequence such that $s_n > 0$ for all n and $\lim s_n = l$, then* $\lim (s_1\ s_2 \ldots s_n)^{1/n} = l$.

Proof: Let us define a sequence $< t_n >$ such that

$$t_n = \log s_n \text{ for all } n.$$

Since $\lim s_n = l$, therefore $\lim t_n = \log l$.

By Cauchy's first theorem on limits, we have

$$\lim \frac{t_1 + t_2 + \ldots + t_n}{n} = \log l$$

i.e., $$\lim \frac{\log s_1 + \log s_2 + \ldots + \log s_n}{n} = \log l$$

i.e., $$\lim \log (s_1\ s_2 \ldots s_n)^{1/n} = \log l$$

and hence, $\lim (s_1\ s_2 \ldots s_n)^{1/n} = l$.

Note: While proving this theorem, we have used the following fact :

$$\lim s_n = l \Leftrightarrow \lim \log s_n = \log l,$$

provided $s_n > 0$ for all n and $l > 0$.

Theorem 12: *If $< s_n >$ is a sequence such that $s_n > 0$ for all $n \in \mathbf{N}$ and $\lim \frac{s_{n+1}}{s_n} = l$, then* $\lim \sqrt[n]{s_n} = l$.

Proof: Let us define a sequence $< t_n >$ such that

$$t_1 = s_1, t_2 = \frac{s_2}{s_1}, t_3 = \frac{s_3}{s_2}, \ldots, t_n = \frac{s_n}{s_{n-1}}, \ldots.$$

$\therefore$ $$t_1\ t_2 \ldots t_n = s_n .$$

Also $\lim \frac{s_{n+1}}{s_n} = l \Rightarrow \lim \frac{s_n}{s_{n-1}} = l \Rightarrow \lim t_n = l.$.

Since $s_n > 0$ for all n, hence $t_n > 0$ for all n.

Thus we have a sequence $< t_n >$ such that $t_n > 0$ for all n and $\lim t_n = l$.

Hence by therorem 11 of article 2.9 , we have $\lim (t_1\ t_2 \ldots t_n)^{1/n} = l$

i.e., $\lim (s_n)^{1/n} = l$.

Theorem 13: (Cesaro's theorem)

If $\lim s_n = l$ *and* $\lim t_n = l'$, *then*

$$\lim \frac{s_1 t_n + s_2 t_{n-1} + \ldots + s_n\ t_1}{n} = ll'.$$

Proof: Let $s_n = l + x_n$ and $|x_n| = X_n$. Then $\lim x_n = 0$ and hence $\lim X_n = 0$.

Therefore by theorem 10 of article 2.9, we have

$$\lim \frac{1}{n}(X_1 + X_2 + \ldots + X_n) = 0.$$

Now
$$\frac{1}{n}(s_1\ t_n + s_2\ t_{n-1} + \ldots + s_n\ t_1)$$

$$= \frac{l}{n}(t_1 + t_2 + \ldots + t_n) + \frac{1}{n}(x_1\ t_n + x_2\ t_{n-1} + \ldots + x_n\ t_1), \qquad \ldots(1)$$

on substituting for $s_1, s_2, \ldots, s_n$.

Now the sequence $< t_n >$ is convergent and every convergent sequence is bounded.

Therefore there exists a positive real number k, such that

$$|t_n| < k\ \forall n.$$

$\therefore$
$$0 \le \left|\frac{1}{n}(x_1\ t_n + x_2\ t_{n-1} + \ldots + x_n\ t_1)\right|$$

$$\le \frac{1}{n}[|x_1|.|t_n| + |x_2|.|t_{n-1}| + \ldots + |x_n|.|t_1|]$$

$$< \frac{k}{n}(|x_1| + |x_2| + \ldots + |x_n|)$$

$$= \frac{k}{n}(X_1 + X_2 + \ldots + X_n)$$

$$\to 0, \text{ since } k \text{ is fixed for all } n.$$

$\therefore$ $\lim \frac{1}{n}(x_1 t_n + x_2 t_{n-1} + \ldots + x_n\ t_1) = 0$, by Sandwich theorem.

Now since $\lim t_n = l'$, we have by theorem 10 of article 2.9,

$$\lim \frac{1}{n}(t_1 + t_2 + \ldots + t_n) = l'.$$

Hence finally, we get from (1)

$$\lim \frac{1}{n}(s_1 t_n + s_2\ t_{n-1} + \ldots + s_n\ t_1) = ll'.$$

Note: Theorems of article 2.9 provide an easier method for evaluating the limits of sequences than the method for evaluating these limits directly by definition. Later on we shall illustrate the use of the theorems of this section to evaluate the limits of sequences.

2.10 Monotonic Sequences

Definition 1: *A sequence* $< s_n >$ *is said to be monotonically increasing (or non-decreasing), if* $s_n \leq s_{n+1}$ *for all* n *i.e.,* $s_n \leq s_m$ *for* $n < m$.

Definition 2: *A sequence* $< s_n >$ *is said to be strictly increasing if* $s_n < s_{n+1}$ *for all* $n \in \mathbb{N}$.

Definition 3: *A sequence* $< s_n >$ *is said to be monotonically decreasing (or non-increasing) if* $s_n \geq s_{n+1}$ *for all* n *i.e.,* $s_n \geq s_m$ *for* $n < m$.

Definition 4: *A sequence* $< s_n >$ *is said to be strictly decreasing if* $s_n > s_{n+1}$ *for all* $n \in \mathbb{N}$.

Definition 5: *A sequence* $< s_n >$ *is said to be monotonic if it is either monotonically increasing or monotonically decreasing.*

Illustrations:

1. The sequence $< 1, 2, 3, \ldots, n, \ldots >$ is strictly increasing.
2. $< 2, 2, 4, 4, 6, 6, \ldots >$ is monotonically increasing.
3. $< -\frac{1}{n} >$ is strictly increasing.
4. $< 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots >$ is strictly decreasing.
5. $< 1, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \ldots >$ is monotonically decreasing.
6. $< -2, -4, -6, -8, \ldots >$ is strictly decreasing.
7. $< 0, 1, 0, 1, 0, 1, \ldots >$ is not monotonic.
8. $< -2, 2, -4, 4, -6, 6, \ldots >$ is not monotonic.
9. $< 1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \ldots >$ is not monotonic.

Note: If $< s_n >$ is a sequence of positive terms, then $< s_n >$ is increasing $\Leftrightarrow < 1 / s_n >$ is decreasing.

Theorem 1: (Monotone convergence theorem). *Every bounded monotonically increasing sequence converges.*

Proof: Let $< s_n >$ be a bounded monotonically increasing sequence.

Let $S = \{ s_n : n \in \mathbb{N} \}$ be the range of the sequence $< s_n >$. Then S is a non-empty set which is bounded above. Hence by the completeness axiom for R, there exists a number $l = \sup S$. We shall show that $< s_n >$ converges to l.

Let $\varepsilon > 0$ be given. Then $l - \varepsilon < l$, so that $l - \varepsilon$ is not an upper bound of S. Hence there exists a positive integer m such that $s_m > l - \varepsilon$. Since $< s_n >$ is monotonically increasing, therefore,

$$s_n \geq s_m > l - \varepsilon \text{ for all } n \geq m. \quad \ldots(1)$$

Also, since l is the supremum of S, therefore,

$$s_n \leq l < l + \varepsilon \text{ for all } n. \quad \ldots(2)$$

From (1) and (2), we get $l - \varepsilon < s_n < l + \varepsilon$ for all $n \geq m$

i.e., $|s_n - l| < \varepsilon$ for all $n \geq m$.

$\therefore$ $\lim s_n = l$.

Corollary 1: *Every bounded monotonically decreasing sequence converges.*

Proof: Let $< s_n >$ be a bounded monotonically decreasing sequence. Define a sequence $< t_n >$ such that $t_n = -s_n$, for all $n \in \mathbb{N}$. Then $< t_n >$ is a bounded monotonically increasing sequence and hence by the above theorem, it converges. If $\lim t_n = l$, then $\lim s_n = \lim (-t_n) = -\lim t_n = -l$.

Note: We can prove this result independently by taking infimum of the set S.

Corollary 2: *Every bounded monotonic sequence converges.* **(Kanpur 2012)**

Proof: In order to prove this theorem we are to prove the following two results.

(i) Every bounded monotonically increasing sequence is convergent. (Give complete proof of theorem 1)

(ii) Every bounded monotonically decreasing sequence is convergent. (Deduce it from the result of theorem 1 as deduced in corollary 1)

Theorem 2: *A non-decreasing (i.e., monotonically increasing) sequence which is not bounded above diverges to infinity.*

Proof: Let $< s_n >$ be a non-decreasing sequence which is not bounded above.

Take any real number $k > 0$.

Now $< s_n >$ is not bounded above $\Rightarrow$ there exists $m \in \mathbb{N}$ such that $s_m > k$.

Also $< s_n >$ is non-decreasing $\Rightarrow s_n \geq s_m$ for $n > m$.

$\therefore$ $s_n \geq s_m > k$ for $n > m$ or $s_n > k$ for $n > m$.

Hence $< s_n >$ diverges to infinity.

Theorem 3: *A non-increasing (i.e., monotonically decreasing) sequence which is not bounded below diverges to minus infinity.*

Proof: Let $< s_n >$ be a non-increasing sequence which is not bounded below.

Take any real number $k < 0$.

Now $< s_n >$ is not bounded below $\Rightarrow$ there exists $m \in \mathbb{N}$ such that $s_m < k$.

Also $< s_n >$ is non-increasing $\Rightarrow s_n \leq s_m$ for $n > m$.

$\therefore$ $s_n \leq s_m < k$ for $n > m$ or $s_n < k$ for $n > m$.

Hence $< s_n >$ diverges to $-\infty$.

Theorem 4: *Every sequence has a monotonic subsequence.*

Proof: Consider the sequence $a_0 = < s_n >$. Let $a_1, a_2, a_3, \ldots$ denote the subsequences

$$< s_2, s_3, s_4, \ldots >, < s_3, s_4, s_5, \ldots >, < s_4, s_5, s_6, \ldots >, \ldots \text{ respectively.}$$

There arise two different cases.

(i) **Each of the sequences $a_0, a_1, a_2, \ldots$ has a greatest term.** Let $s_{n_1}, s_{n_2}, s_{n_3}, \ldots$ denote the greatest terms of $a_0, a_1, a_2, \ldots$ respectively.

Then $n_1 \le n_2 \le n_3 \le \ldots$ and $s_{n_1} \ge s_{n_2} \ge s_{n_3} \ge \ldots$.

Consequently $< s_{n_1}, s_{n_2}, s_{n_3}, \ldots >$ is a monotonically decreasing subsequence of $< s_n >$.

(ii) **At least one of the sequences $a_0, a_1, a_2, \ldots$ has no greatest term.** Suppose a_m has no greatest term. Then each term of a_m is ultimately followed by some term of a_m that exceeds it. For, if there is a term of a_m which exceeds all the terms following it, then it can be exceeded by finitely many terms at the most and hence, a_m must have a greatest term. Now s_{m+1} is the first term of a_m. Let s_{n_2} be the first term of a_m exceeding s_{m+1}, s_{n_3} the first term of a_m that follows s_{n_2} and exceeds it, s_{n_4} the first term of a_m that follows s_{n_3} and exceeds it, and so on.

Thus $< s_{m+1}, s_{n_2}, s_{n_3}, s_{n_4}, \ldots >$ is a monotonically increasing subsequence of $< s_n >$.

Note: In the above proof, we have used the concept of a greatest term of a sequence.

A term s_k of a sequence $< s_n >$ is said to be a greatest term of $< s_n >$ if $s_n \le s_k$ for all n.

It is not necessary for a sequence to have a greatest term; *e.g.*, the sequence $< 1, 3, 5, \ldots >$ has no greatest term. Also, it is not necessary that a greatest term of a sequence be unique; *e.g.*, for the sequence $< s_n >$ defined by $s_n = (-1)^{n-1}$, each of the terms $s_1, s_3, s_5, \ldots$ is a greatest term.

If $< s_{n_k} >$ be a subsequence of $< s_n >$, s_{n_p} be a greatest term of $< s_{n_k} >$ and s_m be a greatest term of $< s_n >$, then $s_{n_p} \le s_m$, because s_{n_p} is also a term of $< s_n >$, and s_m is a greatest term of $< s_n >$.

2.11 Limit Points of a Sequence

Definition: *A real number p is said to be a limit point (or a cluster point) of a sequence $< s_n >$ if every neighbourhood of p contains infinite number of terms of the sequence.*

Since every open interval $]\, p - \varepsilon, p + \varepsilon\, [$, $\varepsilon > 0$, is a neighbourhood of p and also every neighbourhood of p contains an open interval $]\, p - \varepsilon, p + \varepsilon\, [$ for some $\varepsilon > 0$, therefore we can say that a real number p is a limit point of a sequence $< s_n >$ iff given any $\varepsilon > 0$, $s_n \in]\, p - \varepsilon, p + \varepsilon\, [$ for infinitely many values of n *i.e.*, $| s_n - p | < \varepsilon$ for infinitely many values of n.

It can be easily seen that a real number p is a limit point of a sequence $< s_n >$ iff given any neighbourhood N of p and $m \in \mathbb{N}$ we can find $k \in \mathbb{N}$ such that $k > m$ and $s_k \in N$.

Some Remarks 1: Limit point of a sequence is different from the limit of a sequence. The limit of a sequence is a limit point of the sequence, while a limit point of a sequence need not be the limit of the sequence.

2. Limit point of a sequence need not be a term of the sequence.

3. If $s_n = l$ for infinitely many values of n then l is a limit point of $< s_n >$.

4. A real number p is not a limit point of $< s_n >$ if there exists even one neighbourhood of p containing finite number of terms of the sequence.

Illustrations:

1. The sequence $< (-1)^n >$ has 1 and -1 as limit points. Here $s_n = -1$, if n is odd and $s_n = 1$, if n is even. Any neighbourhood of -1 will contain all the odd terms of the sequence hence -1 is a limit point.

Similarly any neighbourhood of 1 will contain all the even terms of the sequence, so 1 is a limit point.

2. The sequence $< \frac{1}{n} >$ has only one limit point, namely 0.

Given $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$.

For $n \geq m$, $0 < \frac{1}{n} \leq \frac{1}{m} < \varepsilon$ *i.e.*, $-\varepsilon < 0 < \frac{1}{n} < \varepsilon$ for all $n \geq m$.

Hence $\frac{1}{n} \in]-\varepsilon, \varepsilon[$ for all $n \geq m$. Thus every ε-nhd of 0 contains infinitely many points of the sequence. Hence 0 is a limit point of this sequence.

3. The sequence $< 1, 2, 3, \ldots, n, \ldots >$ has no limit point.

Let $p \in \mathbb{R}$. Whatever ε we take, the neighbourhood $]p - \varepsilon, p + \varepsilon[$ of p contains at the most a finite number of terms of this sequence. Hence p is not a limit point of this sequence.

Theorem 1: *If l is a limit point of the range of a sequence $< s_n >$, then l is a limit point of the sequence $< s_n >$.*

Proof: Let S be the range set of the sequence $< s_n >$. Since l is a limit point of S, therefore, every nhd. of l contains infinite number of distinct elements of the set S. But each element of the set S is a term of the sequence $< s_n >$. Hence every nhd of l contains infinite number of terms of the sequence $< s_n >$. Thus l is a limit point of the sequence $< s_n >$.

Note 1: The converse of the above theorem need not be true.

Consider the sequence $< s_n >$ where $s_n = 1 + (-1)^n$.

We have $s_n = 0$, if n is odd and $s_n = 2$, if n is even.

$\therefore$ 0, 2 are limit points of the sequence $< s_n >$. But the range of this sequence is the set $\{0, 2\}$, which is a finite set.

Now a finite set has no limit points, hence the range of $< s_n >$ has no limit points.

2. If a sequence has all its terms distinct, then the limit points of the sequence and the limit points of the range set are same.

Theorem 2: *If $s_n \to l$, then l is the only limit point of $< s_n >$.*

Proof: First we shall show that l is a limit point of $< s_n >$. Let $\varepsilon > 0$ be given. Since $s_n \to l$, therefore, there exists a positive integer m such that

$$|s_n - l| < \varepsilon \text{ for all } n \geq m$$

i.e., $|s_n - l| < \varepsilon$ for infinitely many values of n.

This shows that l is a limit point of $< s_n >$.

Now we shall show that if l' be any limit point of $< s_n >$, then we must have $l' = l$.

Let $\varepsilon > 0$ be arbitrary. Since l is the limit of $< s_n >$, therefore, there exists a positive integer p such that

$$|s_n - l| < \varepsilon / 2 \text{ for all } n \geq p. \quad \ldots(1)$$

Since l' is a limit point of $< s_n >$, therefore, there must exist a positive integer $q > p$ such that $|s_q - l'| < \varepsilon / 2$. $\ldots(2)$

Putting $n = q$ in (1), $|s_q - l| < \varepsilon / 2$. $\ldots(3)$

Now $|l - l'| = |(s_q - l') - (s_q - l)| \leq |s_q - l'| + |s_q - l|$

$< \varepsilon / 2 + \varepsilon / 2$, from (2) and (3)

e., $|l - l'| < \varepsilon$.

Since ε is aribtrary, hence we must have $|l - l'| = 0$ *i.e.,* $l = l'$.

Note: The converse of the above theorem need not be true; *i.e.,* a sequence having only one limit point may not converge.

Consider the sequence $< s_n >$, where s_n is given by

$$s_n = \begin{cases} \frac{1}{n}, & \text{if } n \text{ is even;} \\ n, & \text{if } n \text{ is odd.} \end{cases}$$

There is only one limit point of $< s_n >$, namely 0, and yet the sequence does not converge.

Theorem 3: (Bolzano-Weierstrass Theorem for sequences): *Every bounded sequence has at least one limit point.*

Proof: Let $< s_n >$ be a bounded sequence and S be its range set. Then $S = \{s_n : n \in N\}$. Since the sequence $< s_n >$ is bounded, therefore, S is a bounded set.

Case I: Let S be a finite set. Then for infinitely many indices n, $s_n = p$, where p is some real number. Obviously p is a limit point of $< s_n >$.

Case II: Let S be an infinite set. Since S is bounded, by Bolzano-Weierstrass theorem for sets of real numbers, S has a limit point, say p. Hence every nbd of p contains infinitely many distinct points of S or every nbd of p contains infinitely many terms of the sequence $< s_n >$ and consequently p is a limit point of the sequence $< s_n >$.

Corollary 1: *If F is a closed and bounded set of real numbers, then every sequence in F has a limit point in F.*

Proof: Let $< s_n >$ be a sequence in F. Then $< s_n >$ is bounded and hence, it has a limit point, say p. Also $p \in F$, since p cannot belong to $R - F$. If $p \in R - F$, then $R - F$ is an open set containing p. Thus it is a neighbourhood of p that contains no term of the sequence $< s_n >$ and hence we get a contradiction to the fact that p is a limit point of the sequence $< s_n >$.

Corollary 2: *If I is a closed interval, then every sequence in I has a limit point in I.*

Proof: Since every closed interval is a closed and bounded set, hence the result follows from corollary 1.

We have proved earlier that every convergent sequence is bounded and also it has only one limit point. Now we shall prove the converse.

Theorem 4: *If a sequence $< s_n >$ is bounded and has only one limit point, say l, then $s_n \to l$.*

Proof: Since $< s_n >$ is bounded, so it has at least one limit point. But l is the only limit point of $< s_n >$. Hence, for any $\varepsilon > 0$, $]\, l - \varepsilon, l + \varepsilon\, [$ contains s_n for all except a finite number of values of n.

Let $s_{m_1}, s_{m_2}, s_{m_3}, \ldots, s_{m_p}$ be the finite number of terms of the sequence $< s_n >$ that lie outside $]\, l - \varepsilon, l + \varepsilon [$.

If $m - 1 = \max. \{m_1, m_2, \ldots, m_p\}$, then $s_n \in]\, l - \varepsilon, l + \varepsilon\, [\ \forall n \geq m$.

Hence for any $\varepsilon > 0$, there exists $m \in N$ such that

$$| s_n - l | < \varepsilon \quad \forall n \geq m.$$

$\therefore$ the sequence $< s_n >$ converges to l.

Theorem 5: *A real number p is a limit point of a sequence $< s_n >$ iff there exists a subsequence of $< s_n >$ converging to p.*

Proof: First, let p be a limit point of a sequence $< s_n >$.

We shall use the result that a real number p is a limit point of a sequence $< s_n >$ if given any $\varepsilon > 0$ and any positive integer m, there exists a positive integer $k > m$ such that $s_k \in]\, p - \varepsilon, p + \varepsilon\, [$.

By choosing $\varepsilon = 1$ and $m = 1$, there must exist a positive integer $n_1 > 1$, such that

$$| s_{n_1} - p | < 1. \qquad \ldots (1)$$

Choosing $\varepsilon = \frac{1}{2}$, $m = n_1$, there must exist a positive integer $n_2 > n_1$, such that

$$|s_{n_2} - p| < \frac{1}{2}. \qquad \ldots(2)$$

Continuing in this way, we can inductively define a subsequence $< s_{n_1}, s_{n_2}, \ldots, s_{n_k}, \ldots >$ such that $|s_{n_k} - p| < \frac{1}{k}$.

In fact, if we assume that $s_{n_1}, s_{n_2}, \ldots, s_{n_k}$ have been obtained, by choosing

$$\varepsilon = \frac{1}{k+1}, m = n_k,$$

we can get a positive integer $n_{k+1} > n_k$ such that

$$|s_{n_{k+1}} - p| < \frac{1}{k+1}.$$

But s_{n_1} has already been obtained. Thus the construction of $< s_{n_k} >$ is complete by induction.

We now claim that the sequence $< s_{n_k} > \to p$. In fact, for any $\varepsilon > 0$, we can choose a positive integer j, such that $1/j < \varepsilon$. For this choice of j, we get $|s_{n_k} - p| < 1/j < \varepsilon \forall$ $k \geq j$. This shows that the sequence $< s_{n_k} > \to p$.

Conversely, let $< s_{n_k} >$ be a subsequence of $< s_n >$ converging to p. We have to show that p is a limit point of $< s_n >$.

Since $s_{n_k} \to p$, therefore, given any $\varepsilon > 0$ there exists positive integer j such that

$$|s_{n_k} - p| < \varepsilon \text{ for all } k \geq j.$$

Thus every neighbourhood of p contains infinitely many terms of $< s_{n_k} >$ *i.e.*, infinitely many terms of $< s_n >$ and consequently p is a limit point of $< s_n >$.

Theorem 6: *The set of limit points of a bounded sequence is bounded.*

Proof: Let $< s_n >$ be a bounded sequence. Then there exist $k_1, k_2 \in \mathbf{R}$ such that

$$k_1 \leq s_n \leq k_2 \text{ for all } n \in \mathbf{N}$$

i.e., $s_n \notin]-\infty, k_1[$ and $s_n \notin]k_2, \infty[$ for any n. Hence if $l \in \mathbf{R}$ and $l \notin]-\infty, k_1[\cup]k_2, \infty[$, then l is not a limit point of the sequence. Thus if $l \in \mathbf{R}$ is a limit point of the sequence, then $l \in [k_1, k_2]$. Consequently the set of limit points of $< s_n >$ is bounded.

Theorem 7: *Every bounded sequence has the greatest and the least limit points.*

Proof: Let $< s_n >$ be a bounded sequence. Then the set L of limit points of $< s_n >$ is bounded.

Now $L \neq \varnothing$ and L is bounded, hence by completeness axiom the set L has infimum and supremum.

If inf $L = u$ and sup $L = v$, then we have to show that $u, v \in L$.

For $\varepsilon > 0$, $]v - \varepsilon, v + \varepsilon[$ is a neighbourhood of v.

Since $v = \sup L$, therefore, there exists some $x \in L$ such that

$$v - \varepsilon < x \leq v < v + \varepsilon \quad i.e., \quad x \in]v - \varepsilon, v + \varepsilon[$$

or $\quad]\, v - \varepsilon, v + \varepsilon\, [$ is a neighbourhood of x.

Since x is a limit point of $< s_n >$, hence $]\, v - \varepsilon, v + \varepsilon\, [$ contains infinite number of terms of the sequence. It holds for every $\varepsilon > 0$. Thus every neighbourhood of v contains infinite number of terms of the sequence $< s_n >$.

$\therefore \quad v$ is a limit point of the sequence $< s_n >$.

$\therefore \quad v \in L$.

Similarly, we can show that $u \in L$.

2.12 Cauchy Sequences

In this section we shall establish an important criterion, known as *Cauchy Convergence Criterion*, which will help us to decide whether a sequence is convergent or divergent without knowing its limit or limit point. It involves only the elements of the sequence to which we wish to apply it.

Definition: *A sequence* $< s_n >$ *is said to be a Cauchy sequence if given* $\varepsilon > 0$ *there exists* $m \in \mathbf{N}$ *such that*

$$|s_n - s_m| < \varepsilon \text{ for all } n \geq m$$

or $\quad |s_{n+p} - s_n| < \varepsilon$ *for all* $n \geq m$ *and every* $p \geq 0$

or $\quad |s_{m+p} - s_m| < \varepsilon$ *for all* $p \geq 0$

or $\quad |s_p - s_q| < \varepsilon$ *for all* $p, q \geq m$.

Remark: $|s_p - s_q| < \varepsilon$ for all $p, q \geq m$ means that s_p and s_q are arbitrarily close together for large values of p and q.

Illustrations:

1. $< 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots >$ is a Cauchy sequence.

Let the given sequence be $< s_n >$, where $s_n = 1/n$.

Take any given $\varepsilon > 0$.

If $n \geq m$, then

$$|s_n - s_m| = \left| \frac{1}{n} - \frac{1}{m} \right| = \left| \frac{m-n}{nm} \right| = \frac{n-m}{nm} = \frac{n-m}{n} \cdot \frac{1}{m} < \frac{1}{m}.$$

$$\left[\because \ 0 \leq \frac{n-m}{n} < 1 \right]$$

If we take $m \in \mathbf{N}$ such that $m > \frac{1}{\varepsilon}$ *i.e.*, $\frac{1}{m} < \varepsilon$, then

$$|s_n - s_m| = \left| \frac{1}{n} - \frac{1}{m} \right| < \varepsilon, \forall\ n \geq m.$$

Hence the given sequence is a Cauchy sequence.

2. The sequence $< n^2 >$ is not a Cauchy sequence.

If $n > m$, then $n^2 - m^2 = (n - m)(n + m) > 2m > 1$, for any value of m. Taking $\varepsilon = 1$, we cannot find a positive integer m such that $|n^2 - m^2| < \varepsilon$ for all $n \geq m$.

Theorem 1: *If $< s_n >$ is a Cauchy sequence, then $< s_n >$ is bounded.* **(Kanpur 2008)**

Proof: Let $< s_n >$ be a Cauchy sequence. For $\varepsilon = 1$, there exists $m \in N$ such that $|s_n - s_m| < 1$ for all $n \geq m$

i.e., $s_m - 1 < s_n < s_m + 1$ for all $n \geq m$.

Let $k_1 = \min. \{s_1, s_2, \ldots, s_{m-1}, s_m - 1\}$

and $k_2 = \max. \{s_1, s_2, \ldots, s_{m-1}, s_m + 1\}$.

$\therefore \quad k_1 \leq s_n \leq k_2$ for all $n \in N$.

Hence $< s_n >$ is bounded.

Note: The converse of the above theorem need not be true. The sequence $< (-1)^n >$ is bounded but is not a Cauchy sequence.

Theorem 2: (Cauchy Convergence Criterion): *A sequence converges if and only if it is a Cauchy sequence.*

Proof: First, let $< s_n >$ be a convergent sequence which converges to, say, l.

Since $s_n \to l$, therefore, for given $\varepsilon > 0$ there must exist $m \in N$ such that

$$|s_n - l| < \varepsilon / 2 \; \forall \, n \geq m.$$

In particular, $|s_m - l| < \varepsilon / 2$.

Now $$|s_n - s_m| = |(s_n - l) - (s_m - l)| \leq |s_n - l| + |s_m - l|$$
$$< \varepsilon / 2 + \varepsilon / 2 \text{ for all } n \geq m.$$

Thus $|s_n - s_m| < \varepsilon \; \forall \, n \geq m$, showing that $< s_n >$ is a Cauchy sequence.

Conversely, let $< s_n >$ be a Cauchy sequence. Then $< s_n >$ is bounded. By Bolzano-Weierstrass theorem, $< s_n >$ has a limit point, say l. We shall show that $s_n \to l$.

Let $\varepsilon > 0$ be given. Since $< s_n >$ is a Cauchy sequence, there exists $m \in N$ such that

$$|s_n - s_m| < \varepsilon / 3 \; \forall \, n \geq m.$$

Since l is a limit point of $< s_n >$, therefore every nbd of l contains infinite terms of the sequence $< s_n >$. In particular the open interval $\left] l - \frac{1}{3}\varepsilon, l + \frac{1}{3}\varepsilon \right[$ contains infinite terms of $< s_n >$. Hence there exists a positive integer $k > m$ such that

$$l - \frac{1}{3}\varepsilon < s_k < l + \frac{1}{3}\varepsilon \quad i.e., \; |s_k - l| < \varepsilon / 3.$$

Now $$|s_n - l| = |(s_n - s_m) + (s_m - s_k) + (s_k - l)|$$
$$\leq |s_n - s_m| + |s_m - s_k| + |s_k - l|$$
$$< \varepsilon / 3 + \varepsilon / 3 + \varepsilon / 3 \text{ for all } n \geq m.$$

Thus $|s_n - l| < \varepsilon$ for all $n \geq m$.

$\therefore \quad < s_n >$ converges to l.

2.13 Limit Superior and Limit Inferior of a Sequence

Let $< s_n >$ be a sequence which is bounded above. Then, for each fixed $n \in \mathbb{N}$, the set $\{s_n, s_{n+1}, \ldots\}$ is bounded above and hence it must have a supremum. Let

$$\bar{s}_n = \sup \{s_n, s_{n+1}, \ldots\}.$$

Since $\{s_{n+1}, s_{n+2}, \ldots\}$ is a subset of $\{s_n, s_{n+1}, \ldots\}$, therefore, it is obvious that $\bar{s}_n \geq \bar{s}_{n+1}$. Thus the sequence $< \bar{s}_n >$ is a monotonically decreasing sequence and consequently, it either converges or else it diverges to $-\infty$.

Similarly, if the sequence $< s_n >$ is bounded below, then the set $\{s_n, s_{n+1}, \ldots\}$ has an infimum. Let $\underline{s}_n = \inf \{s_n, s_{n+1}, \ldots\}$, then the sequence $< \underline{s}_n >$ is monotonically increasing and hence it either converges or diverges to ∞.

Keeping these notations in mind we now define limit superior and limit inferior.

Definition 1: *Let $< s_n >$ be a sequence of real numbers which is bounded above and let*

$$\bar{s}_n = \sup \{s_n, s_{n+1}, \ldots\}.$$

If $< \bar{s}_n >$ converges we define the **limit superior** *of $< s_n >$ by*

$$\lim_{n \to \infty} \sup s_n = \lim_{n \to \infty} \bar{s}_n$$

If $< \bar{s}_n >$ diverges to $-\infty$, we write $\lim_{n \to \infty} \sup s_n = -\infty$

If a sequence $< s_n >$ is not bounded above, we write

$$\lim_{n \to \infty} \sup s_n = \infty$$

Definition 2: *Let $< s_n >$ be a sequence of real numbers which is bounded below and let*

$$\underline{s}_n = \inf \{s_n, s_{n+1}, \ldots\}.$$

If $< \underline{s}_n >$ converges we define the **limit inferior** *of $< s_n >$ by*

$$\lim_{n \to \infty} \inf s_n = \lim_{n \to \infty} \underline{s}_n .$$

If $< \underline{s}_n >$ diverges to ∞, we write $\lim_{n \to \infty} \inf s_n = \infty.$

If a sequence $< s_n >$ is not bounded below, we write

$$\lim_{n \to \infty} \inf s_n = -\infty.$$

Note 1: The notations $\overline{\lim}\, s_n$ and $\underline{\lim}\, s_n$ are also used for lim sup s_n and lim inf s_n respectively. In future, we shall use these notations.

2. The limit superior and the limit inferior are also called the **upper limit** and the **lower limit** of $< s_n >$ respectively.

3. We have $\overline{\lim}\, s_n = \inf \{\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n, \ldots\}$

and $\underline{\lim}\, s_n = \sup \{\underline{s_1}, \underline{s_2}, \ldots, \underline{s_n}, \ldots\}$

Illustrations:

1. Let $< s_n >$ be the sequence defined by $s_n = (-1)^n \; \forall n \in N$.

It is bounded above by 1 and bounded below by -1. For this sequence, $\overline{s_n} = 1$ and $\underline{s_n} = -1$ for all $n \in N$.

Hence $\overline{\lim}\, s_n = 1$ and $\underline{\lim}\, s_n = -1$.

2. Let $< s_n >$ be the sequence defined by $s_n = -n \forall n \in N$. It is bounded above by -1 but it is not bounded below.

$$\bar{s}_n = \sup \{-n, -n-1, \ldots\} = -n$$

Since $\bar{s}_n \to -\infty$ as $n \to \infty$, hence $\overline{\lim}\, s_n = -\infty$. Also, since $< s_n >$ is not bounded below, by definition $\underline{\lim}\, s_n = -\infty$. Thus in this sequence both the limit superior and the limit inferior are $-\infty$.

3. Let $< s_n >$ be the sequence defined by $s_n = n \; \forall n \in N$. It is bounded below but not bounded above.

$$\underline{s_n} = \inf \{n, n+1, \ldots\} = n.$$

Since $\underline{s_n} \to \infty$ as $n \to \infty$, hence $\underline{\lim}\, s_n = \infty$.

Also, since $< s_n >$ is not bounded above, by definition $\overline{\lim}\, s_n = \infty$. Thus in this sequence both the limit superior and the limit inferior are ∞.

4. Let $< s_n >$ be the sequence defined by $s_n = (-1)^n \left(1 + \frac{1}{n}\right)$.

Then $< s_n > = < -2, \frac{3}{2}, -\frac{4}{3}, \frac{5}{4}, -\frac{6}{5}, \frac{7}{6}, \ldots >$.

In this case $\bar{s}_1 = \frac{3}{2}, \bar{s}_2 = \frac{3}{2}, \bar{s}_3 = \frac{5}{4}, \bar{s}_4 = \frac{5}{4}, \bar{s}_5 = \frac{7}{6}$ etc.

and $\underline{s_1} = -2, \underline{s_2} = -\frac{4}{3}, \underline{s_3} = -\frac{4}{3}, \underline{s_4} = -\frac{6}{5}$ etc.

Hence $\overline{\lim}\, s_n = \inf \left\{\frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \ldots\right\} = 1$

and $\underline{\lim}\, s_n = \sup \left\{-2, -\frac{4}{3}, -\frac{6}{5}, \ldots\right\} = -1$.

Theorem 1: *If $< s_n >$ is a convergent sequence of real numbers and if $\lim s_n = l$, then $\overline{\lim}\, s_n = \underline{\lim}\, s_n = l$. Conversely, if*

$$\overline{\lim}\, s_n = \underline{\lim}\, s_n = l \in R,$$

then $< s_n >$ is convergent and $\lim\limits_{n \to \infty} s_n = l$.

Proof: First suppose that the sequence $< s_n >$ converges with $\lim s_n = l$. Let $\varepsilon > 0$ be given. Since $s_n \to l$, therefore, we can find a positive integer m, such that

$$|s_n - l| < \varepsilon \text{ for all } n \geq m$$

i.e., $l - \varepsilon < s_n < l + \varepsilon$ for all $n \geq m$.

This inequality shows that for all $n \geq m$, $l + \varepsilon$ is an upper bound of $\{s_n, s_{n+1}, \ldots\}$ and $l - \varepsilon$ is not an upper bound of $\{s_n, s_{n+1}, \ldots\}$.

Since $\bar{s}_n = \{s_n, s_{n+1}, \ldots\}$, it follows that

$$l - \varepsilon < \bar{s}_n \leq l + \varepsilon, \; n \geq m.$$

Taking limits as $n \to \infty$, we get

$$l - \varepsilon \leq \lim_{n \to \infty} \bar{s}_n \leq l + \varepsilon.$$

Since ε is arbitrary, it follows that $\overline{\lim}\, s_n = l$.

Similarly, we can show that $\underline{\lim}\, s_n = l$.

Thus $\overline{\lim}\, s_n = \underline{\lim}\, s_n = l$.

Conversely, let $\overline{\lim}\, s_n = \underline{\lim}\, s_n = l$.

Since $l = \lim_{n \to \infty} \bar{s}_n$, given any $\varepsilon > 0$, there exists $m_1 \in \mathbb{N}$ such that

$$|\bar{s}_n - l| < \varepsilon \quad \text{for } n \geq m_1 \qquad \textit{i.e.,} \qquad l - \varepsilon < \bar{s}_n < l + \varepsilon \quad \text{for } n \geq m_1.$$

The definition of $\bar{s}_n$ then gives that

$$s_n < l + \varepsilon \quad \text{for } n \geq m_1. \qquad \ldots(1)$$

Similarly, since $l = \lim_{n \to \infty} \underline{s}_n$, there exists $m_2 \in \mathbb{N}$ such that

$$|\underline{s}_n - l| < \varepsilon \text{ for } n \geq m_2$$

which implies as above that $s_n > l - \varepsilon$ for $n \geq m_2$. $\ldots(2)$

Let $m = \max.\{m_1, m_2\}$. Then from (1) and (2) we find that

$$|s_n - l| < \varepsilon \text{ for } n \geq m.$$

This proves that the sequence $< s_n >$ converges and that

$$\lim_{n \to \infty} s_n = l.$$

Similar results hold good for divergent sequences. Below we state them without proof.

Theorem 2: *A sequence* $< s_n >$ *diverges to* $+\infty$ *iff* $\overline{\lim}\, s_n = \underline{\lim}\, s_n = \infty$.

Theorem 3: *A sequence* $< s_n >$ *diverges to* $-\infty$ *iff* $\overline{\lim}\, s_n = \underline{\lim}\, s_n = -\infty$.

Theorem 4: *If* $< s_n >$ *and* $< t_n >$ *are bounded sequences of real numbers such that* $s_n \leq t_n$ *for all* $n \in \mathbb{N}$, *then* $\overline{\lim}\, s_n \leq \overline{\lim}\, t_n$ *and* $\underline{\lim}\, s_n \leq \underline{\lim}\, t_n$.

Proof: Since $s_n \leq t_n$, therefore it is easy to see that

$$\bar{s}_n \leq \bar{t}_n \quad \text{and} \quad \underline{s}_n \leq \underline{t}_n,$$

where $\bar{s}_n$, $\bar{t}_n$, $\underline{s}_n$, $\underline{t}_n$ have their usual meanings as defined earlier.

Then we have from the corollary to theorem 3 of article 2.9,

$$\lim \bar{s}_n \leq \lim \bar{t}_n \quad \text{and} \quad \lim \underline{s}_n \leq \lim \underline{t}_n$$

or

$$\overline{\lim}\, s_n \leq \overline{\lim}\, t_n \quad \text{and} \quad \underline{\lim}\, s_n \leq \underline{\lim}\, t_n.$$

Theorem 5: *If $< s_n >$ and $< t_n >$ are bounded sequences of real numbers, then*

(*i*) $\overline{\lim}\,(s_n + t_n) \le \overline{\lim}\, s_n + \overline{\lim}\, t_n$; (*ii*) $\underline{\lim}\,(s_n + t_n) \ge \underline{\lim}\, s_n + \underline{\lim}\, t_n$.

Proof: Let $\bar{s}_n = \{s_n, s_{n+1}, \ldots\}$, and $\bar{t}_n = \sup\{t_n, t_{n+1}, \ldots\}$.

Then $s_k \le \bar{s}_n$, $(k \ge n)$, $t_k \le \bar{t}_n$, $(k \ge n)$.

$\therefore$ $s_k + t_k \le \bar{s}_n + \bar{t}_n$ for $k \ge n$.

Thus $\bar{s}_n + \bar{t}_n$ is an upper bound for $\{s_n + t_n , s_{n+1} + t_{n+1}, \ldots\}$.

Hence $\overline{(s_n + t_n)} = \sup\{s_n + t_n, s_{n+1} + t_{n+1}, \ldots\} \le \bar{s}_n + \bar{t}_n$.

$\therefore$ $\lim \overline{(s_n + t_n)} \le \lim (\bar{s}_n + \bar{t}_n) = \lim \bar{s}_n + \lim \bar{t}_n$

i.e., $\overline{\lim}\,(s_n + t_n) \le \overline{\lim}\, s_n + \overline{\lim}\, t_n$.

Thus the result (i) has been proved. Similarly (ii) can be proved.

Note: It can be shown that there exist sequences for which the inequalities in the above theorem are strict inequalities.

Let $s_n = (-1)^n$, $n \in \mathbb{N}$ and $t_n = (-1)^{n+1}$, $n \in \mathbb{N}$.

Then $s_n + t_n = 0$, $n \in \mathbb{N}$. Here $\overline{\lim}\, s_n = 1 = \overline{\lim}\, t_n$ and $\overline{\lim}\,(s_n + t_n) = 0$.

Hence, in this case $\overline{\lim}\,(s_n + t_n) < \overline{\lim}\, s_n + \overline{\lim}\, t_n$.

2.14 Nested Interval Theorem or Cantor's Intersection Theorem

Definition: *A sequence of sets $< A_n >$ is called a nested sequence of sets if* $A_1 \supset A_2 \supset \ldots\ A_n \supset A_{n+1} \supset \ldots$

Theorem: *For each $n \in \mathbb{N}$, let $I_n = [a_n, b_n]$ be a (non-empty) closed and bounded interval on $\mathbb{R}$, such that $< I_n >$ is a nested sequence with $\lim\limits_{n \to \infty}$ (length of I_n) i.e., $\lim\limits_{n \to \infty} (b_n - a_n) = 0$. Then $\bigcap\limits_{n=1}^{\infty} I_n$ contains precisely one point.*

Proof: Since $< I_n >$ is nested, we have

$I_n \supset I_{n+1}$ for all $n \in \mathbb{N}$

i.e., $[a_n , b_n] \supset [a_{n+1}, b_{n+1}]$ for all $n \in \mathbb{N}$. ...(1)

It follows from (1) that $a_n \le a_{n+1} \le b_{n+1} \le b_n \ \forall\ n \in \mathbb{N}$.

This shows that the sequence $< a_n >$ is a monotonically increasing sequence bounded above by b_1, and $< b_n >$ is a monotonically decreasing sequence bounded below by a_1. Hence, $< a_n >$ and $< b_n >$, both converge.

Now we have $\lim\limits_{n \to \infty}$ (length of I_n) $= \lim\limits_{n \to \infty} (b_n - a_n) = 0$

$$\Rightarrow \quad \lim_{n \to \infty} b_n = \lim_{n \to \infty} a_n = l, \text{ (say)}.$$

Since a monotonically increasing bounded sequence converges to its supremum, it follows that l is the supremum of the range set of the sequence $< a_n >$. Also, a monotonically decreasing bounded sequence converges to its infimum, hence l is the infimum of the range set of the sequence $< b_n >$.

Thus $a_n \le l \le b_n \ \forall n$. Hence, $l \in I_n$ for all n. It follows that $l \in \bigcap_{n=1}^{\infty} I_n$.

Clearly no $l' \ne l$ can belong to $\bigcap_{n=1}^{\infty} I_n$.

For let $l' \in \bigcap_{n=1}^{\infty} I_n$, then $0 \le |l' - l| \le |b_n - a_n| \ \forall n$.

Since $|b_n - a_n| \to 0$, hence we get $|l' - l| = 0$ *i.e.*, $l' = l$.

Hence $\bigcap_{n=1}^{\infty}$ consists of exactly one point.

Illustrative Examples

Example 17: *Show that the sequence* $< s_n >$ *defined by*

$$s_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n},$$

converges.

Solution: We have

$$s_{n+1} - s_n = \left(\frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n+2}\right) - \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right)$$

$$= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2n+2} > 0 \text{ for all } n.$$

Hence, the sequence $< s_n >$ is monotonically increasing.

Now $\quad |s_n| = s_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n} < \frac{1}{n} + \ldots + \frac{1}{n}$ (upto n terms)

$$= n \cdot \frac{1}{n} = 1$$

i.e., $\quad |s_n| < 1$ for all n.

Hence, the sequence $< s_n >$ is bounded.

Since $< s_n >$ is a bounded, monotonically increasing sequence, hence it converges.

Example 18: *Show that the sequence* $< s_n >$ *defined by the relation*

$$s_1 = 2, s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{(n-1)!} \quad (n \ge 2), \text{ converges.}$$

Solution: We have $s_{n+1} - s_n = \frac{1}{n!} > 0$ for all n.

Hence, the sequence $< s_n >$ is monotonically increasing.

Now, we shall show that $< s_n >$ is bounded.

For $n \geq 2$, $n! = 1.2.3 \ldots n$ contains $(n-1)$ factors each of which is greater than or equal to 2. Hence $n! \geq 2^{n-1}$ for all $n \geq 2$.

$\therefore \quad \frac{1}{n!} \leq \frac{1}{2^{n-1}}$, for all $n \geq 2$.

Thus $s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{(n-1)!}$

$$\leq 1 + \frac{1}{1!} + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-2}}$$

$$= 1 + \frac{1 - \left(\frac{1}{2}\right)^{n-1}}{1 - \frac{1}{2}} < 3, \text{ for all } n \geq 2.$$

Also $s_1 = 2 < 3$.

$\therefore \quad 2 \leq s_n < 3$ for all $n \in \mathbf{N}$

i.e., $< s_n >$ is bounded.

Since $< s_n >$ is a bounded, monotonically increasing sequence, consequently, it converges.

Example 19: *Show that* $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ *exists and lies between* 2 *and* 3.

Solution: Here $s_n = \left(1 + \frac{1}{n}\right)^n$. Obviously $s_1 = 2$.

By the binomial theorem, we get

$$s_n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \ldots + \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots$$

$$+ \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \ldots \left(1 - \frac{n-1}{n}\right) \quad \ldots(1)$$

Similarly, $s_{n+1} = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n+1}\right) + \frac{1}{3!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) + \ldots$

$$+ \frac{1}{(n+1)!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) \ldots \left(1 - \frac{n}{n+1}\right)$$

$$> 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots$$

$$+ \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{n-1}{n}\right)$$

[$\because$ each term on the R.H.S. of s_{n+1} is $\geq$ the corresponding term on the R.H.S. of s_n and moreover the number of terms in the expansion of s_{n+1} is $n + 2$ *i.e.*, one more than the number of terms $n + 1$ in the expansion of s_n]

$\therefore \quad s_{n+1} > s_n$ for all $n \in \mathrm{N}$.

Hence, the sequence $< s_n >$ is monotonically increasing.

$\therefore \quad s_n \geq s_1 = 2, \forall\ n \in \mathrm{N}$.

From (1), we see that

$$s_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} \qquad \text{[See Ex. 18]}$$

$$= 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} < 1 + \frac{1}{1 - \frac{1}{2}} = 3, \text{ for all } n.$$

Thus $2 \leq s_n < 3$, for all n.

Hence the sequence $< s_n >$ is bounded.

Since $< s_n >$ is a bounded, monotonically increasing sequence, consequently, it converges *i.e.*, $\lim_{n \to \infty} s_n$ exists and $\lim s_n = \sup < s_n >$.

Now $2 \leq s_n < 3$ for all $n \Rightarrow 2 \leq \lim_{n \to \infty} s_n \leq 3$, which shows that the limit lies between 2 and 3.

Note: The actual value of the $\lim \left(1 + \frac{1}{n}\right)^n$ is defined to be equal to e. Hence e lies between 2 and 3. Taking limit of both sides of (1) as $n \to \infty$, we see that

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots \infty.$$

Example 20: *Prove that* $< s_n >$ *is convergent where* $s_n = 2 - \frac{1}{2^{n-1}}$.

Solution: We have

$$s_{n+1} - s_n = \left(2 - \frac{1}{2^n}\right) - \left(2 - \frac{1}{2^{n-1}}\right) = \frac{1}{2^{n-1}} - \frac{1}{2^n} > 0 \text{ for all } n.$$

Hence, the sequence $< s_n >$ is monotonically increasing.

Also, $s_n = 2 - \frac{1}{2^{n-1}} < 2$ for all n. $\left[\because \frac{1}{2^{n-1}} > 0\right]$

$\therefore$ $< s_n >$ is bounded above.

Since $< s_n >$ is a bounded above, monotonically increasing sequence, hence it converges.

We have $\lim_{n \to \infty} s_n = \lim_{n \to \infty} \left(2 - \frac{1}{2^{n-1}}\right) = 2.$

Example 21: *Show that the sequence $< s_n >$ defined by*

$$s_1 = \frac{1}{2}, s_{n+1} = \frac{2s_n + 1}{3} \forall\ n \in \mathbb{N}$$ *is convergent. Also find its limit.*

Solution: We have $s_1 = \frac{1}{2}$ and $s_{n+1} = \frac{2s_n + 1}{3}$ $\forall\ n \in \mathbb{N}$.

First applying mathematical induction we shall show that

$$s_{n+1} > s_n \ \forall\ n \in \mathbb{N}.$$

We have $s_2 = \frac{2.s_1 + 1}{3} = \frac{2.\left(\frac{1}{2}\right) + 1}{3} = \frac{2}{3}.$

$\therefore$ $s_2 > s_1$.

Now assume as our induction hypothesis that $s_{n+1} > s_n$ for some positive integer n.

Then $s_{n+1} > s_n \Rightarrow 2s_{n+1} > 2s_n \Rightarrow 2s_{n+1} + 1 > 2s_n + 1$

$\Rightarrow$ $\frac{2s_{n+1} + 1}{3} > \frac{2s_n + 1}{3} \Rightarrow s_{n+2} > s_{n+1}.$

Thus $s_2 > s_1$ and if $s_{n+1} > s_n$, then $s_{n+2} > s_{n+1}$.

$\therefore$ by induction $s_{n+1} > s_n \ \forall\ n \in \mathbb{N}$.

Thus the sequence $< s_n >$ is monotonic increasing.

Now we shall show that the sequence $< s_n >$ is also bounded above.

We have $s_{n+1} > s_n \ \forall\ n \in \mathbb{N}$

$\Rightarrow$ $\frac{2s_n + 1}{3} > s_n \ \forall\ n \in \mathbb{N} \Rightarrow 2s_n + 1 > 3s_n \ \forall\ n \in \mathbb{N}$

$\Rightarrow$ $s_n < 1 \ \forall n \in \mathbb{N}$.

$\therefore$ the sequence $< s_n >$ is bounded above by 1.

Since the sequence $< s_n >$ is monotonic increasing and bounded above, therefore by monotone convergence theorem $< s_n >$ converges to its supremum.

Let $\lim_{n \to \infty} s_n = l$. Then $\lim_{n \to \infty} s_{n+1} = l$.

Now $s_{n+1} = \frac{2s_n + 1}{3} \Rightarrow \lim s_{n+1} = \lim \frac{2s_n + 1}{3}$

$\Rightarrow$ $l = \frac{2 \lim s_n + 1}{3} \Rightarrow l = \frac{2l + 1}{3} \Rightarrow l = 1.$

Hence $s_n \to 1$. We have $\inf < s_n > = s_1 = \frac{1}{2}$ and $\sup < s_n > = \lim s_n = 1$.

Example 22: *Show that the sequence* $< s_n >$ *defined by*

$$s_1 = \sqrt{2}\,,\ s_{n+1} = \sqrt{(2\,s_n)}$$

converges to 2.

Solution: We have $s_2 = \sqrt{(2\sqrt{2})}$.

Since $1 < \sqrt{2} \Rightarrow 2 < 2\sqrt{2} \Rightarrow \sqrt{2} < \sqrt{(2\sqrt{2})}$,

$\therefore \quad s_1 < s_2$.

Let us suppose that $s_m < s_{m+1}$.

Then $\sqrt{(2s_m)} < \sqrt{(2s_{m+1})} \quad \Rightarrow \quad s_{m+1} < s_{m+2}$.

Hence, by mathematical induction, we have

$$s_n < s_{n+1} \text{ for all } n \in \mathrm{N}$$

i.e., $< s_n >$ is monotonically increasing.

Also, we have $s_1 = \sqrt{2} < 2$.

Let us suppose that $s_m < 2$. Then $\sqrt{(2s_m)} < \sqrt{(2.2)} = 2 \Rightarrow s_{m+1} < 2$.

Hence, by mathematical induction, we have $s_n < 2$ for all $n \in \mathrm{N}$

i.e., $< s_n >$ is bounded above by 2.

Thus $< s_n >$ is a monotonically increasing sequence bounded above by 2, hence, it converges.

Let $\lim\limits_{n \to \infty} s_n = l$. Then $\lim\limits_{n \to \infty} s_{n+1} = l$.

Now $s_{n+1} = \sqrt{(2s_n)} \Rightarrow \lim s_{n+1} = \lim \sqrt{(2s_n)}$

$\Rightarrow \quad l = \sqrt{(2l)} \Rightarrow l\,(l-2) = 0 \Rightarrow l = 0, 2.$

But $s_n \geq s_1 = \sqrt{2}\ \forall\ n \in \mathrm{N} \Rightarrow s_n - \sqrt{2} \geq 0\ \forall\ n \in \mathrm{N}.$

$\therefore \quad \lim (s_n - \sqrt{2}) \geq 0 \quad$ *i.e.*, $\lim s_n \geq \sqrt{2}$.

Hence l cannot be zero. Therefore $l = 2$.

Example 23: *Show that the sequence* $< s_n >$ *defined by* $s_1 = 1$ *and* $s_{n+1} = \sqrt{(2 + s_n)}, \forall\ n \in \mathrm{N}$ *is monotonically increasing and bounded. Also find its limit.*

Solution: We have $s_1 = 1$ and $(s_{n+1})^2 = 2 + s_n, \forall n \in \mathrm{N}$.

$\therefore \quad s_2 = \sqrt{3}, s_3 = \sqrt{(2 + \sqrt{3})}, \ldots\ldots$

Now $1 < \sqrt{3} \quad \Rightarrow \quad s_1 < s_2$.

Let us suppose that $s_m < s_{m+1}$. Then $\sqrt{(2 + s_m)} < \sqrt{(2 + s_{m+1})}$

$\Rightarrow \quad s_{m+1} < s_{m+2}$.

Hence, by mathematical induction, we have

$$s_n < s_{n+1} \text{ for all } n \in \mathrm{N}$$

i.e., $< s_n >$ is monotonically increasing.

Again, $s_{n+1} > s_n \quad \Rightarrow \quad \sqrt{(2 + s_n)} > s_n$

$\Rightarrow \quad 2 + s_n - s_n^2 > 0 \quad \Rightarrow \quad (2 - s_n)(1 + s_n) > 0$

$\Rightarrow \quad (2 - s_n) > 0 \qquad [\because \ 1 + s_n > 0]$

or $\quad s_n < 2 \ \forall \ n \in \mathbf{N}.$

Hence $< s_n >$ is bounded.

Thus $< s_n >$ is a monotonically increasing sequence bounded above by 2; consequently it converges.

Let $\lim s_n = l$. Then $\lim s_{n+1} = l$.

Now $\quad s_{n+1} = \sqrt{(2 + s_n)} \quad \Rightarrow \quad \lim s_{n+1} = \lim \sqrt{(2 + s_n)}$

$\Rightarrow \quad l = \sqrt{(2 + l)} \quad \Rightarrow \quad l^2 - l - 2 = 0 \quad \Rightarrow \quad (l + 1)(l - 2) = 0$

$\Rightarrow \quad l = -1, 2.$

But l cannot be -1 since all terms of the sequence are positive. Hence $l = 2$.

Note: If a sequence $< s_n >$ is monotonically increasing then there is no need to show that $< s_n >$ has a lower bound because s_1 is always its lower bound.

Similarly, for a monotonically decreasing sequence there is no need to find an upper bound, because s_1 will always be its upper bound.

Example 24: *Show that the sequence $< s_n >$ defined by*

$$s_1 = 1, s_{n+1} = \frac{4 + 3s_n}{3 + 2s_n}, \ n \in \mathbf{N}$$

is convergent and find its limit.

Solution: We observe that all the terms of the given sequence are positive.

First by mathematical induction we shall show that

$$s_{n+1} > s_n \ \forall \ n \in \mathbf{N}.$$

We have $s_1 = 1, s_2 = \dfrac{4 + 3s_1}{3 + 2s_1} = \dfrac{4 + 3 \cdot 1}{3 + 2 \cdot 1} = \dfrac{7}{5}$.

$\therefore \quad s_2 > s_1$.

Now assume as our induction hypothesis that for some positive integer n,

$$s_{n+1} > s_n. \qquad \ldots(1)$$

Then $\quad s_{n+2} - s_{n+1} = \dfrac{4 + 3\,s_{n+1}}{3 + 2\,s_{n+1}} - \dfrac{4 + 3s_n}{3 + 2s_n}$

$$= \frac{(4 + 3s_{n+1})(3 + 2s_n) - (4 + 3s_n)(3 + 2\,s_{n+1})}{(3 + 2\,s_{n+1})(3 + 2\,s_n)}$$

$$= \frac{s_{n+1} - s_n}{(3 + 2\,s_{n+1})(3 + 2s_n)} > 0, \text{ by } (1).$$

$\therefore \quad s_{n+2} > s_{n+1}$.

Thus $s_2 > s_1$ and if $s_{n+1} > s_n$, then we have also $s_{n+2} > s_{n+1}$.

$\therefore$ by mathematical induction $s_{n+1} > s_n$, $\forall \ n \in \mathbf{N}$.

Thus the sequence $< s_n >$ is monotonic increasing.

Now we shall show that the sequence $< s_n >$ is also bounded above.

We have $s_{n+1} = \dfrac{3s_n + 4}{2\,s_n + 3} = \dfrac{\frac{3}{2}(2\,s_n + 3) - \frac{1}{2}}{2\,s_n + 3} = \dfrac{3}{2} - \dfrac{1}{2\,(2\,s_n + 3)}$,

showing that $s_{n+1} < \dfrac{3}{2}, \forall\ n \in \mathrm{N}$.

Also $s_1 = 1 < \dfrac{3}{2}$.

Thus $s_n < \dfrac{3}{2}, \forall\, n \in \mathrm{N}$. Therefore the sequence $< s_n >$ is bounded above by $\dfrac{3}{2}$.

Since the sequence $< s_n >$ is monotonic increasing and bounded above, therefore by monotone convergence theorem it converges to its supremum.

Let $\lim s_n = l$. Then $\lim s_{n+1} = l$.

Now $s_{n+1} = \dfrac{4 + 3s_n}{3 + 2s_n} \Rightarrow \lim s_{n+1} = \dfrac{4 + 3 \lim s_n}{3 + 2 \lim s_n}$

$\Rightarrow \quad l = \dfrac{4 + 3l}{3 + 2l} \Rightarrow l^2 = 2 \Rightarrow l = \pm\sqrt{2}$.

Since all terms of the sequence are positive so l cannot be negative. Hence $l = \sqrt{2}$.

We have $\inf < s_n > = s_1 = 1$ and $\sup < s_n > = \lim s_n = \sqrt{2}$.

Example 25: *A sequence $< s_n >$ of positive terms is defined by*

$$s_1 = k > 0\ ;\ s_{n+1} = \frac{3 + 2s_n}{2 + s_n}, \forall n \in \mathrm{N}.$$

Show that the sequence converges to a limit independent of k and find the limit.

Solution: We have $s_1 = k > 0$, and $s_{n+1} = \dfrac{3 + 2\,s_n}{2 + s_n}, \forall\ n \in \mathrm{N}$.

Then $s_2 > 0, s_3 > 0$ and so on.

Therefore the terms of the sequence are all positive.

Now first by mathematical induction we shall show that

$$s_{n+1} > s_n \quad \forall\ n \in \mathrm{N}.$$

We have $s_2 - s_1 = \dfrac{3 + 2s_1}{2 + s_1} - s_1 = \dfrac{3 + 2k}{2 + k} - k = \dfrac{3 - k^2}{2 + k} > 0$ if $0 < k < \sqrt{3}$.

Thus $s_2 > s_1$ if $0 < k < \sqrt{3}$.

Now assume as our induction hypothesis that for some positive integer n,

$$s_{n+1} > s_n. \qquad \ldots(1)$$

Then $s_{n+2} - s_{n+1} = \dfrac{3 + 2\,s_{n+1}}{2 + s_{n+1}} - \dfrac{3 + 2\,s_n}{2 + s_n}$

$$= \frac{s_{n+1} - s_n}{(2 + s_n)(2 + s_{n+1})} > 0, \text{ by (1).}$$

$\therefore \quad s_{n+2} > s_{n+1}$.

Thus $s_2 > s_1$ and if $s_{n+1} > s_n$, then we have also $s_{n+2} > s_{n+1}$.

$\therefore$ by induction $s_{n+1} > s_n$, $\forall\ n \in N$.

Thus the sequence $< s_n >$ is monotonic increasing.

Now we shall show that the sequence $< s_n >$ is also bounded above.

We have $s_{n+1} > s_n\ \forall\ n \in N$

$$\Rightarrow \quad \frac{3 + 2s_n}{2 + s_n} > s_n \quad \Rightarrow \quad \frac{3 + 2s_n}{2 + s_n} - s_n > 0$$

$$\Rightarrow \quad \frac{3 - s_n^2}{2 + s_n} > 0 \quad \Rightarrow 3 - s_n^2 > 0 \quad \Rightarrow \quad s_n^2 < 3$$

$$\Rightarrow \quad s_n < \sqrt{3}, \forall n \in N.$$

$\therefore \quad < s_n >$ is bounded above by $\sqrt{3}$.

Thus $< s_n >$ is a bounded monotonically increasing sequence. Hence it converges.

Let $\lim s_n = l$.

Then $\lim s_{n+1} = l$.

Now $\quad s_{n+1} = \dfrac{3 + 2s_n}{2 + s_n} \quad \Rightarrow \quad \lim s_{n+1} = \dfrac{3 + 2 \lim s_n}{2 + \lim s_n}$

$$\Rightarrow \quad l = \frac{3 + 2l}{2 + l} \quad \Rightarrow \quad l^2 = 3 \quad \Rightarrow \quad l = \pm\sqrt{3}.$$

But l cannot be negative because the terms of the sequence $< s_n >$ are all positive.

Hence $l = \sqrt{3}$ which is independent of k.

Example 26: *If u_1, v_1 are given unequal numbers and*

$$u_n = \frac{1}{2}(u_{n-1} + v_{n-1}),\ v_n = \sqrt{(u_{n-1}\ v_{n-1})},\ \textit{where } n \geq 2,$$

prove that (i) u_n decreases, and v_n increases as n increases,

(ii) $< u_n >, < v_n >$ are both convergent and have the same limit, where $u > v > 0$, and $u_1 = \dfrac{1}{2}(u + v)$ and $v_1 = \sqrt{(uv)}$.

Solution: Since $u > v > 0$, $u_1 = \dfrac{1}{2}(u + v)$ and $v_1 = \sqrt{(uv)}$, therefore, u_1 and v_1 are positive and $u_1 > v_1$ since A.M. > G.M.

$\therefore \quad v_1 < u_2 < u_1 \qquad [\because u_2 = \dfrac{1}{2}(u_1 + v_1)$ so that v_1, u_2, u_1 are in A.P.]

and $\quad v_1 < v_2 < u_1. \qquad [\because\ v_2 = \sqrt{(u_1\ v_1)}$ so that v_1, v_2, u_1 are in G.P.]

Since u_2 is the A.M. and v_2 is the G.M. of u_1 and v_1, therefore, we have $u_2 > v_2$.

Hence as above, we get

$v_2 < u_3 < u_2$ and $v_2 < v_3 < u_2$, and so on.

Thus $v_1 < v_2 < v_3 < \ldots < \ldots < u_3 < u_2 < u_1$.

Therefore the sequences $< u_n >$ and $< v_n >$ are monotonically decreasing and monotonically increasing respectively. Obviously both are bounded so that they are convergent.

Now we have to show that $< u_n >$ and $< v_n >$ have the same limits.

Let $\lim u_n = A$ and $\lim v_n = B$.

Now $u_n = \frac{1}{2}(u_{n-1} + v_{n-1}) \Rightarrow \lim u_n = \lim \frac{1}{2}(u_{n-1} + v_{n-1})$

or $A = \frac{1}{2}(A + B)$ *i.e.*, $A = B$.

Example 27: *If x_1, x_2 are + ive and $x_{n+2} = \sqrt{(x_{n+1} \cdot x_n)}$, prove that the sequences $x_1, x_3, x_5, \ldots$ and $x_2, x_4, x_6, \ldots$ are one an increasing and the other a decreasing sequence and show that their common limit is $(x_1 x_2^2)^{1/3}$.*

Solution: Let $x_1 > x_2$. Then since $x_3 = \sqrt{(x_2 x_1)}$, we have

$x_1 > x_3 > x_2$,

and since $x_3 > x_2$, we have $x_3 > x_4 > x_2$. $[\because x_4 = \sqrt{(x_3 x_2)}]$.

Similarly, we have

$x_3 > x_5 > x_4$; $x_5 > x_6 > x_4$;

$x_5 > x_7 > x_6$; $x_7 > x_8 > x_6$;

and so on.

Thus $x_2 < x_4 < x_6 < \ldots < \ldots < x_5 < x_3 < x_1$.

Hence $< x_1, x_3, x_5, \ldots >$ is monotonic decreasing and

$< x_2, x_4, x_6, \ldots >$

is monotonic increasing and both being bounded, are convergent.

Let $\lim x_n = A$, if n is even and $\lim x_n = B$, if n is odd.

Now $x_{n+2} = \sqrt{(x_{n+1} \cdot x_n)} \Rightarrow \lim x_{n+2} = \lim \sqrt{(x_{n+1} \cdot x_n)}$.

$\therefore$ $B = \sqrt{(A \cdot B)}$ or $A = B$, if n is odd

and $A = \sqrt{(B \cdot A)}$ or $A = B$, if n is even.

Hence in either case $A = B$.

Now $\frac{x_3}{x_2} = \frac{1}{x_2}\sqrt{(x_1 x_2)} = \sqrt{\left(\frac{x_1}{x_2}\right)}$,

$$\frac{x_4}{x_2} = \frac{x_4}{x_3} \cdot \frac{x_3}{x_2} = \sqrt{\left(\frac{x_2}{x_3}\right)}\sqrt{\left(\frac{x_1}{x_2}\right)} = \sqrt{\left(\frac{x_1}{x_3}\right)},$$

$$\frac{x_5}{x_2} = \frac{x_5}{x_4} \cdot \frac{x_4}{x_3} \cdot \frac{x_3}{x_2} = \sqrt{\left(\frac{x_3}{x_4}\right)}\sqrt{\left(\frac{x_2}{x_3}\right)}\sqrt{\left(\frac{x_1}{x_2}\right)} = \sqrt{\left(\frac{x_1}{x_4}\right)}.$$

In a like manner, $\frac{x_n}{x_2} = \sqrt{\left(\frac{x_1}{x_{n-1}}\right)}$

or $\lim (x_n \sqrt{x_{n-1}}) = \sqrt{(x_1 x_2{}^2)}$

i.e., $A \cdot A^{1/2} = \sqrt{(x_1 x_2{}^2)}$ or $A = (x_1 x_2{}^2)^{1/3}$.

Example 28: *If k is positive and $\alpha, -\beta$ are the positive and negative roots of $x^2 - x - k = 0$, prove that if $u_n = \sqrt{(k + u_{n-1})}$ and $u_1 > 0$, then $u_n \to \alpha$.*

Solution: Since u_1 is positive, hence by virtue of the relation

$u_n = \sqrt{(k + u_{n-1})}$, $u_2, u_3, \ldots, u_n, \ldots$ are all positive.

Thus $u_n > 0, \forall\ n \in \mathbf{N}$.

We have $u_n^2 - u_{n-1}^2 = (k + u_{n-1}) - (k + u_{n-2}) = u_{n-1} - u_{n-2}$ so that $u_n >$ or $< u_{n-1}$ according as $u_{n-1} >$ or $< u_{n-2}$ and hence $< u_n >$ is a monotonic sequence, it is an increasing or a decreasing sequence according as $u_2 >$ or $< u_1$.

Now $x^2 - x - k \equiv (x - \alpha)(x + \beta)$ ...(1)

$\Rightarrow$ $u_1{}^2 - u_1 - k = (u_1 - \alpha)(u_1 + \beta)$. ...(2)

Let $u_1 > \alpha$. Then from (2), we have

$u_1{}^2 - u_1 - k > 0$

$\Rightarrow$ $u_1 + k < u_1{}^2 \Rightarrow \sqrt{(u_1 + k)} < u_1 \Rightarrow u_2 < u_1$.

Hence in this case $< u_n >$ is a decreasing sequence.

Since $u_n > 0, \forall\ n \in \mathbf{N}$, therefore $< u_n >$ is bounded below by 0.

Thus $< u_n >$ is a monotonically decreasing sequence and is bounded below and hence $< u_n >$ is convergent.

Again let $u_1 < \alpha$. Then from (2), we have

$u_1{}^2 - u_1 - k < 0 \Rightarrow u_1 + k > u_1{}^2 \Rightarrow \sqrt{(u_1 + k)} > u_1 \Rightarrow u_2 > u_1$.

Hence is this case $< u_n >$ is an increasing sequence.

Now $u_n{}^2 = u_{n-1} + k < u_n + k$, $[\because u_{n-1} < u_n]$

i.e., $u_n{}^2 - u_n - k < 0$ or $(u_n - \alpha)(u_n + \beta) < 0$, using (1).

$\therefore$ $u_n < \alpha$. $[\because\ u_n > 0$ and $\beta > 0 \Rightarrow u_n + \beta > 0]$

Thus in this case $u_n < \alpha, \forall\ n \in \mathbf{N}$.

Hence in this case $< u_n >$ is a monotonically increasing sequence and is bounded above by α and so $< u_n >$ is convergent.

Thus $< u_n >$ is convergent whether $u_1 >$ or $< \alpha$.

Let $\lim u_n = l$.

Now $(u_n - \alpha)(u_n + \beta) = u_n^2 - u_n - k = (u_{n-1} + k) - u_n - k = u_{n-1} - u_n$.

Taking limits, we get $(l - \alpha)(l + \beta) = l - l = 0$.

This gives $l = \alpha$ or $l = -\beta$.

Since the terms of the sequence $< u_n >$ are all positive, so its limit cannot be negative. Therefore we cannot have $l = -\beta$. Therefore we must have $l = \alpha$.

In case $u_1 = \alpha$, then from (2), we have

$$u_1^2 - u_1 - k = 0 \Rightarrow u_2 = \sqrt{(u_1 + k)} = u_1.$$

Now repeatedly using the relation

$$u_n^2 - u_{n-1}^2 = u_{n-1} - u_{n-2},$$

we observe that $u_3 = u_2$, $u_4 = u_3$, $u_5 = u_4$, and so on.

Thus in this case $u_n = u_1 = \alpha, \forall\ n \in N$.

So in this case also the sequence $< u_n >$ converges to α.

Hence in all cases $< u_n >$ converges to α which is the positive root of the equation $x^2 - x - k = 0$.

Example 29: *Show that the sequence* $< s_n >$ *defined by*

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \text{ does not converge.}$$

Solution: We shall show that the given sequence is not a Cauchy sequence. For this we shall show that if we take $\varepsilon = \frac{1}{2} > 0$, then there exists no positive integer m such that

$$|s_n - s_m| < \varepsilon \ \forall\ n \geq m.$$

Whatever positive integer m may be, if we take $n = 2m$, then $n > m$ and we have

$$|s_n - s_m| = |s_{2m} - s_m|$$

$$= \left|\left(1 + \frac{1}{2} + \ldots + \frac{1}{m} + \frac{1}{m+1} + \ldots + \frac{1}{2m}\right) - \left(1 + \frac{1}{2} + \ldots + \frac{1}{m}\right)\right|$$

$$= \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{2m}$$

$$> \frac{1}{2m} + \frac{1}{2m} + \frac{1}{2m} + \ldots \text{upto } m \text{ terms}$$

$$= m \cdot \frac{1}{2m} = \frac{1}{2}.$$

Thus if we take $\varepsilon = \frac{1}{2}$, then whatever positive integer m we take, we have $n = 2m > m$ and

$$|s_n - s_m| = |s_{2m} - s_m| > \frac{1}{2} \quad i.e.,\ |s_n - s_m| > \varepsilon.$$

In this way for $\varepsilon = \frac{1}{2} > 0$, there exists no positive integer m such that

$$|s_n - s_m| < \varepsilon\ \forall n \geq m.$$

$\therefore$ the given sequence is not a Cauchy sequence.

Hence by Cauchy convergence criterion $< s_n >$ is not convergent.

Example 30: *If* $< s_n >$ *be a sequence of positive numbers such that*

$$s_n = \frac{1}{2}(s_{n-1} + s_{n-2}), \text{ for all } n > 2,$$

then show that $< s_n >$ *converges and find* $\lim s_n$.

Solution: In case $s_1 = s_2$, it can be easily seen that $s_n = s_1$ for all n, therefore $< s_n >$ converges to s_1. Now we consider the case $s_1 \neq s_2$.

We first find that

$$|s_n - s_{n-1}| = \left|\frac{1}{2}(s_{n-1} + s_{n-2}) - s_{n-1}\right| = \frac{1}{2}|s_{n-1} - s_{n-2}|$$

$$= \frac{1}{2}\cdot\frac{1}{2}|s_{n-2} - s_{n-3}| = \frac{1}{2^2}|s_{n-2} - s_{n-3}|$$

$$= \frac{1}{2^{n-2}}|s_2 - s_1|, \text{ for } n \geq 2. \qquad \ldots(1)$$

Now for $n \geq m$, we have

$$|s_n - s_m| = |(s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + \ldots + (s_{m+1} - s_m)|$$

$$\leq |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \ldots + |s_{m+1} - s_m|$$

$$= \left(\frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + \ldots + \frac{1}{2^{m-1}}\right)|s_2 - s_1| \text{ by (1)}$$

$$= \frac{1}{2^{m-1}}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-m-1}}\right)|s_2 - s_1|$$

$$< \frac{1}{2^{m-2}}|s_2 - s_1|. \qquad \ldots(2)$$

$$\left[\because\ 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots + \left(\frac{1}{2}\right)^{n-m-1} < 2\right]$$

Let $\varepsilon > 0$ be given. We can choose a positive integer m such that $\frac{1}{2^{m-2}}|s_2 - s_1| < \varepsilon$. For this value of m, we have from (2)

$$|s_n - s_m| < \varepsilon \text{ for } n \geq m.$$

Hence $< s_n >$ is a Cauchy sequence and therefore by Cauchy's convergence criterion it converges.

Let $\lim s_n = l$. Putting $n = 3, 4, \ldots, k$ in the relation $s_n = \frac{1}{2}(s_{n-1} + s_{n-2})$, we get

$$\left.\begin{aligned} s_3 &= \frac{1}{2}(s_2 + s_1) \\ s_4 &= \frac{1}{2}(s_3 + s_2) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ s_{k-1} &= \frac{1}{2}(s_{k-2} + s_{k-3}) \\ s_k &= \frac{1}{2}(s_{k-1} + s_{k-2}). \end{aligned}\right\} \qquad \ldots(3)$$

Adding the corresponding sides of the relations in (3), we get

$$s_k + \frac{1}{2} s_{k-1} = \frac{1}{2}(s_1 + 2s_2).$$

Proceeding to the limit as $k \to \infty$, we get

$$\frac{3}{2} l = \frac{1}{2}(s_1 + 2s_2) \quad i.e., \quad l = \frac{1}{3}(s_1 + 2s_2).$$

Example 31: *Let $< u_n >$ be a sequence and $s_n = u_1 + u_2 + \ldots + u_n$.*

If $t_n = |u_1| + |u_2| + \ldots + |u_n|$ for each $n \in \mathbb{N}$ and $< t_n >$ is a Cauchy sequence, then $< s_n >$ is also a Cauchy sequence.

Solution: Let $\varepsilon > 0$ be given. Since $< t_n >$ is a Cauchy sequence, therefore, for given $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that

$$|t_n - t_m| < \varepsilon \; \forall \; n \geq m$$

$\Rightarrow$ $|u_{m+1}| + |u_{m+2}| + \ldots + |u_n| < \varepsilon \; \forall n \geq m.$

But $|u_{m+1}| + |u_{m+2}| + \ldots + |u_n| \geq |u_{m+1} + u_{m+2} + \ldots + u_n|.$

$\therefore$ $|u_{m+1} + u_{m+2} + \ldots + u_n| < \varepsilon \; \forall n \geq m$

or $|s_n - s_m| < \varepsilon \; \forall n \geq m.$

Hence $< s_n >$ is a Cauchy sequence.

Example 32: *Find* (i) $\lim \sqrt{\left(\frac{n+1}{n}\right)}$ (ii) $\lim \frac{\sin(n\pi/3)}{\sqrt{n}}$.

Solution: (i) We have $\sqrt{\left(\frac{n+1}{n}\right)} = \sqrt{\left(1 + \frac{1}{n}\right)} < 1 + \frac{1}{2n}$.

$$\left[\because \left(1 + \frac{1}{2n}\right)^2 = 1 + \frac{1}{n} + \frac{1}{4n^2} > 1 + \frac{1}{n}\right].$$

Since $1 + \frac{1}{n} > 1$, hence $\sqrt{\left(1 + \frac{1}{n}\right)} > 1$.

$\therefore$ $1 < \sqrt{\left(\frac{n+1}{n}\right)} < 1 + \frac{1}{2n}$ for all $n \in \mathbb{N}$.

But $\lim 1 = 1$ and $\lim \left(1 + \frac{1}{2n}\right) = 1$.

Hence, by Sandwich theorem, $\lim \sqrt{\left(\frac{n+1}{n}\right)} = 1$.

(ii) Let $s_n = \sin(n\pi/3)$, $t_n = 1/\sqrt{n}$.

We have $-1 \leq \sin \frac{n\pi}{3} \leq 1$ for all $n \in \mathbb{N}$.

$\therefore$ $< s_n >$ is a bounded sequence.

Also $\lim t_n = \lim \frac{1}{\sqrt{n}} = 0.$

Hence by theorem 4 of article 2.9, $\lim (s_n \; t_n) = 0$ *i.e.*, $\lim \frac{\sin(n\pi/3)}{\sqrt{n}} = 0.$

Example 33: *Show that* $\lim\left[\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\ldots+\frac{1}{(n+n)^2}\right]=0.$

Solution: Let $s_n=\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\ldots+\frac{1}{(n+n)^2}.$

For $1\le m\le n$, we have $(n+1)^2\le(n+m)^2\le(n+n)^2$

or $$\frac{1}{(n+1)^2}\ge\frac{1}{(n+m)^2}\ge\frac{1}{(n+n)^2}.$$

Putting $m=1,2,\ldots,n$ and adding the corresponding sides of the n inequalities thus obtained, we get

$$\frac{n}{(n+1)^2}\ge s_n\ge\frac{n}{(n+n)^2}$$

i.e., $$\frac{n}{4n^2}\le s_n\le\frac{n}{(n+1)^2}<\frac{n}{n^2}\quad\text{for all } n\in\mathbb{N}$$

i.e., $$\frac{1}{4n}\le s_n<\frac{1}{n}\text{ for all } n\in\mathbb{N}.$$

But $\lim\frac{1}{4n}=0$ and $\lim\frac{1}{n}=0.$

Hence, by Sandwich theorem, $\lim s_n=0.$

Example 34: *If* $r>0$, *show that* $\lim r^{1/n}=1.$

Solution: **Case 1.** When $r>1.$

Let $s_n=r^{1/n}-1$. Then $s_n>0$, for all n.

Now $$s_n=r^{1/n}-1\in r^{1/n}=1+s_n\Rightarrow r=(1+s_n)^n$$

$\Rightarrow$ $$r=1+n\,s_n+\ldots+s_n{}^n\ge1+n\,s_n\text{ for all } n\in\mathbb{N}$$

$\Rightarrow$ $$\frac{r-1}{n}\le s_n\ \forall n\in\mathbb{N}.$$

$\therefore$ $$0<s_n\le\frac{r-1}{n}\ \forall n\in\mathbb{N}.$$

Hence, by Sandwich theorem, $\lim s_n=0$ *i.e.,* $\lim(r^{1/n}-1)=0$ or $\lim r^{1/n}=1.$

Case 2: When $r=1.$

In this case $r^{1/n}=1\ \forall n$ and hence $<r^{1/n}>$ converges to 1.

Case 3. When $0<r<1.$

Since $\frac{1}{r}>1$, therefore, by Case 1, $\lim\left(\frac{1}{r}\right)^{1/n}=1$ *i.e.,* $\lim\frac{1}{r^{1/n}}=1.$

$\therefore$ By theorem 7 of article 2.9, $\lim r^{1/n}=1.$

Example 35: *Show that* $\lim \frac{1}{n}\left(1+\frac{1}{3}+\ldots+\frac{1}{2n-1}\right)=0.$

Solution: Let $s_n=\frac{1}{2n-1}$. Then $\lim s_n=\lim \frac{1}{2n-1}=0.$

$\therefore$ By Cauchy's first theorem on limits, $\lim \frac{s_1+s_2+\ldots+s_n}{n}=0$.

Since $s_1=1,\ s_2=\frac{1}{3},\ldots,s_n=\frac{1}{2n-1}$,

$\therefore$ $\lim \frac{1}{n}\left\{1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}\right\}=0$.

Example 36: *Prove that* $\lim \frac{1}{n}\left(1+2^{1/2}+3^{1/3}+\ldots+n^{1/n}\right)=1$.

Solution: Let $s_n=n^{1/n}$. Then we know that $\lim n^{1/n}=1$.

Hence, by Cauchy's first theorem on limits,

$$\lim \frac{1}{n}\left(s_1+s_2+\ldots+s_n\right)=1$$

or $$\lim \frac{1}{n}\left(1+2^{1/2}+3^{1/3}+\ldots+n^{1/n}\right)=1.$$

Example 37: *Prove that* $\lim\left[\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\ldots\left(\frac{n+1}{n}\right)^n\right]^{1/n}=e.$

Solution: Let $s_n=\left(\frac{n+1}{n}\right)^n=\left(1+\frac{1}{n}\right)^n$. Then $\lim s_n=e$.

Clearly $s_n>0$ for all n.

Hence, by theorem 11 of article 2.9, $\lim\left(s_1\ s_2\ldots s_n\right)^{1/n}=e$.

Since $s_1=\frac{2}{1},\ s_2=\left(\frac{3}{2}\right)^2,\ s_3=\left(\frac{4}{3}\right)^3,\ldots s_n=\left(\frac{n+1}{n}\right)^n$,

$\therefore$ $\lim\left[\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\ldots\left(\frac{n+1}{n}\right)^n\right]^{1/n}=e.$

Example 38: *Show that*

(*i*) $\lim \frac{n}{(n!)^{1/n}}=e$,

(*ii*) $\lim\left[\{(n+1)(n+2)\ldots(n+n)\}^{1/n}/n\right]=4/e.$

Solution: (i) Let $s_n=\frac{n^n}{n!}$, then $s_{n+1}=\frac{(n+1)^{n+1}}{(n+1)!}$.

$\therefore$ $\frac{s_{n+1}}{s_n}=\frac{(n+1)^{n+1}}{n+1}\cdot\frac{1}{n^n}=\left(\frac{n+1}{n}\right)^n=\left(1+\frac{1}{n}\right)^n$.

Also $s_n > 0$ for all $n \in N$.

Hence by theorem 12 of article 2.9, we have

$$\lim s_n^{1/n} = \lim \frac{s_{n+1}}{s_n} = \lim \left(1 + \frac{1}{n}\right)^n = e$$

i.e., $$\lim \frac{n}{(n\,!)^{1/n}} = e.$$

(ii) Let $s_n = (n+1)(n+2)\ldots(n+n)/n^n$.

Then $$\frac{s_{n+1}}{s_n} = \frac{2(2n+1)}{(n+1)} \left(\frac{n}{n+1}\right)^n,$$

so that $$\lim \frac{s_{n+1}}{s_n} = \lim \left[\frac{2(2n+1)}{n+1} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n}\right] = 4 \cdot \frac{1}{e} = \frac{4}{e}.$$

Also $s_n > 0$ for all $n \in N$.

Hence by theorem 12 of article 2.9, we have

$$\lim s_n^{1/n} = \lim [\{(n+1)(n+2)\ldots(n+n)\}^{1/n} / n] = 4/e.$$

Example 39: *If $p > 0$ and c is real, then find the* $\lim \dfrac{n^c}{(1+p)^n}$.

Solution: Let k be an integer such that $k > c$, $k > 0$.

We have, for $n > 2k$

$$(1+p)^n > {}^nC_k\, p^k = \frac{n(n-1)\ldots(n-k+1)}{k\,!} p^k > \frac{n^k}{2^k} \cdot \frac{p^k}{k\,!}$$

$$\left[\because\ n > 2k \Rightarrow (n-r+1) > \frac{n}{2} \text{ for } r = 1, 2, \ldots, k\right]$$

or $$0 < \frac{1}{(1+p)^n} < \frac{2^k\, k\,!}{n^k\, p^k}, \text{ for } n > 2k.$$

$\therefore$ $$0 < \frac{n^c}{(1+p)^n} < \frac{2^k\, k\,!}{p^k} \cdot \frac{1}{n^{k-c}} \quad \text{for } n > 2k. \qquad \ldots(1)$$

Since $k - c > 0$, therefore, $\dfrac{1}{n^{k-c}} \to 0$ as $n \to \infty$.

Hence from (1), we get $$\lim_{n \to \infty} \frac{n^c}{(1+p)^n} = 0.$$

Example 40: *The usual definition of e is given by* $e = \sum_{n=0}^{\infty} \dfrac{1}{n\,!}$. *Show that e is irrational.*

Solution: Let $s_n = 1 + \dfrac{1}{1\,!} + \dfrac{1}{2\,!} + \dfrac{1}{3\,!} + \ldots + \dfrac{1}{n\,!}$.

Then $e - s_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \ldots$

$$< \frac{1}{(n+1)!}\left[1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \ldots\right]$$

$$= \frac{1}{(n+1)!} \cdot \frac{1}{1 - \frac{1}{n+1}} = \frac{1}{n\,(n!)}.$$

Thus $0 < e - s_n < \frac{1}{n\,(n!)}$. ...(1)

Let, if possible, e be rational. Then e can be put in the form p / q where p and q are positive integers.

From (1), we have $0 < e - s_q < \frac{1}{q\,(q!)}$

or $0 < (q!)\,(e - s_q) < \frac{1}{q}$ ...(2)

Now $(q!)\,s_q = (q!)\left\{1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{q!}\right\}$

is an integer. Also, by our assumption $e\,(q!)$ is an integer. It follows that $e\,(q!) - s_q\,(q!) = q!\,(e - s_q)$ is an integer. Since $q \geq 1$, therefore, (2) shows the existence of an integer between 0 and 1 which is absurd. Hence our initial assumption is wrong. So e is irrational.

Comprehensive Exercise 1

1. Write a formula or formulae for the nth term s_n for each of the following sequences :
   (a) $1, -4, 9, -16, 25, -36, \ldots$ (b) $1, 0, 1, 0, 1, 0, \ldots$
   (c) $1, 3, 6, 10, 15, \ldots$
2. Which of the sequences (a), (b), (c) in the above problem are subsequences of the sequence $< s_n >$ defined by $s_n = n$?
3. Find whether the following sequences are bounded above or below :
   (i) $< \frac{(-1)^n}{n} >$ (ii) $< 2^n >$ (iii) $< n! >$.
4. Are the sequences $< s_n >$ defined as follows, bounded ?
   (i) $s_n = 1 + \frac{(-1)^n}{n}$ (ii) $s_n = \left(1 + \frac{1}{n}\right)^n$

(iii) $s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$.

(iv) $s_n = 1$, if n is divisible by 3 and $s_n = 0$, otherwise.

5. Use the definition of the limit of a sequence to show that the limit of the sequence $< s_n >$ where $s_n = 2n / (n + 3)$, is 2.

6. Show that the sequence $< s_n >$ where $s_n = n / (n + 1)$ converges to 1.

7. If the sequence $< s_n >$ converges to l, then prove that the sequence $|s_n|$ converges to $|l|$.

8. Show by considering the sequence $< s_n = (-1)^n >$ that $< |s_n| >$ may converge but $< s_n >$ may not.

9. If $\lim s_n = l$ and $s_n \leq m$ for all $n \in N$, prove that $l \leq m$.

10. Let $< s_n >$ be a sequence such that $< s_n^2 >$ converges to zero. Is it necessary that $< s_n >$ should converge to zero ?

11. If $s_n = \frac{2n}{n + 4n^{1/2}}$, prove that $< s_n >$ is convergent.

12. If $s_n = \frac{n}{2^n}$, prove that $< s_n > \to 0$.

13. If $< s_n >$ converges to l $(\neq 0)$, prove that $< (-1)^n s_n >$ oscillates.

14. If $< s_n >$ diverges and c $(\neq 0) \in R$, prove that $< cs_n >$ diverges.

15. Show that the sequence $< n + (-1)^n n >$ oscillates infinitely.

16. Show that $< s_n >$ converges to e, where s_n is

(i) $\left(1 + \frac{1}{n}\right)^{n+1}$ (ii) $\left(1 + \frac{1}{n+1}\right)^n$ (iii) $\left(1 - \frac{1}{n}\right)^{-n}$

17. Show that the sequence $< s_n >$, where $s_n = \left(1 + \frac{2}{n}\right)^n$, converges to e^2.

18. If $s_n = \frac{n^3 - 2n + 1}{n^3 + 2n^2 - 1}$, prove that $\lim s_n = 1$.

19. If $s_n = \frac{(3n - 1)(n^4 - n)}{(n^2 + 2)(n^3 + 1)}$, prove that $\lim s_n = 3$.

20. Prove that the sequence $< \frac{n^2 + 3n + 5}{2n^2 + 5n + 7} >$ converges to $\frac{1}{2}$.

21. Prove that the sequence $< s_n >$ where $s_n = \frac{n}{n^2 + 1}$ is convergent.

22. Show that the sequence $< s_n >$ defined by

$$s_n = \frac{1}{\sqrt{(n^2+1)}} + \frac{1}{\sqrt{(n^2+2)}} + \ldots + \frac{1}{\sqrt{(n^2+n)}}$$

converges to 1.

23. Prove that $\lim \left[\frac{1}{\sqrt{(2n^2+1)}} + \frac{1}{\sqrt{(2n^2+2)}} + \ldots + \frac{1}{\sqrt{(2n^2+n)}} \right] = \frac{1}{\sqrt{2}}$.

24. Show that $\lim \left[\frac{1}{(n+1)^{\lambda}} + \frac{1}{(n+2)^{\lambda}} + \ldots + \frac{1}{(2n)^{\lambda}} \right] = 0, \lambda > 1$.

25. A sequence $< s_n >$ is defined as follows :

$$s_1 = a > 0, s_{n+1} = \sqrt{\{(ab^2 + s_n^2)/(a+1)\}}, b > a, n \geq 1.$$

Show that $< s_n >$ is a bounded monotonically increasing sequence and $\lim s_n = b$.

26. Prove that $\lim_{n \to \infty} \left\{ \frac{(3n)!}{(n!)^3} \right\}^{1/n} = 27$.

27. Prove that $\lim \left[\frac{(n!)^{1/n}}{n} \right] = \frac{1}{e}$.

28. (i) Let $< s_n >$ be a sequence defined as follows :

$$s_1 = \frac{3}{2}; s_{n+1} = 2 - \frac{1}{s_n}, n \geq 1.$$

Show that $< s_n >$ is monotonic and bounded. Find the limit of the sequence.

(ii) Show that the sequence $< s_n >$ defined by the formula $s_1 = 1$, $s_{n+1} = \sqrt{(3s_n)}$ converges to 3.

29. If $< s_n >$ is a sequence such that $s_n > 0$ and $s_{n+1} \leq k\, s_n$ for all $n \geq m$ and $0 < k < 1$, m being a fixed positive integer, then $\lim s_n = 0$.

30. Prove that if x be any real number, then $\lim_{n \to \infty} \frac{x^n}{n!} = 0$.

31. If $s_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \ldots + \frac{1}{3^n}$, prove that $< s_n >$ converges. Also find $\sup < s_n >$ and $\inf < s_n >$.

32. What do you understand by a monotonic sequence ? Prove that the sequence $< s_n >$, where $s_n = \frac{2n-7}{3n+1}$, is :

(i) monotonic increasing (ii) bounded above.

(iii) bounded below.

Also show that the sequence $< s_n >$ is convergent and find $\sup < s_n >$ and $\inf < s_n >$.

33. If $s_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n+1)}$, prove that $< s_n >$ is increasing and convergent. Find also $\sup < s_n >$ and $\inf < s_n >$.

34. If $s_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n$, prove that $< s_n >$ is decreasing and bounded.

35. Test the sequence $< s_n >$, defined by $s_n = (-1)^n n$, for limit points.

36. Find the limit points of the sequence $< s_n >$ defined by

$$s_n = (-1)^n \left(1 + \frac{1}{n}\right).$$

37. State and prove Cauchy's general principle of convergence for real sequences. Hence prove that the sequence $< s_n >$, where $s_n = \frac{n+1}{n}$ converges.

38. Show, by applying Cauchy's convergence criterion, that the sequence $< s_n >$ defined by

$$s_n = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} \text{ does not converge.}$$

39. If the sequence $< s_n >$ converges and if $< t_n >$ is a sequence such that

$$|t_n - t_m| \leq |s_n - s_m|,$$

for all positive integers m and n, then prove that $< t_n >$ converges.

40. If $< s_n >$ is a Cauchy sequence of real numbers which has a subsequence converging to l, prove that $< s_n >$ itself converges to l.

41. Show that $\lim [(n!)\,(a/n)^n] = 0$ or $+\infty$ according as $a < e$ or $a > e$, where a is any non-negative real number.

42. If $0 < u_1 < u_2$ and $u_n = \frac{2u_{n-1}\,u_{n-2}}{u_{n-1} + u_{n-2}}$ (*i.e.*, u_n is the harmonic mean of u_{n-1} and u_{n-2}), show that

$$\lim u_n = 3u_1\,u_2 / (2u_1 + u_2).$$

43. If the sequences $< s_n >$ and $< t_n >$ converge to zero and if $< t_n >$ is a strictly decreasing sequence so that $t_{n+1} < t_n \ \forall n \in N$, then

$$lim \frac{s_n}{t_n} = lim \frac{s_n - s_{n+1}}{t_n - t_{n+1}}$$

provided that the limit on the right exists, whether finite or infinite.

44. If $s_n = \frac{a}{1 + s_{n-1}}$, where a, s_n are positive, show that the sequence $< s_n >$ tends to a definite limit l, the positive root of the equation $x^2 + x = a$.

45. Prove that the set of limit points of every sequence is a closed set.

Answers 1

1. (a) $s_n = (-1)^{n-1} n^2$ (b) $s_n = 1$ if n is odd, $s_n = 0$ if n is even,
   (c) $s_n = \dfrac{n(n+1)}{2}$
2. (c).
3. (i) Bounded above as well as bounded below
   (ii) Bounded below but not above
   (iii) Bounded below but not above
4. (i) Yes (ii) Yes (iii) Yes (iv) Yes

10. Yes 28. (i) 1

35. No limit point 36. 1 and – 1

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. Every bounded monotonically increasing sequence converges to
   (a) its supremum (b) its infimum
   (c) 0 (d) 1
2. If $< s_n >$ is a sequence of non-negative numbers such that $\lim s_n = l$, then
   (a) $l < 0$ (b) $l > 0$ (c) $l \geq 0$ (d) $l = 1$
3. Every subsequence of a convergent sequence is
   (a) divergent (b) convergent
   (c) may be convergent or divergent (d) oscillatory
4. The sequence $< s_n >$ where $s_n = \dfrac{5n}{n + 3n^{1/2}}$ has the limit
   (a) 3 (b) 1 (c) $\dfrac{1}{3}$ (d) 5
5. The sequence $< 1, -1, 1, -1, 1, -1, \ldots >$ has
   (a) no limit point
   (b) only one limit point
   (c) two limit points
   (d) an infinite number of limit points

6. The sequence $< s_n >$, where $s_n = \left(1 + \frac{2}{n}\right)^{n+3}$, converges to

(a) e (b) e^2 (c) $e + 3$ (d) $e^2 + 3$

7. If $s_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}$, then the sequence $< s_n >$ is

(a) unbounded (b) convergent
(c) divergent (d) oscillatory

8. The sequence $< s_n >$, where $s_n = \frac{3n^2 + 1}{3n^2 - 1}$, converges to

(a) 1 (b) 3 (c) -1 (d) 0

9. Every Cauchy sequence is

(a) oscillatory (b) divergent
(c) unbounded (d) convergent

10. The value of the limit $\lim_{n \to \infty} \frac{1}{n}(1 + 2^{1/2} + 3^{1/3} + \ldots + n^{1/n})$ is equal to

(a) 1 (b) 0 (c) 2 (d) 3

11. Every convergent sequence is :

(a) oscillatory (b) unbounded
(c) bounded (d) oscillates finitely

(Rohilkhand 2011)

Fill In The Blank(s)

Fill in the blanks "....." so that the following statements are complete and correct.

1. The range of the sequence $< (-1)^n >$ is the set ... .
2. The sequence $< 1/n >$ converges to ... .
3. If $s_n = \frac{1}{2^n}$ for all $n \in N$, then

(i) $\sup < s_n > =$ ..... and (ii) $\inf < s_n > = \ldots$.

4. The supremum of the sequence $< \frac{n}{n+1} >$ is ... .

(Rohilkhand 2012)

5. The nth term of the sequence $< 1, -1, 1, -1, \ldots >$ is ... .
6. If the subsequences $< s_{2n-1} >$ and $< s_{2n} >$ of the sequence $< s_n >$ converge to the same limit l, then $\lim_{n \to \infty} s_n = \ldots$.
7. If $s_n = n\sqrt{n}$, then $\lim_{n \to \infty} s_n = \ldots$.
8. The sequence $< -2, -4, -6, \ldots, -2n, \ldots >$ diverges to ... .
9. If $\lim s_n = l$ and $\lim t_n = l'$, then $\lim (s_n\, t_n) = \ldots$.

10. If $\lim_{n \to \infty} s_n = l$, then $\lim_{n \to \infty} \frac{s_1 + s_2 + \ldots + s_n}{n} = \ldots$.

11. Every bounded monotonic sequence is ... .

12. The sequence $< \frac{n^2 + 4n + 7}{3n^2 + 5n - 9} >$ converges to ... .

13. If $s_n = \left(1 + \frac{1}{n}\right)^{n+2}$, then $\lim_{n \to \infty} s_n = \ldots$.

14. If $s_n = \sqrt{(n+1)} - \sqrt{n}$, then $\lim_{n \to \infty} s_n = \ldots$.

15. If $s_n = \left(1 + \frac{1}{n}\right)^n$, then $\lim_{n \to \infty} (s_1 \, s_2 \ldots s_n)^{1/n} = \ldots$.

True or False

Write 'T' for true and 'F' for false statement.

1. The sequence $< 2, -2, 2, -2, 2, -2, \ldots >$ is bounded.
2. The sequence $< 1, -1, 1, -1, 1, -1, \ldots >$ is convergent.
3. The sequence $< 3, -3, 3, -3, 3, -3, \ldots >$ is a Cauchy sequence.
4. Every Cauchy sequence is always convergent.
5. Every subsequence of a divergent sequence is always divergent.
6. A sequence $< s_n >$ is said to converge to a number l, if for any given $\varepsilon > 0$ there exists a positive integer m such that $|s_n - l| > \varepsilon$ for all $n \geq m$.
7. If a sequence $< s_n >$ is convergent, then $\lim_{n \to \infty} s_n$ is unique.
8. Every convergent sequence is always bounded.
9. Every bounded sequence is always convergent.
10. The sequence $< 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots >$ is a Cauchy sequence.
11. If $\lim s_n = l$ and $\lim t_n = l'$, then $\lim (s_n + t_n) = l + l'$.
12. Every bounded monotonically decreasing sequence converges to its infimum.
13. A monotonically decreasing sequence which is not bounded below diverges to minus infinity.
14. A monotonically increasing sequence which is not bounded above is a convergent sequence.
15. Every bounded sequence has at least one limit point.
16. The sequence $< 1, -1, 1, -1, \ldots >$ has no limit points.
17. If a sequence $< s_n >$ is convergent, then it may or may not be a Cauchy sequence.

18. The sequence $< s_n >$, where $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$, is a Cauchy sequence.

19. The sequence $< s_n >$, where $s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{(n-1)!}$, is a convergent sequence.

20. If a sequence $< s_n >$ is convergent, then the sequence $< |s_n| >$ is also convergent.

21. If a sequence $< |s_n| >$ is convergent, then the sequence $< s_n >$ is also convergent.

22. The sequence $< s_n >$, where $s_n = \frac{n}{n^3 + 1}$, converges to 0.

23. The sequence $< s_n >$, where $s_n = \left(1 + \frac{1}{n}\right)^n$, is a bounded sequence.

24. The sequence $< 1, -1, 1, -1, 1, -1, \ldots >$ is a monotonic sequence.

25. The sequence $< s_n >$, where $s_n = 3 - \frac{1}{3^{n-1}}$, converges to 2.

Answers

Multiple Choice Questions

1.	(a)	2.	(c)	3.	(b)	4.	(d)	5.	(c)
6.	(b)	7.	(b)	8.	(a)	9.	(d)	10.	(a)
11.	(c)								

Fill in the Blank(s)

1.	$\{1, -1\}$	2.	0	3.	(i) $\frac{1}{2}$ (ii) 0	4.	1	5.	$(-1)^{n-1}$
6.	l	7.	1	8.	$-\infty$	9.	ll'	10.	l
11.	convergent	12.	$\frac{1}{3}$	13.	e	14.	0	15.	e

True or False

1.	T	2.	F	3.	F	4.	T	5.	T
6.	F	7.	T	8.	T	9.	F	10.	T
11.	T	12.	T	13.	T	14.	F	15.	T
16.	F	17.	F	18.	F	19.	T	20.	T
21.	F	22.	T	23.	T	24.	F	25.	F

Chapter

3 Uniform Convergence of Sequences and Series of Functions

3.1 Some Definitions

In the present chapter we shall study the uniform convergence of sequences and series of real-valued functions defined on an interval.

Sequence of real-valued functions: *Let f_n be a real-valued function defined on the interval I for all positive integral values n. Then $< f_1, f_2, \ldots, f_n, \ldots >$ or simply $< f_n >$ is called a sequence of real valued functions on I.*

Uniformly bounded sequence: *A sequence $< f_n >$ is said to be uniformly bounded on an interval I if*

$$|f_n(x)| < M \text{ for every } x \in I$$

and for every positive integer n.

Examples: (i) The sequence $< f_n >$ where $f_n(x) = 1/nx$ is bounded but not uniformly bounded.

(ii) The sequence defined by $< \sin nx >$ is uniformly bounded on R since $|\sin nx| \leq 1$ for all $x \in R$ and for all $n \in N$.

Limit function of a sequence of functions:

Let f_n be a real valued function defined on the interval I for all positive integral values n. Suppose that the function f_n tends to a definite limit for all values of x in I as $n \to \infty$.

This means that, for $c \in I$, the sequence $< f_n(c) >$ *i.e.*, $< f_1(c), f_2(c), \ldots, f_n(c), \ldots >$ of real numbers is convergent. The limiting values of each of the sequences of real numbers for various points in I will define a function of x, say f, such that the limiting value of the convergent sequence $< f_n(c) >$ is equal to the value $f(c)$ of the function f at c. This function f is called the **limit function or the limit of the convergent sequence** $< f_n >$ of functions on I.

Sum function of a series

Let $$u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots \quad \ldots(1)$$

be the series of real valued functions defined on the interval I. Corresponding to this series, we find a sequence $< f_n >$ of functions where

$$f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x).$$

We say that the series (1) is convergent if the sequence $< f_n >$ is convergent and the limit function f of the sequence is called the **sum function** of the series.

3.2 Uniform Convergence of Sequences

Let the sequence $< f_n(x) >$ converge for every point x in I *i.e.*, the function f_n tends to a definite limit as $n \to \infty$ for every x in I. This limit will be a function of x, say f. It follows from the definition of a limit that for every $\varepsilon > 0$, there exists a positive integer m such that

$$|f_n(x) - f(x)| < \varepsilon \text{ whenever } n \geq m.$$

This integer m will depend upon x as well as ε and so we can write it as $m(x, \varepsilon)$. Now if we keep ε fixed and vary x, then for a given point x in I there corresponds a value of $m(x, \varepsilon)$. Thus, we shall find a set of values of $m(x, \varepsilon)$. This set may or may not have an upper bound. If this set has an upper bound, say M, then for every point x in I, we get

$$|f_n(x) - f(x)| < \varepsilon \text{ whenever } n \geq M.$$

In this case, the sequence $< f_n >$ is said to converge uniformly to f on I.

Definition: *Let $< f_n >$ be a sequence of functions defined on an interval I. The sequence $< f_n >$ is said to converge uniformly to the function f on I if for every $\varepsilon > 0$, there can be found a positive integer m such that*

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq m \text{ and for all } x \in I. \quad \text{(Avadh 2009)}$$

The function f is called uniform limit of the sequence $< f_n >$ on I.

Note 1: It should be noted carefully that in the above definition the emphasis is on the phrase "one m for all x" whereas for ordinary convergence it is "one m for each x".

Note 2: The sequence $< f_n >$ does not converge uniformly to f on an interval I iff there exists some $\varepsilon > 0$ such that there is no positive integer m for which

$$|f_n(x) - f(x)| < \varepsilon \quad \forall \; n \geq m \text{ and } x \in I.$$

Note 3: It can be easily observed that uniform convergence of a sequence $< f_n >$ on I implies pointwise convergence of the sequence $< f_n >$ at every point of I but point-wise convergence does not necessarily ensure its uniform convergence on I. A sequence of functions may be convergent at every point on I and yet may not be uniformly convergent on I. For example, consider the sequence $< f_n >$ defined by

$$f_n(x) = x^n \; (0 \leq x \leq 1).$$

We have $\lim_{n \to \infty} x^n = 0$ if $0 \leq x < 1$

and $\lim_{n \to \infty} x^n = 1$ if $x = 1$.

Thus the limit function f is defined as follows :

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Since the function f_n has a definite limit for every value of x in $[0, 1]$ as $n \to \infty$ so the sequence $< f_n(x) >$ converges for every $x \in [0, 1]$.

Now we shall check whether the convergence is uniform or not. For this we take the interval $[0, 1[$. Let $\varepsilon > 0$ be given. Then

$$|f_n(x) - f(x)| < \varepsilon \Rightarrow |x^n - 0| < \varepsilon \Rightarrow x^n < \varepsilon \Rightarrow \left(\frac{1}{x}\right)^n > \frac{1}{\varepsilon}$$

$$\Rightarrow \quad n \log \frac{1}{x} > \log \frac{1}{\varepsilon} \Rightarrow n > \frac{\log (1/\varepsilon)}{\log (1/x)} \qquad \ldots(1)$$

Thus $m(x, \varepsilon)$ is an integer just greater than $\log (1/\varepsilon) / \log (1/x)$ for $x \neq 1$. In particular, when $x = 0$, $m(x, \varepsilon) = 1$.

Now from (1), we see that $n \to \infty$ as x, starting from 0, increases and approaches 1 and hence it is not possible to find a positive integer m such that

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq m \text{ and } \forall \; x \in [0, 1[.$$

Consequently $< f_n >$ is not uniformly convergent in $[0, 1[$.

However, if we take the interval $[0, k]$, where $0 < k < 1$, we find that the greatest value of $\log (1/\varepsilon) / \log (1/x)$ is $\log (1/\varepsilon) / \log (1/k)$ so that taking m equal to any positive integer greater than this greatest value, we get

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq m \text{ and } \forall \; x \in [0, k].$$

Hence $< f_n(x) >$ converges uniformly on $[0, k]$.

Point of non-uniform convergence of the sequence.

It is a point, scuh that the sequence does not converge uniformly in any neighbourhood of it, however small.

In the above example, the point $x = 1$ is such a point. (Avadh 2009)

Uniform convergence of a series of functions.

Definition: *Let* $\sum_{n=1}^{\infty} u_n(x)$ *be a series of functions defined on the interval* I *and let*

$$f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) \text{ for } n \in \mathbb{N}.$$

The series Σu_n *is said to converge uniformly on* I *if the sequence* $< f_n >$ *converges uniformly on* I.

3.3 Cauchy's General Principle of Uniform Convergence or A Necessary and Sufficient Condition for Uniform Convergence

It helps us in proving the uniform convergence of $< f_n >$ without having any idea about the limit function f.

Theorem: *Let* $< f_n >$ *be a sequence of real-valued functions defined on an interval* I. *Then* $< f_n >$ *converges uniformly on* I *if and only if for every* $\varepsilon > 0$, *there exists a positive integer* m *such that*

$$|f_n(x) - f_p(x)| < \varepsilon \text{ for all } n, p \geq m \text{ and } \forall\, x \in I. \quad \ldots(1)$$

(Rohilkhand 2007)

Proof: The only if part (Necessary part). Let the sequence $< f_n >$ converge uniformly to f on I. By definition, given $\varepsilon > 0$, there exists a positive integer m such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2} \; \forall\, n \geq m, \text{ and } \forall\, x \in I.$$

Hence, if $n, p \in \mathbb{N}$ are such that $n, p \geq m$ then $\forall\, x \in I$, we have

$$\begin{aligned} |f_n(x) - f_p(x)| &= |f_n(x) - f(x) + f(x) - f_p(x)| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_p(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus (1) holds for this m.

The if part (Sufficient part): Let $< f_n >$ be any sequence of functions on I such that, given $\varepsilon > 0$, there exists a positive integer m such that the result (1) holds. Now we shall show the existence of a function f on I such that the sequence $< f_n >$ converges uniformly to f on I.

We see, from (1) that for each fixed $x \in I$, the sequence $< f_n(x) >$ of real numbers is a Cauchy sequence. Since every Cauchy sequence of real numbers is convergent so $\lim_{n \to \infty} f_n(x)$ exists for every $x \in I$.

Let us define a function f by $f(x) = \lim_{n \to \infty} f_n(x) \; \forall \, x \in I$

Keeping p fixed in (1) and letting $n \to \infty$, we have

$$|f(x) - f_p(x)| < \varepsilon \quad \forall \quad p \geq m \text{ and } \forall \quad x \in I.$$

$\therefore$ $< f_n(x) >$ converges uniformly to f on I.

Note: The above theorem can be also stated as below :

A sequence $< f_n >$ is uniformly convergent on I if given $\varepsilon > 0$ there exists a positive integer m such that

$$|f_{n+p}(x) - f_n(x)| < \varepsilon \; \forall \quad n \geq m, \; \forall \, x \in I \text{ and } \forall \, p \in \mathbb{N}.$$

Cor: *A series $\Sigma \, u_n(x)$ will converge uniformly on I iff for every $\varepsilon > 0$, there exists a positive integer m such that*

$$|u_{n+1}(x) + u_{n+2}(x) + \ldots + u_{n+p}(x)| < \varepsilon$$

for all $n \geq m$, for all $x \in I$, and for all $p = 1, 2, \ldots .$

This result follows by the above theorem since the uniform convergence of a series $\Sigma \, u_n(x)$ depends upon the uniform convergence of the sequence $< f_n >$ where

$$f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x).$$

3.4 Tests for Uniform Convergence

Theorem 1: (M_n-test). *Let $< f_n >$ be a sequence of functions defined on an interval I.*

Let $\lim_{n \to \infty} f_n(x) = f(x)$ *for all* $x \in I$.

Set $M_n = \sup \{ |f_n(x) - f(x)| : x \in I \}$.

Then $< f_n >$ converges uniformly to f if and only if $M_n \to 0$ as $n \to \infty$.

Proof: The only if part: Let $< f_n >$ converge uniformly to f on I. Then for every $\varepsilon > 0$, there exists a positive integer m, independent of x, such that

$$|f_n(x) - f(x)| < \varepsilon \; \forall \quad n \geq m, \forall \, x \in I.$$

Since $M_n = \sup \{ |f_n(x) - f(x)| : x \in I \}$, we have

$M_n \leq \varepsilon$ for all $n \geq m$.

$\therefore$ $M_n \to 0$ as $n \to \infty$.

The if part: Let $M_n \to 0$ as $n \to \infty$. Then for given $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $M_n < \varepsilon \; \forall \, n \geq m$.

Since $M_n = \sup \{ |f_n(x) - f(x)| : x \in I \}$,

$\therefore$ $|f_n(x) - f(x)| \leq M_n < \varepsilon \quad \forall \, n \geq m$ and $\forall \, x \in I$

i.e., $|f_n(x) - f(x)| < \varepsilon \; \forall \quad n \geq m$ and $\forall \, x \in I$.

$\therefore$ $< f_n >$ converges uniformly to f on I.

Note: The M_n-test is a very powerful test. Therefore reader is advised to use it whenever it is applicable.

Theorem 2: (Weierstrass's M-test). *A series* $\sum_{n=1}^{\infty} u_n(x)$ *of functions will converge uniformly on* I *if there exists a convergent series* $\sum_{n=1}^{\infty} M_n$ *of positive constants such that*

$$|u_n(x)| \leq M_n$$

for all n *and for all* $x \in I$. (Rohilkhand 2007; Kanpur 09)

Proof: Since ΣM_n is convergent, by Cauchy's principle, for given $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$M_{n+1} + M_{n+2} + \ldots + M_{n+p} < \varepsilon \quad \ldots(1)$$

for all $n \geq m$, $p = 1, 2, \ldots$.

By hypothesis $|u_n(x)| \leq M_n \ \forall\, n \in \mathbb{N}$ and $\forall\, x \in I$. $\ldots(2)$

From (1) and (2) we conclude that

$$\begin{aligned} &|u_{n+1}(x) + u_{n+2}(x) + \ldots + u_{n+p}(x)| \\ &\leq |u_{n+1}(x)| + |u_{n+2}(x)| + \ldots + |u_{n+p}(x)| \\ &\leq M_{n+1} + M_{n+2} + \ldots + M_{n+p} < \varepsilon \end{aligned}$$

for all $n \geq m$ and for all $x \in I$.

$\therefore$ $\Sigma u_n(x)$ converges uniformly and absolutely on I.

Note: It can be easily seen that the same proof would hold if M_n were a function of x and if the series $\Sigma M_n(x)$ were uniformly convergent on I.

In the next two results we shall use Abel's Lemma which is stated below :

Abel's Lemma: *If the sequence* $< v_n >$ *of positive terms is monotonic decreasing and numbers* $u_1, u_2, \ldots, u_n$ *and* k_1, k_2 *are such that*

$$k_1 < u_1 + u_2 + \ldots + u_r < k_2 \quad \text{for } 1 \leq r \leq n$$

then $$k_1 v_1 \leq \sum_{n=1}^{\infty} u_r v_r \leq k_2 v_1.$$

Theorem 3: (Abel's Test). *The series* $\Sigma u_n(x)\, v_n(x)$ *will converge uniformly on* $[a, b]$ *if*

(*i*) $\Sigma u_n(x)$ *is uniformly convergent on* $[a, b]$

(*ii*) *the sequence* $< v_n(x) >$ *is monotonic for every* x *in* $[a, b]$

(*iii*) $< v_n(x) >$ *is uniformly bounded in* $[a, b]$ *i.e., there is a positive number* k, *independent of* x *and* n, *such that* $|v_n(x)| < k$ *for every value of* x *in* $[a, b]$ *and every positive integer* n.

Proof: Let $f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$, $n \in \mathbb{N}$.

$\Sigma u_n(x)$ is uniformly convergent on $[a, b]$

$\Rightarrow$ by Cauchy's principle for given $\varepsilon > 0$ there exists a positive integer m such that

$$|f_q(x) - f_p(x)| < \frac{\varepsilon}{k} \quad \forall\, q, p \in \mathbb{N}, q > p > m \text{ and } \forall\, x \in [a, b]$$

$$\left| \sum_{n=p+1}^{q} u_n(x) \right| < \frac{\varepsilon}{k} \quad \forall\, q > p > m \text{ and } \forall\, x \in [a, b].$$

$\therefore$ ε / k is an upper bound of $\sum_{n=p+1}^{q} u_n(x)$.

By hypothesis, the sequence $< v_n(x) >$ is monotonic in $[a, b]$. Hence by Abel's lemma, we have

$$\left| \sum_{n=p+1}^{q} u_n(x)\, v_n(x) \right| \le \frac{\varepsilon}{k} v_{p+1}(x) < \frac{\varepsilon}{k} \cdot k = \varepsilon$$

$\forall\, p, q \in \mathbb{N}, q > p > m$ and $\forall\, x \in [a, b]$.

Consequently, by Cauchy's principle, $\Sigma\, u_n(x)\, v_n(x)$ is uniformly convergent on $[a, b]$.

Theorem 4: (Dirichlet's Test): *The sereis* $\Sigma\, u_n(x)\, v_n(x)$ *will be uniformly convergent on* $[a, b]$ *if*

(*i*) $< v_n(x) >$ *is a positive monotonic decreasing sequence converging uniformly to zero on* $[a, b]$

(*ii*) $|f_n(x)| = \left| \sum_{r=1}^{n} u_r(x) \right| < k,$

for every value of x in $[a, b]$ and for all integral values of n, where k is a fixed number independent of x.

Proof: We have $|f_n(x)| < k\ \forall\, x \in [a, b]$ and $\forall\, n \in \mathbb{N}$. (Avadh 2008)

$\therefore$ for all $x \in [a, b]$ and for all $p, q \in \mathbb{N}, q > p > m_1$, we get

$$|f_q(x) - f_p(x)| \le |f_q(x)| + |f_p(x)| < k + k = 2k$$

i.e. $$\left| \sum_{n=p+1}^{q} u_n(x) \right| < 2k \quad \forall\, x \in [a, b] \text{ and } \forall\, q > p > m_1.$$

$\therefore$ $2k$ is an upper bound of $\sum_{n=p+1}^{q} u_n(x)$.

Also sequence $< v_n(x) >$ is positive monotonic decreasing, so by Abel's lemma

$$\left| \sum_{n=p+1}^{q} u_n(x)\, v_n(x) \right| < 2k\, v_{p+1}(x). \qquad \ldots(1)$$

Again $< v_n(x) >$ converges uniformly to zero on $[a, b]$

$\Rightarrow$ given $\varepsilon > 0$ there exists $m_2 \in \mathbb{N}$ such that

$$|v_n(x)| < \frac{\varepsilon}{2k} \quad \forall\, n \ge m_2 \text{ and } \forall\, x \in [a, b]. \qquad \ldots(2)$$

Let $m = \max.\{m_1, m_2\}$. Then (1) and (2) hold for all $n > m$.

$$\therefore \quad \left| \sum_{n=p+1}^{q} u_n(x)\, v_n(x) \right| < 2k \cdot \frac{\varepsilon}{2k} = \varepsilon$$

for all $x \in [a, b]$, and for all $q > p > m$.

Hence $\Sigma\, u_n(x)\, v_n(x)$ is uniformly convergent on $[a, b]$.

Illustrative Examples

Example 1: *Let* $f_n(x) = \dfrac{nx}{1 + n^2 x^2}$ *when* $0 \le x \le 1$ *and* $n = 1, 2, 3, \ldots$. *Examine as to whether the sequence* $< f_n >$ *is uniformly convergent on* R. **(Avadh 2008; Rohilkhand 09, 12)**

Solution: We have

$$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{nx}{1 + n^2 x^2} = 0 \;\forall\, x \in \mathbf{R}.$$

Let, if possible, the sequence converge uniformly on R. Then for a given $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$|f_n(x) - f(x)| = \frac{n|x|}{1 + n^2 x^2} < \varepsilon \quad \forall \;\; n \ge m \text{ and } \forall\, x \in \mathbf{R} \qquad \ldots(1)$$

If we take $x = \dfrac{1}{n}$ $(n = 1, 2, 3, \ldots)$, then

$$|f_n(x) - f(x)| = \frac{n \cdot \frac{1}{n}}{1 + n^2 \cdot \frac{1}{n^2}} = \frac{1}{2}.$$

Hence, we note that for $\varepsilon = \dfrac{1}{2}$, there is no single m such that (1) holds simultaneously $\forall\, x \in \mathbf{R}$.

For if such an m existed, we would have

$$|f_m(x) - f(x)| < \frac{1}{2} \;\forall\, x \in \mathbf{R}.$$

But if $x = \dfrac{1}{m}$, this is a contradiction because we would have $\dfrac{1}{2} < \dfrac{1}{2}$ in that case. Thus convergence is non-uniform. In fact 0 is a point of non-uniform convergence.

Note: It can be easily seen that this sequence will converge uniformly on any interval which does not include 0.

Example 2: *The sum to n terms of a series is*

$$f_n(x) = \frac{n^2 x}{1 + n^4 x^2}.$$

Show that it converges non-uniformly in the interval $[0, 1]$.

Solution: We have $f(x) = \lim_{n \to \infty} \frac{n^2 x}{1 + n^4 x^2} = 0 \ \forall\, x \in [0, 1]$

$$\therefore \qquad |f_n(x) - f(x)| = \frac{n^2 |x|}{1 + n^4 x^2}.$$

Take $x = 1/n^2$ and proceed as in above example.

Another Method: M_n-test readily proves the non-uniform convergence. We have

$$M_n = \sup \{|f_n(x) - f(x)| : x \in \mathrm{R}\} = \sup \left\{ \frac{n^2 |x|}{1 + n^4 x^2} : x \in \mathrm{R} \right\}$$

$$\geq \frac{n^2 \cdot \frac{1}{n^2}}{1 + n^4 \cdot \frac{1}{n^4}} = \frac{1}{2} \left(\text{Taking } x = \frac{1}{n^2} \in \mathrm{R} \right).$$

Since M_n cannot tend to zero as $n \to \infty$ hence by M_n-test the sequence is non-uniformly convergent.

Example 3: *Show that* 0 *is a point of non-uniform convergence of the sequence* $< f_n >$ *where* $f_n(x) = 1 - (1 - x^2)^n$.

Solution: We have

$$f(x) = \lim_{n \to \infty} f_n(x) = \begin{cases} 0 & \text{when } x = 0 \\ 1. & \text{when } 0 < |x| < \sqrt{2} \end{cases}$$

$$\therefore \qquad M_n = \sup \{|f_n(x) - f(x)| : x \in]0, \sqrt{2}[\}$$

$$= \sup \{(1 - x^2)^n : x \in]0, \sqrt{2}[\}$$

$$\geq \left(1 - \frac{1}{n}\right)^n \qquad \left\{ \text{taking } x = \frac{1}{\sqrt{n}} \in]0, \sqrt{2}[\right\}$$

$$\to \frac{1}{e} \text{ as } n \to \infty.$$

Since M_n does not tend to zero as $n \to \infty$, the sequence is non-uniformly convergent. Also as $n \to \infty$, $x \to 0$ and hence 0 is a point of non-uniform convergence.

Example 4: *Prove that the exponential series*

$$\Sigma\, u_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

converges uniformly on every bounded subset of R.

Solution: Let $[-k, k]$ be a bounded subset of R, for $k > 0$. Let $\varepsilon > 0$ be given. Then there exists $m \in \mathrm{N}$ such that for any $p \in \mathrm{N}$ and $x \in [-k, k]$, we get

$$\left| \frac{x^n}{n!} \right| \leq \left| \frac{k^n}{n!} \right| < \frac{\varepsilon}{p} \ \forall\ n \geq m.$$

$\therefore$ for all $n \geq m$ and for all $p \in \mathbb{N}$, we have

$$\left| \frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+2)!} + \ldots + \frac{x^{n+p}}{(n+p)!} \right|$$

$$\leq \left| \frac{x^{n+1}}{(n+1)!} \right| + \ldots + \left| \frac{x^{n+p}}{(n+p)!} \right| < \frac{\varepsilon}{p} \cdot p = \varepsilon.$$

Thus for any $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that for any $p \in \mathbb{N}$ and for all $x \in [-k, k]$, we get

$$|f_{n+p} - f_n| < \varepsilon \text{ where } f_n = 1 + \frac{x}{1!} + \ldots + \frac{x^n}{n!}.$$

Hence, by Cauchy's principle $\Sigma\, u_n(x)$ converges uniformly on $[-k, k]$.

Since $k > 0$ is arbitrary, $\Sigma\, u_n(x)$ converges uniformly on any bounded subset of R.

Note: This series is not uniformly convergent on R. Since $\forall\ m \in \mathbb{N}$ there exists $x \in \mathbb{R}$ such that $\frac{x^m}{m!} > 1$ and hence for any $\varepsilon > 0$,

$$1 < \frac{x^m}{m!} < \varepsilon \text{ is not true.}$$

Example 5: *Show that the series* $\cos x + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots$ *converges uniformly on* R.

Solution: Here, we have

$$\left| \frac{\cos nx}{n^2} \right| \leq \frac{1}{n^2} \ \forall\, x \in \mathbb{R}.$$

But we know that the series $\Sigma \frac{1}{n^2}$ is convergent.

Hence by Weierstrass's *M*-test the given series is uniformly convergent on R.

Example 6: *Prove that* $\Sigma\, a_n\, n^{-x}$ *is uniformly convergent on* $[0, 1]$ *if* $\Sigma\, a_n$ *converges uniformly on* $[0, 1]$.

Solution: Take $v_n(x) = n^{-x} = \frac{1}{n^x}$ and $u_n(x) = a_n$.

The sequence $< n^{-x} >$ is monotonic decreasing on $[0, 1]$.

Since $< n^{-x} >$ decreases on $[0, 1]$,

$\therefore$ $\frac{1}{n^x} \leq \frac{1}{n^0} = 1\ \forall\, n \in \mathbb{N}$ and $\forall\, x \in [0, 1]$.

$\therefore$ $|v_n(x)| = |n^{-x}| \leq 1\ \forall\, n \in \mathbb{N}$ and $\forall\, x \in [0, 1]$.

Thus $< v_n(x) >$ is uniformly bounded and monotonic decreasing sequence on $[0, 1]$.

Also $\Sigma\, u_n(x) = \Sigma\, a_n$ is uniformly convergent on $[0, 1]$.

Hence by Abel's test, $\Sigma\, u_n(x)\, v_n(x) = \Sigma\, a_n\, n^{-x}$ is uniformly convergent on $[0, 1]$.

Example 7: *Test for uniform convergence the series*

$$\sum_{n=0}^{\infty} xe^{-nx} \text{ in the closed interval } [0, 1].$$

(Rohilkhand 2012)

Solution: Here $f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$

$$= \sum_{n=0}^{n-1} xe^{-nx} = \frac{x(1 - 1/e^{nx})}{1 - 1/e^{x}} = \frac{xe^{x}}{e^{x} - 1}\left(1 - \frac{1}{e^{nx}}\right).$$

$$\therefore \quad f(x) = \lim_{n \to \infty} f_n(x) = \begin{cases} 0 & \text{when } x = 0 \\ \dfrac{xe^{x}}{e^{x} - 1} & \text{when } 0 < x \leq 1. \end{cases}$$

Let us consider $0 < x \leq 1$. We have

$$M_n = \sup \{ |f_n(x) - f(x)| : x \in]0, 1] \}$$

$$= \sup \left\{ \frac{xe^{x}}{(e^{x} - 1)\, e^{nx}} : x \in]0, 1] \right\}$$

$$\geq \frac{(1/n) \cdot e^{1/n}}{(e^{1/n} - 1)\, e} \qquad \left(\text{Taking } x = \frac{1}{n} \in]0, 1]\right).$$

Now $$\lim_{n \to \infty} \frac{(1/n) \cdot e^{1/n}}{e(e^{1/n} - 1)} \qquad \left(\text{Form } \frac{0}{0}\right)$$

$$= \lim_{n \to \infty} \frac{(1/n) \cdot e^{1/n}(-1/n^2) + (-1/n^2)\, e^{1/n}}{e \cdot e^{1/n}(-1/n^2)}$$

$$= \lim_{n \to \infty} \frac{\{(1/n) + 1\}}{e} = \frac{1}{e}.$$

Hence M_n cannot tend to zero as $n \to \infty$ and consequently the sequence is non-uniformly convergent by M_n-test. Here 0 is a point of non-uniform convergence.

Example 8: *Test for uniform convergence the series*

(*i*) $\Sigma \dfrac{x}{(n + x^2)^2}$ **(Purvanchal 2008)** (*ii*) $\Sigma \dfrac{x}{n(1 + nx^2)}$.

Solution: (i) Here $u_n(x) = \dfrac{x}{(n + x^2)^2}$.

For maximum or minimum of $u_n(x)$, $\dfrac{du_n(x)}{dx} = 0$

or $(n + x^2)^2 - 4x^2(n + x^2) = 0$ or $3x^4 + 2nx^2 - n^2 = 0$

or $x^2 = \dfrac{n}{3}$, *i.e.*, $x = \sqrt{\left(\dfrac{n}{3}\right)}$.

It can be easily seen that $\dfrac{d^2 u_n(x)}{dx^2}$ is – ive when $x = \sqrt{\left(\dfrac{n}{3}\right)}$.

$$\therefore \quad M_n = \max.\, u_n(x) = \frac{\sqrt{(n/3)}}{\left(n+\frac{n}{3}\right)^2} = \frac{3\sqrt{3}}{16n^{3/2}}.$$

Thus $|u_n(x)| \le M_n$. But ΣM_n is convergent.

Hence, by Weierstrass's M-test, the given series is uniformly convergent for all values of x.

(ii) Here $u_n(x) = \dfrac{x}{n(1+nx^2)}$ is max. or min. when

$$n(1+nx^2) - 2n^2x^2 = 0 \quad \text{or} \quad x = \pm 1/\sqrt{n}.$$

It can be easily seen that $u_n(x)$ is maximum for $x = 1/\sqrt{n}$.

$$\therefore \quad M_n = \max.\, u_n(x) = \frac{1/\sqrt{n}}{n(1+1)} = \frac{1}{2n^{3/2}}.$$

But ΣM_n is convergent. Hence, by Weierstrass's M-test, the given series is uniformly convergent for all values of x.

Comprehensive Exercise 1

1. Prove that the sequence $< f_n >$ defined by $f_n(x) = \dfrac{x}{1+nx}$ $(0 \le x < \infty)$ is uniformly convergent to 0 on $[0, \infty[$.
2. Show that the sequence $< f_n >$ where
$$f_n(x) = nx(1-x)^n$$
does not converge uniformly on $[0, 1]$.
3. Show that 0 is a point of non-uniform convergence of the sequence $< f_n(x) >$, where $f_n(x) = nxe^{-nx^2}$, $x \in \mathbb{R}$. **(Rohilkhand 2007)**
4. Show that the sequence $< f_n >$ where $f_n(x) = x^{n-1}(1-x)$ converges uniformly in the interval $[0, 1]$.
5. If $\sum_{n=0}^{\infty} a_n$ converges absolutely prove that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[0, 1]$.
6. Show that $\sum_{n=1}^{\infty} \dfrac{1}{n^2+x^2}$ $(0 \le x < \infty)$ is uniformly convergent on $[0, \infty[$.
7. Prove that $\sum_{n=1}^{\infty} \dfrac{x}{n(n+1)}$ is uniformly convergent on $]0, k[$, $k > 0$ but not on $]0, \infty[$.

8. Prove that if δ is any fixed positive number less than unity, the series $\Sigma \frac{x^n}{n+1}$ is uniformly convergent in $[-\delta, \delta]$.

9. Show that the series $\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots$ converges uniformly in $0 < a \le x \le b < 2\pi$.

10. Show that the series
$$\frac{x^2}{1+x} + \left(\frac{2x^2}{1+2x} - \frac{x^2}{1+x}\right) + \ldots + \left(\frac{nx^2}{1+nx} - \frac{(n-1)x^2}{1+(n-1)x}\right) + \ldots$$
converges uniformly on $[0, 1]$.

11. Consider the series $\Sigma \frac{(-1)^{n-1}}{n} x^n$ for uniform convergence in $[0, 1]$.

12. Consider the series $\Sigma \frac{(-1)^{n-1}}{(n+x^2)}$ for uniform convergence for all values of x.

13. Show that the series $\sum_{n=1}^{\infty} (-1)^{n-1} x^n$ converges uniformly in $0 \le x \le k < 1$.

14. Show that the series $\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$ converges uniformly in $[1, \infty[$.

3.5 Uniform Convergence and Continuity

Let us consider the sequence $< f_n >$ defined on $[0, 1]$ where $f_n(x) = x^n$ $(0 \le x \le 1)$.

Since $f_n(x) = x^n \ \forall \ n \in \mathbb{N}$ is a polynomial function, f_n is continuous on $[0, 1]$ $\forall\, n \in \mathbb{N}$. The limit function f is defined by
$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x = 1. \end{cases}$$

The sequence $< f_n >$ converges pointwise to the function f which is discontinuous at $x = 1$ *i.e.* discontinuous on $[0, 1]$.

Hence we see that a sequence $< f_n >$ of continuous functions may converge pointwise to discontinuous function.

This is also true for a series of continuous functions $\Sigma u_n(x)$ converging pointwise to $f(x)$.

However, in the following theorem we prove that a sequence $< f_n >$ of continuous functions on $[a, b]$ converging uniformly to f on $[a, b]$ is such that the uniform limit funciton f is also continuous on $[a, b]$.

Theorem 1: *Let $< f_n >$ be a sequence of real-valued functions on $[a, b]$ which converges uniformly to the function f on $[a, b]$. If each f_n $(n = 1, 2, 3, \ldots)$ is continuous on $[a, b]$, then f is also continuous on $[a, b]$.*

Proof: Let c be an arbitrary element of $[a, b]$. It is sufficient to prove that f is continuous at c. Since each f_n is continuous on $[a, b]$, it implies it is continuous at c. Now let $\varepsilon > 0$ be given. Since $< f_n >$ converges uniformly to f on $[a, b]$, there exists $m \in N$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3} \; \forall\, n \geq m \text{ and } \forall\, x \in [a, b].$$

In particular, we have

$$|f_m(x) - f(x)| < \frac{\varepsilon}{3} \qquad \ldots(1)$$

and

$$|f_m(c) - f(c)| < \frac{\varepsilon}{3}. \qquad \ldots(2)$$

Also since f_m is continuous at c, there exists $\delta > 0$ such that

$$|f_m(x) - f_m(c)| < \frac{\varepsilon}{3} \text{ whenever } |x - c| < \delta. \qquad \ldots(3)$$

Hence if $|x - c| < \delta$, we get

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_m(x) + f_m(x) - f_m(c) + f_m(c) - f(c)| \\ &\leq |f(x) - f_m(x)| + |f_m(x) - f_m(c)| + |f_m(c) - f(c)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \text{ by (1), (2) and (3).} \end{aligned}$$

Thus for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon \text{ whenever } |x - c| < \delta.$$

It follows that f is continuous at c, which proves the theorem.

Cor. *Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of real valued continuous functions defined on $[a, b]$. If the series converges uniformly to the function f on $[a, b]$, then f is continuous on $[a, b]$. In other words, the sum function of a uniformly convergent series of continuous functions is itself continuous.*

Proof: We know that the uniform convergence of the series $\sum_{n=1}^{\infty} u_n(x)$ is the same thing as the uniform convergence of the sequence $< f_n(x) >$ where

$$f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x).$$

Also the sum $f_m(x)$ of the finite number of continuous functions $u_1(x), u_2(x), \ldots, u_m(x)$ is continuous. For the rest of the proof we can proceed as in the above theorem.

Note 1: The reader should note that uniform convergence is only a sufficient but not a necessary condition for the continuity of the sum function *i.e.*, if the sum

function is continuous on $[a, b]$, it is not necessary that the series is uniformly convergent on $[a, b]$. For example, let us consider the series

$$\sum_{n=1}^{\infty} \left[\frac{n^2 x}{1 + n^3 x^2} - \frac{(n-1)^2 x}{1 + (n-1)^3 x^2} \right] \text{ defined on } [0, 1].$$

Here, we have $f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$

$$= \left(\frac{x}{1 + x^2} - 0 \right) + \left(\frac{2^2 x}{1 + 2^3 x^2} - \frac{x}{1 + x^2} \right) + \ldots$$

$$+ \left(\frac{n^2 x}{1 + n^3 x^2} - \frac{(n-1)^2 x}{1 + (n-1)^3 x^2} \right) = \frac{n^2 x}{1 + n^3 x^2}.$$

$$\therefore \quad f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{n^2 x}{1 + n^3 x^2} = 0 \quad \forall \ x.$$

Hence $f(x)$ is continuous on $[0, 1]$.

Now $M_n = \sup \{ | f_n(x) - f(x) : x \in [0, 1] \}$

$$= \sup \left\{ \frac{n^2 x}{1 + n^3 x^2} : x \in [0, 1] \right\}$$

$$\geq \frac{n^2 \cdot \frac{1}{n^{3/2}}}{1 + n^3 \cdot \frac{1}{n^3}} = \frac{\sqrt{n}}{2} \qquad \left(\text{taking } x = \frac{1}{n^{3/2}} \right)$$

$$\to \infty \text{ as } n \to \infty.$$

Thus M_n cannot tend to zero as $n \to \infty$. Hence the series is non-uniformly convergent on $[0, 1]$ by M_n-test. Here 0 is a point of non-uniform convergence.

Note 2: From the above theorem we also conclude that if the series of continuous functions defined on $[a, b]$ has discontinuous sum, it cannot be uniformly convergent on a subset A of $[a, b]$ which contains a point of discontinuity. For, if the series were uniformly convergent, we have seen above that its sum must be continuous on the domain of uniform convergence.

Consequently the theorem provides a very good negative test for uniform convergence.

For example, consider the series

$$\sum_{n=0}^{\infty} x^n (1 - x) \text{ in } [0, 1]. \quad \text{Here} \quad f_n(x) = 1 - x^n.$$

$$\therefore \quad f(x) = \begin{cases} 0, & \text{if } x = 1 \\ 1, & \text{if } 0 \leq x < 1. \end{cases}$$

For each value of n, $x^n (1 - x)$ is continuous on $[0, 1]$ although $f(x)$ is discontinuous at 1. It gives that the series cannot be uniformly convergent on $[0, 1]$.

Illustrative Examples

Example 9: *Test for uniform convergence and continuity of the sum function of the series for which*

(*i*) $f_n(x) = \dfrac{1}{1+nx}$ $(0 \le x \le 1)$, (*ii*) $f_n(x) = nx(1-x)^n$ $(0 \le x \le 1)$.

Solution: (i) We have $f_n(x) = \dfrac{1}{1+nx}$.

$$\therefore \quad f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0, & \text{if } 0 < x \le 1 \\ 1, & \text{if } x = 0. \end{cases}$$

The sum function f is discontinuous at 0. Hence the series will be non-uniformly convergent in $[0, 1]$.

By the previous methods it can be verified that zero is a point of non-uniform convergence.

(ii) Here $f_n(x) = nx(1-x)^n, (0 \le x \le 1)$.

If $0 < x < 1$, $\lim\limits_{n\to\infty} nx(1-x)^n$ [Form: $\infty \times 0$]

$$= \lim_{n\to\infty} \frac{nx}{(1-x)^{-n}} \qquad \left[\text{Form } \frac{\infty}{\infty}\right]$$

$$= \lim_{n\to\infty} \frac{x}{-(1-x)^{-n}\log(1-x)}$$

$$= \lim_{n\to\infty} -\frac{x(1-x)^n}{\log(1-x)} = 0.$$

Also $f_n(x) = 0$ if $x = 0$ or 1.

Hence $f(x) = \lim\limits_{n\to\infty} f_n(x) = 0 \quad \forall x \in [0, 1]$.

The sum function $f(x)$ is continuous for all $x \in [0, 1]$.

However the sequence $\langle f_n(x) \rangle$ is not uniformly convergent on $[0, 1]$.

Example 10: *Test the series* $\Sigma\, xe^{-nx}$ *for uniform convergence and continuity of its sum function near* $x = 0$.

Solution: We have $f_n(x) = x\dfrac{1-e^{-nx}}{1-e^{-x}}$.

$$\therefore \quad f(x) = \begin{cases} \lim\limits_{n\to\infty} f_n(x) = \dfrac{x}{1-e^{-x}}, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Now $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x}{1 - e^{-x}}$

$$= \lim_{x \to 0} \frac{x}{1 - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right)}$$

$$= \lim_{x \to 0} \frac{1}{1 - \frac{x}{2!} + \ldots} = 1.$$

Also $f(0) = 0$.

Since $\lim_{x \to 0} f(x) \neq f(0)$, the function $f(x)$ is not continuous at $x = 0$. It implies that the series does not converge uniformly in any interval containing 0.

3.6 Uniform Convergence and Integration

Theorem 1: *Let $< f_n >$ be a sequence of real valued functions defined on the closed and bounded interval $[a, b]$ and let $f_n \in R[a, b]$, for $n = 1, 2, 3, \ldots$. If $< f_n >$ converges uniformly to the function f on $[a, b]$, then $f \in R[a, b]$ and*

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \int_a^b f_n(x)\, dx.$$

Proof: Let $\varepsilon > 0$ be given.

The sequence $< f_n >$ converges uniformly to f on $[a, b]$

$\Rightarrow$ there exists $m \in N$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3(b-a)} \quad \forall\, n \geq m \text{ and } \forall\, x \in [a, b]. \qquad \ldots(1)$$

In particular, for $n = m$, $|f_m(x) - f(x)| < \frac{\varepsilon}{3(b-a)}$

i.e., $$f_m(x) - \frac{\varepsilon}{3(b-a)} < f(x) < f_m(x) + \frac{\varepsilon}{3(b-a)}. \qquad \ldots(2)$$

Since $f_m \in R[a, b]$, there exists a partition

$$P = \{a = x_0, x_1, x_2, \ldots, x_n = b\} \text{ of } [a, b] \text{ such that}$$

$$U(P, f_m) - L(P, f_m) < \frac{1}{3}\varepsilon \qquad \ldots(3)$$

Let $m_r^{(m)}$, $M_r^{(m)}$ and m_r, M_r be the infima and suprema of f_m and f respectively on $[x_{r-1}, x_r]$.

Now from (2), we get

$$f_m(x) < f(x) + \frac{\varepsilon}{3(b-a)} \quad \forall\, x \in [a, b].$$

In particular, $f_m(x) < f(x) + \dfrac{\varepsilon}{3(b-a)} \; \forall x \in [x_{r-1}, x_r]$.

It follows that $m_r^{(m)} \leq m_r + \dfrac{\varepsilon}{3(b-a)}$

$$\Rightarrow \quad \sum_{r=1}^{n} m_r^{(m)} \Delta x_r \leq \sum_{r=1}^{n} m_r \Delta x_r + \frac{\varepsilon}{3(b-a)} \sum_{r=1}^{n} \Delta x_r$$

$$\Rightarrow \quad L(P, f_m) \leq L(P, f) + \frac{1}{3}\varepsilon. \qquad \ldots(4)$$

Again from (2), we also have

$$f(x) < f_m(x) + \frac{\varepsilon}{3(b-a)} \quad \forall x \in [x_{r-1}, x_r].$$

It follows that $M_r \leq M_r^{(m)} + \dfrac{\varepsilon}{3(b-a)}$

$$\Rightarrow \quad \sum_{r=1}^{n} M_r \Delta x_r \leq \sum_{r=1}^{n} M_r^{(m)} \Delta x_r + \frac{\varepsilon}{3(b-a)} \sum_{r=1}^{n} \Delta x_r$$

$$\Rightarrow \quad U(P, f) \leq U(P, f_m) + \frac{1}{3}\varepsilon. \qquad \ldots(5)$$

Adding (4) and (5), we have

$$U(P, f) + L(P, f_m) \leq L(P, f) + U(P, f_m) + \frac{2\varepsilon}{3}$$

or
$$U(P, f) - L(P, f) \leq U(P, f_m) - L(P, f_m) + \frac{2\varepsilon}{3}$$

$$< \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon, \text{ using (3).}$$

Thus for a given $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$.

Hence $f \in R[a, b]$. This proves the first result.

To prove the second result, we have for all $n \geq m$,

$$\left| \int_a^b f_n - \int_a^b f \right| = \left| \int_a^b (f_n - f) \right| \leq \int_a^b |f_n - f|$$

$$< \int_a^b \frac{\varepsilon}{3(b-a)} dx, \text{ using (1)}$$

$$= \frac{\varepsilon}{3(b-a)} \int_a^b dx = \frac{\varepsilon}{3(b-a)} (b-a) = \frac{\varepsilon}{3}.$$

It follows that $\lim_{n \to \infty} \left[\int_a^b f_n(x)\, dx \right] = \int_a^b f(x)\, dx$.

This completes the proof of the theorem.

Corollary 1: *Let $< f_n >$ be a sequence of real valued continuous functions defined on $[a, b]$ such that $f_n \to f$ uniformly on $[a, b]$. Then $f \in R[a, b]$ and*

$$\lim_{n \to \infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Proof: Since every continuous function is Riemann integrable so $f_n \in R[a, b]$ for $n = 1, 2, 3, \ldots$ and hence the symbol $\int_a^b f_n(x)\,dx$ is meaningful.

Again if the sequence $< f_n >$ converges uniformly to f on $[a, b]$ and f_n is continuous on $[a, b]$ for each $n \in N$ then (by theorem 1 of article 3.5) f is continuous on $[a, b]$ and hence $f \in R[a, b]$. Thus the first result is proved. For the second result the proof is the same as that given in the above theorem.

Term by term integration

Corollary 2: *Let* $\sum_{n=1}^{\infty} u_n(x)$ *be a series of real valued functions defined on* $[a, b]$ *such that* $u_n(x) \in R[a, b]$, *for* $n = 1, 2, 3, \ldots$ *If the series converges uniformly to* f *on* $[a, b]$, *then*

$$f \in R[a, b]$$

and

$$\int_a^b \left[\sum_{n=1}^{\infty} u_n(x)\right] dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx.$$

Proof: Let $u_1(x) + u_2(x) + \ldots + u_n(x) = f_n(x)$. Then $f_n \in R[a, b]$ for each fixed n because the sum of a finite number of R-integrable functions is R-integrable. Also we know that the uniform convergence of the series $\Sigma u_n(x)$ is the same thing as the uniform convergence of the sequence $< f_n >$ so that $< f_n >$ converges uniformly to f on $[a, b]$. Hence $f \in R[a, b]$, by the above theorem.

Now $$\int_a^b \left[\sum_{n=1}^{\infty} u_n(x)\right] dx = \int_a^b f(x)\,dx = \lim_{n \to \infty} \int_a^b f_n(x)\,dx$$

$$= \lim_{n \to \infty} \int_a^b \left[\sum_{m=1}^{n} u_m(x)\right] dx$$

$$= \lim_{n \to \infty} \sum_{m=1}^{n} \int_a^b u_m(x)\,dx$$

$$= \sum_{n=1}^{\infty} \left[\int_a^b u_n(x)\,dx\right].$$

Note: The condition of uniform convergence of the series $\Sigma u_n(x)$ is only sufficient but not necessary for the validity of term by term integration. For example, consider the series

$$\sum_{n=1}^{\infty} u_n(x), \quad \text{where} \quad u_n(x) = \frac{nx}{1 + n^2 x^2} - \frac{(n-1)x}{1 + (n-1)^2 x^2}.$$

We have $$f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) = \frac{nx}{1 + n^2 x^2}.$$

$\therefore$ $$f(x) = \lim_{n \to \infty} f_n(x) = 0 \text{ for all } x.$$

Now $\int_0^1 \left[\sum_{n=1}^{\infty} u_n(x) \right] dx = \int_0^1 f(x)\, dx = \int_0^1 0\, dx = 0,$

and $$\sum_{n=1}^{\infty} \int_0^1 u_n(x)\, dx = \lim_{n \to \infty} \sum_{m=1}^{n} \int_0^1 u_m(x)\, dx$$

$$= \lim_{n \to \infty} \int_0^1 \left[\sum_{m=1}^{n} u_m(x) \right] dx$$

$$= \lim_{n \to \infty} \int_0^1 f_n(x)\, dx$$

$$= \lim_{n \to \infty} \int_0^1 \frac{nx}{1 + n^2 x^2}\, dx$$

$$= \lim_{n \to \infty} \frac{1}{2n} \log(1 + n^2) = 0.$$

Hence $\sum_{1}^{\infty} \int_0^1 u_n(x)\, dx = \int_0^1 \sum_{1}^{\infty} u_n(x)\, dx$, although 0 is a point of non-uniform convergence of the series as we have seen earlier.

Illustrative Examples

Example 11: *Show that*

$$\int_0^1 \left(\sum_{1}^{\infty} \frac{x^n}{n^2} \right) dx = \sum_{1}^{\infty} \frac{1}{n^2 (n+1)}.$$

Solution: By Weierstrass's M-test, the series $\Sigma \frac{x^n}{n^2}$ is uniformly convergent for $0 \le x \le 1$. Therfore it can be integrated term by term.

Hence $\int_0^1 \left(\sum_{1}^{\infty} \frac{x^n}{n^2} \right) dx = \sum_{1}^{\infty} \int_0^1 \frac{x^n}{n^2}\, dx = \sum_{1}^{\infty} \left[\frac{x^{n+1}}{(n+1)\, n^2} \right]_0^1 = \sum_{1}^{\infty} \frac{1}{n^2 (n+1)}.$

Example 12: *Examine for term by term integration the series the sum of whose first n terms is* $n^2 x (1-x)^n, (0 \le x \le 1)$.

Solution: Here $f_n(x) = n^2 x (1-x)^n$.

Obviously $f_n(x) = 0$ when $x = 0$ or 1.

When $0 < x < 1$, we have

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{n^2 x}{(1-x)^{-n}} \qquad \left[\text{Form } \frac{\infty}{\infty} \right]$$

$$= \lim_{n \to \infty} \frac{2nx}{-(1-x)^{-n} \log(1-x)} \qquad \left[\text{Form } \frac{\infty}{\infty} \right]$$

$$= \lim_{n \to \infty} \frac{2x}{(1-x)^{-n} [\log (1-x)]^2} = 0.$$

Hence $f(x) = \lim_{n \to \infty} f_n(x) = 0$ for $0 \le x \le 1$.

$\therefore \quad \int_0^1 f(x)\, dx = 0.$

But $\int_0^1 f_n(x)\, dx = \int_0^1 n^2 x (1-x)^n\, dx$

$$= n^2 \left[-\frac{x(1-x)^{n+1}}{n+1} - \frac{(1-x)^{n+2}}{(n+1)(n+2)} \right]_0^1$$

$$= \frac{n^2}{(n+1)(n+2)} \to 1 \text{ as } n \to \infty.$$

Consequently term by term integration over the interval $[0, 1]$ is not justified. In fact, 0 is a point of non-uniform convergence of the series *i.e.* the series is non-uniformly convergent in $0 \le x \le 1$.

For, let if possible, the series be uniformly convergent.

Then for a given $\varepsilon > 0$, we have

$$|f_n(x) - f(x)| = n^2 x (1-x)^n < \varepsilon \qquad \therefore (1)$$

for all values of $n \ge m$ and $\forall\ x \in [0, 1]$.

Taking $x = 1/n$, we get

$$|f_n(x) - f(x)| = n^2 \cdot \frac{1}{n} \left(1 - \frac{1}{n}\right)^n = n \left(1 - \frac{1}{n}\right)^n$$

$$\to \infty \text{ as } n \to \infty, \text{ i.e., as } x \to 0.$$

This contradicts (1). Hence the series does not converge uniformly in $[0, 1]$. Here 0 is a point of non-uniform convergence of the series.

Example 13: *Show that the series*

$$2xe^{-x^2} = \sum_1^\infty 2x \left[\frac{1}{n^2} e^{-x^2/n^2} - \frac{1}{(n+1)^2} e^{-x^2/(n+1)^2} \right]$$

can be integrated term by term between any two finite limits. Can the function defined by the series be integrated between the limits 0 and ∞? If so, what is the value of this integral given by integrating the series term by term between these limits ?

Solution: Here $u_n(x) = 2x \left[\frac{e^{-x^2/n^2}}{n^2} - \frac{e^{-x^2/(n+1)^2}}{(n+1)^2} \right]$.

$\therefore \quad f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$

$$= 2x\left[e^{-x^2} - \frac{e^{-x^2/2^2}}{2^2}\right] + 2x\left[\frac{e^{-x^2/2^2}}{2^2} - \frac{e^{-x^2/3^2}}{3^2}\right]$$

$$+ \ldots + 2x\left[\frac{e^{-x^2/n^2}}{n^2} - \frac{e^{-x^2/(n+1)^2}}{(n+1)^2}\right]$$

$$= 2x\left[e^{-x^2} - \frac{e^{-x^2/(n+1)^2}}{(n+1)^2}\right].$$

Then $f(x) = \lim_{n \to \infty} f_n(x) = 2xe^{-x^2}$ for all values of x.

Now $\int_a^b f(x)\,dx = \int_a^b 2xe^{-x^2}\,dx = e^{-a^2} - e^{-b^2}$

and $\int_a^b f_n(x)\,dx = e^{-a^2} - e^{-b^2} + \left[e^{-b^2/(n+1)^2} - e^{-a^2/(n+1)^2}\right]$

$\to e^{-a^2} - e^{-b^2}$ as $n \to \infty$.

Since $\int_a^b f(x)\,dx = \lim_{n \to \infty} \int_a^b f_n(x)\,dx,$

the series can be integrated term by term between any two finite limits.

Again $\int_0^\infty f(x)\,dx = \int_0^\infty 2xe^{-x^2}\,dx = 1$

and $\int_0^\infty f_n(x)\,dx = \left[-e^{-x^2} + e^{-x^2/(n+1)^2}\right]_0^\infty = 0.$

Since $\int_0^\infty f(x)\,dx \neq \lim_{n \to \infty} \int_0^\infty f_n(x)\,dx,$

the series cannot be integrated term by term between the limits 0 and ∞.

3.7 Uniform Convergence and Differentiation

Theorem 1: *Let $< f_n >$ be a sequence of real valued functions defined on $[a, b]$ which converges uniformly to the function f on $[a, b]$. Let c be any point of $[a, b]$ and suppose that*

$$\lim_{x \to c} f_n(x) = L_n \quad (n = 1, 2, 3, \ldots).$$

Then the sequence $< L_n >$ of real constants converges, and

$$\lim_{x \to c} f(x) = \lim_{n \to \infty} L_n.$$

In other words, the conclusion of the theorem is that

$$\lim_{x \to c} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to c} f_n(x).$$

Proof: Let $\varepsilon > 0$ be given. Since $< f_n >$ converges uniformly to f on $[a, b]$, there exists $m \in N$ such that

$$|f_n(x) - f_p(x)| < \varepsilon \text{ for all } n, p \geq m \text{ and } \forall x \in [a, b] \quad ...(1)$$

In (1), letting $x \to c$, we have $|L_n - L_p| < \varepsilon$ for all $n \geq m$, $p \geq m$. Hence by Cauchy's general principle of convergence for real sequences, the sequence $< L_n >$ converges. This proves the first assertion.

Let $\lim_{n \to \infty} L_n = L$.

Now since $< L_n >$ converges to L and $< f_n >$ converges uniformly to f on $[a, b]$, there exists a positive integer k such that

$$|L_k - L| < \frac{1}{3}\varepsilon \quad ...(2)$$

and

$$|f_k(x) - f(x)| < \frac{1}{3}\varepsilon, \quad ...(3)$$

for all $x \in [a, b]$.

Also since $\lim_{n \to c} f_k(x) = L_k$, there exists $\delta > 0$ such that

$$|f_k(x) - L_k| < \frac{1}{3}\varepsilon \text{ whenever } |x - c| < \delta. \quad ...(4)$$

Thus for all x satisfying $|x - c| < \delta$, we have

$$\begin{aligned} |f(x) - L| &= |f(x) - f_k(x) + f_k(x) - L_k + L_k - L| \\ &\leq |f(x) - f_k(x)| + |f_k(x) - L_k| + |L_k - L| \\ &< \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon, \text{ by (2), (3) and (4).} \end{aligned}$$

Hence for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ whenever } |x - c| < \delta.$$

By the definition of limit, it follows that

$$\lim_{x \to c} f(x) = L = \lim_{n \to \infty} L_n$$

or

$$\lim_{n \to c} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{n \to c} f_n(x).$$

The theorem is now completely established.

Corollary: *Let* $\sum_{n=1}^{\infty} u_n(x)$ *be a series of real valued functions defined on* $[a, b]$ *and let* $\lim_{x \to c} u_n(x)$ *exist* $(n = 1, 2, 3, \ldots)$ *where* c *is any point of* $[a, b]$. *If the series* $\Sigma\, u_n(x)$ *converges uniformly on* $[a, b]$, *then*

$$\lim_{x \to c} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \left[\lim_{x \to c} u_n(x) \right].$$

Proof: Let $f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$. Then

$$\lim_{x \to c} \sum_{n=1}^{\infty} u_n(x) = \lim_{x \to c} \lim_{n \to \infty} \sum_{m=1}^{n} u_m(x) = \lim_{x \to c} \lim_{n \to \infty} f_n(x) \quad \ldots(5)$$

and

$$\sum_{n=1}^{\infty} \left[\lim_{x \to c} u_n(x) \right]$$

$$= \lim_{n \to \infty} \sum_{m=1}^{n} \left[\lim_{x \to c} u_m(x) \right] = \lim_{n \to \infty} \lim_{x \to c} \sum_{m=1}^{n} u_m(x).$$

[$\because$ the limit of the sum of a finite number of functions is the sum of their limits]

$$= \lim_{n \to \infty} \lim_{x \to c} f_n(x). \quad \ldots(6)$$

Now the result follows from (5), (6) and the above theorem.

Theorem 2: *Let $< f_n >$ be a sequence of real valued functions defined on $[a, b]$ such that*

(i) f_n is differentiable on $[a, b]$ for $n = 1, 2, 3, \ldots$,

(ii) the sequence $< f_n(c) >$ converges for some point c of $[a, b]$

(iii) the sequence $< f_n >$ converges uniformly on $[a, b]$. Then the sequence $< f_n >$ converges uniformly to a differentiable limit f and

$$\lim_{n \to \infty} f_n'(x) = f'(x) \quad (a \le x \le b).$$

Proof: Let $\varepsilon > 0$ be given. Since $< f_n(c) >$ converges and $< f_n' >$ converges uniformly on $[a, b]$, there exists a positive integer m such that for all $n, p \ge m$,

$$|f_n(c) - f_p(c)| < \frac{1}{2}\varepsilon \quad \ldots(1)$$

and

$$|f_n'(x) - f_p'(x)| < \frac{\varepsilon}{2(b-a)} \quad (a \le x \le b). \quad \ldots(2)$$

Applying the mean value theorem of differential calculus to the function $f_n - f_p$ where $n, p \ge m$, for any x, y in $[a, b]$ there exists some ξ between x and y such that

$$\frac{(f_n - f_p)(x) - (f_n - f_p)(y)}{x - y} = (f_n - f_p)'(\xi).$$

$$\therefore \quad |f_n(x) - f_p(x) - f_n(y) + f_p(y)| = |x - y| |f_n'(\xi) - f_p'(\xi)|$$

$$< \frac{|x - y|\varepsilon}{2(b-a)}, \text{ using (2)} \quad \ldots(3)$$

$$< \frac{1}{2}\varepsilon \quad [\because |x - y| \le (b - a)] \quad \ldots(4)$$

for all $n, p \ge m$ and $\forall\, x, y \in [a, b]$.

Hence, for all $n, p \ge m$ and $\forall\, x \in [a, b]$,

$$|f_n(x) - f_p(x)| = |f_n(x) - f_p(x) - f_n(c) + f_p(c) + f_n(c) - f_p(c)|$$
$$\leq |f_n(x) - f_p(x) - f_n(c) + f_p(c)| + |f_n(c) - f_p(c)|$$
$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, \text{ using (1) and (4).}$$

Thus for a given $\varepsilon > 0$, there exists a positive integer m such that $|f_n(x) - f_p(x)| < \varepsilon$ for all $n, p \geq m$ and $\forall\, x \in [a, b]$.

It follows that the sequence $< f_n >$ converges uniformly on $[a, b]$, say to a function f,

i.e., $$f(x) = \lim_{n \to \infty} f_n(x), (a \leq x \leq b).$$

Thus the first result is proved.

Now for an arbitrary but for the moment a fixed $x \in [a, b]$, define

$$F_n(y) = \frac{f_n(y) - f_n(x)}{y - x}, \; F(y) = \frac{f(y) - f(x)}{y - x} \qquad \ldots(5)$$

for $a \leq y \leq b$, $y \neq x$. Then

$$\lim_{y \to x} F_n(y) = \lim_{y \to x} \frac{f_n(y) - f_n(x)}{y - x} = f_n'(x) \qquad \ldots(6)$$

for $n = 1, 2, 3, \ldots$.

Also for $n, p \geq m$, we have

$$|F_n(y) - F_p(y)| = \left| \frac{f_n(y) - f_n(x) + f_p(y) - f_p(x)}{y - x} \right|$$

$$< \frac{\varepsilon}{2(b - a)}, \text{ using (3).}$$

It implies that $< F_n >$ converges uniformly for $a \leq y \leq b$, $y \neq x$. Since $< f_n >$ converges to f, it follows from (5) that

$$\lim_{n \to \infty} F_n(y) = \lim_{n \to \infty} \frac{f_n(y) - f_n(x)}{y - x}$$

$$= \frac{f(y) - f(x)}{y - x} = F(y). \qquad \ldots(7)$$

Now applying theorem 1 of article 3.7 to $< F_n >$ with $L_n = f_n'(x)$, and using (6), (7), we get

$$\lim_{y \to x} F(y) = \lim_{n \to \infty} f_n'(x)$$

or $$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = \lim_{n \to \infty} f_n'(x), \text{ using (5)}$$

or $$f'(x) = \lim_{n \to \infty} f_n'(x), \forall\, x \in [a, b].$$

Thus the theorem is completely established.

Term by term differentiation.

Corollary: *Let* $\sum_{n=1}^{\infty} u_n(x)$ *be a series of real valued differentiable functions on* $[a, b]$ *such that* $\sum_{n=1}^{\infty} u_n(c)$ *converges for some point* c *of* $[a, b]$ *and* $\sum_{n=1}^{\infty} u_n'(x)$ *converges uniformly on* $[a, b]$. *Then the series* $\sum_{n=1}^{\infty} u_n(x)$ *converges uniformly on* $[a, b]$ *to a differentiable sum function* f *and*

$$f'(x) = \lim_{n \to \infty} \sum_{m=1}^{n} u_m'(x), \quad (a \le x \le b)$$

i.e.,

$$\frac{d}{dx}\left[\sum_{n=1}^{\infty} u_n(x)\right] = \sum_{n=1}^{\infty}\left[\frac{d}{dx} u_n(x)\right], \quad (a \le x \le b).$$

(Purvanchal 2009)

Proof: Let $f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$.

Since the differential coefficient of the sum of a finite number of differentiable functions is equal to the sum of the their differential coefficients, hence we have

$$f_n'(x) = u_1'(x) + u_2'(x) + \ldots + u_n'(x).$$

Thus the series $\sum_{n=1}^{\infty} u_n(x)$ and $\sum_{n=1}^{\infty} u_n'(x)$ are equivalent to the sequences $< f_n >$ and $< f_n' >$ respectively.

Now the proof is the same as in the above theorem.

Note: If in addition to the hypothesis of the above theorem, the continuity of f_n' is assumed then a much shorter proof based on theorem 1 of article 3.5, theorem 1 of article 3.6 and the fundamental theorem of integral calculus is given in the following theorem.

Theorem 3: *Let* $< f_n >$ *be a sequence of real valued functions defined on* $[a, b]$ *such that*

(*i*) f_n *is differentiable on* $[a, b]$ *for* $n = 1, 2, 3, \ldots$

(*ii*) *the sequence* $< f_n >$ *converges to* f *on* $[a, b]$,

(*iii*) *the sequence* $< f_n' >$ *converges uniformly to* g *on* $[a, b]$,

(*iv*) *each* f_n' *is continuous on* $[a, b]$.

Then $g(x) = f'(x) \quad (a \le x \le b)$

i.e., $\lim_{n \to \infty} f_n'(x) = f'(x), \quad (a \le x \le b).$

Proof: Since the sequence $< f_n' >$ of continuous functions converges uniformly to g on $[a, b]$, it follows by theorem 1 of article 3.5 that g is continuous on $[a, b]$. Also $< f_n' >$ converges uniformly to g on $[a, x]$, where x is any point of $[a, b]$. It follows, by theorem 1 of article 3.6, that

$$\lim_{n \to \infty} \int_a^x f_n'(t)\, dt = \int_a^x g(t)\, dt. \qquad \ldots(1)$$

By the fundamental theorem of integral calculus, we have

$$\int_a^x f_n'(t)\,dt = f_n(x) - f_n(a). \qquad \ldots(2)$$

But by hypothesis,

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ and } \lim_{n\to\infty} f_n(a) = f(a). \qquad \ldots(3)$$

Combining (1), (2) and (3), we get

$$f(x) - f(a) = \int_a^x g(t)\,dt, \qquad (a \le x \le b).$$

It implies that

$$f'(x) = g(x), \qquad (a \le x \le b)$$

or $$f'(x) = \lim_{n\to\infty} f_n'(x), \qquad (a \le x \le b).$$

Corollary: *Let* $\sum_{n=1}^{\infty} u_n(x)$ *be a series of functions defined on* $[a, b]$ *such that*

(*i*) $u_n(x)$ *is differentiable on* $[a, b]$ *for* $n = 1, 2, 3, \ldots$

(*ii*) *the series* $\sum_{n=1}^{\infty} u_n(x)$ *converges to* f *on* $[a, b]$

(*iii*) *the series* $\sum_{n=1}^{\infty} u_n'(x)$ *converges uniformly to* g *on* $[a, b]$

(*iv*) *each* u_n' *is continuous on* $[a, b]$.

Then $$f'(x) = g(x), \qquad (a \le x \le b)$$

i.e., $$f'(x) = \sum_{n=1}^{\infty} u_n'(x), \qquad (a \le x \le b)$$

Note: For term by term differentiation, the derived series must be uniformly convergent and the original series need only be simply convergent.

Illustrative Examples

Example 14: *Show that*

$$\lim_{x\to 1} \sum_{n=1}^{\infty} \frac{nx^2}{n^3 + x^3} = \sum_{n=1}^{\infty} \frac{n}{n^3 + 1}.$$

Solution: First we shall show that the series $\sum_{n=1}^{\infty} \frac{nx^2}{n^3 + x^3}$ is uniformly convergent on $[0, k]$ for any $k > 0$.

Let $$u_n(x) = \frac{1}{n^3 + x^3} \text{ and } v_n(x) = nx^2.$$

Then $$|u_n(x)| \le \frac{1}{n^3} \quad \forall\, x \in [0, k].$$

But $\Sigma \frac{1}{n^3}$ is a convergent series. Hence by Weierstrass's M-test, the series $\Sigma\, u_n\,(x)$ is uniformly convergent on $[0, k]$. Also, for every $x \in [0, k]$, the sequence $< v_n\,(x) >$ is monotonically increasing.

It follows (by Abel's test) that the series

$$\Sigma\, u_n\,(x)\, v_n\,(x) = \Sigma\, \frac{nx^2}{n^3 + x^3}$$

converges uniformly on $[0, k]$. Hence by the above corollary, we get

$$\lim_{x \to 1} \left(\sum_{n=1}^{\infty} \frac{nx^2}{n^3 + x^3} \right) = \sum_{n=1}^{\infty} \left(\lim_{x \to 1} \frac{nx^2}{n^3 + x^3} \right) = \sum_{n=1}^{\infty} \frac{n}{n^3 + 1}.$$

Example 15: *Show that the series* $\Sigma\, u_n\,(x)$ *for which*

$$f_n\,(x) = \frac{1}{2n^2} \log\,(1 + n^4 x^2)$$

can be differentiated term by term although the series $\Sigma\, u_n'\,(x)$ *does not converge uniformly.*

Solution: Here

$$f\,(x) = \lim_{n \to \infty} f_n\,(x) = \lim_{n \to \infty} \frac{\log\,(1 + n^4 x^2)}{2n^2} \qquad \left[\text{Form } \frac{\infty}{\infty}\right]$$

$$= \lim_{n \to \infty} \frac{\frac{4n^3 x^2}{1 + n^4 x^2}}{4n} = \lim_{n \to \infty} \frac{n^2 x^2}{1 + n^4 x^2} = 0 \text{ for } 0 \le x \le 1.$$

Hence $f'\,(x) = 0$.

Also, $$\lim_{n \to \infty} f_n'\,(x) = \lim_{n \to \infty} \frac{n^2 x}{1 + n^4 x^2} = 0 \text{ for } 0 \le x \le 1.$$

Since $$f'\,(x) = \lim_{n \to \infty} f_n'\,(x),$$

therefore, the given series can be differentiated term by term. But the series $\Sigma\, u_n'\,(x)$ does not converge uniformly in $[0, 1]$ since the sequence $< f_n'\,(x) >$ has 0 as a point of non-uniform convergence as we have seen earlier.

This example shows that term by term differentiation holds even though $\Sigma\, u_n'\,(x)$ is non-uniformly convergent.

Example 16: *Show that, if* $f\,(x) = \sum_{1}^{\infty} \frac{1}{n^3 + n^4 x^2}$, *then it has a differential coefficient equal to* $-2x \sum_{1}^{\infty} \frac{1}{n^2\,(1 + nx^2)^2}$ *for all values of* x.

Solution: Here $u_n\,(x) = \frac{1}{n^3 + n^4 x^2}$,

so th$_n$ $'(x) = -\dfrac{2x}{n^2\,(1+nx^2)^2}$.

Nc maximum when $\dfrac{d}{dx}\,u_n'(x) = 0$,

i.e $(1+nx^2)^2 - 4nx^2\,(1+nx^2) = 0$

o $1 - 3nx^2 = 0$ or $x = \pm 1/\sqrt{(3n)}$.

ily seen that $\dfrac{d^2}{dx^2}\,u_n'(x)$ is – ive when

$x = -1/\sqrt{(3n)}$.

Max. $|u_n'(x)| = \dfrac{2}{\sqrt{3}\,.\,n^{5/2}\,(1+\frac{1}{3})^2} = \dfrac{3\sqrt{3}}{8n^{5/2}}$

$|u_n'(x)| < \dfrac{1}{n^{5/2}}$ for all values of x.

$\frac{1}{5/2}$ is convergent, it follows by Weierstrass's M-test that the series

onverges uniformly for all values of x. Hence the series $\Sigma\, u_n(x)$ can be ited term by term.

$$f'(x) = \sum_{n=1}^{\infty} u_n'(x) = -2x \sum_{1}^{\infty} \frac{1}{n^2\,(1+nx^2)^2}.$$

Comprehensive Exercise 2

mine for term by term integration the series for which

$$f_n(x) = nxe^{-nx^2}$$

icating the interval over which your conclusion holds. **(Purvanchal 2009)**

ow that the series for which

$$f_n(x) = \frac{1}{1+nx}, \qquad \text{(ii)}\ f_n(x) = nx\,(1-x)^n$$

n be integrated term by term in $0 \le x \le 1$, although they are not uniformly nvergent in this interval. **(Purvanchal 2008; Rohilkhand 10)**

amine for the continuity the sum function and for term by term egration the series whose nth term is

$$n^2 xe^{-n^2x^2} - (n-1)^2\ xe^{-(n-1)^2\,x^2}$$

aving all values in the interval $[0, 1]$.

ow that near $x = 0$, the series $u_1(x) + u_2(x) + u_3(x) + \ldots$ where $(x) = x$, $u_n(x) = x^{1/(2n-1)} - x^{1/(2n-3)}$ and real values of x are concerned, is

discontinuous and non-uniformly convergent. Can the series be term by term ?

5. Examine for term by term integration the series Σx^{n-1} ($0 \le x \le 2$) interval $0 \le x \le 1$.

6. Show that the series, for which $f_n(x) = \dfrac{nx}{1 + n^2 x^2}$, $0 \le x \le 1$ cannot be differentiated term by term at $x = 0$.

7. Show that the function represented by $\sum_{n=1}^{\infty} \dfrac{\sin nx}{n^3}$ is differentiable x and its derivative is $\sum_{1}^{\infty} \dfrac{\cos nx}{n^2}$.

8. Let $u_n(x) = x^2 \left[x^{1/(2n-1)} - x^{1/(2n-3)}\right] \sin \dfrac{1}{x}$ for $x \neq 0$, $u_n(0) = 0$ for any positive integer greater than unity and

$$u_1(x) = x^3 \sin(1/x) \text{ for } x \neq 0, u_1(0) = 0.$$

Show that $\sum_{1}^{\infty} u_n(x)$ converges for all values of x to $f(x)$, $f(x) = x^3 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Also that f' is discontinuous $x = 0$, that $\sum_{1}^{\infty} u_n'(x)$ is not uniformly convergent in any interval in the origin, and that

$$f'(x) = \sum_{1}^{\infty} u_n'(x) \text{ for all values of } x.$$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding (a), (b), (c) *and* (d).

1. On every bounded subset of **R**, the exponential series :

$$\Sigma u_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

(a) converges uniformly
(b) converges non-uniformly
(c) diverges uniformly
(d) none of these.

2. For all real values of x, the series $\cos x + \dfrac{\cos 2x}{2^2} + \dfrac{\cos 3x}{3^2} + \ldots$

(a) converges uniformly
(b) diverges uniformly
(c) converges non uniformly
(d) none of these.

3. The series $\Sigma \frac{x}{n(1+nx^2)}$:

(a) converges uniformly (b) diverges uniformly
(c) converges non uniformly (d) none of these.

4. The sum function $f(x)$, for which

$$f_n(x) = nx(1-x)^n, 0 \le x \le 1, \text{ is}$$

(a) not continuous (b) continuous
(c) uniformly convergent (d) none of these.

5. $\lim_{x \to 1} \sum_{n=1}^{\infty} \frac{nx^2}{n^3 + x^3} =$

(a) $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ (b) $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$

(c) $\sum_{n=1}^{\infty} \frac{1}{2n^3}$ (d) none of these.

Fill in the Blank(s)

Fill in the blanks "....." so that the following statements are complete and correct.

1. $\Sigma a_n n^{-x}$ is uniformly convergent on [0, 1] if Σa_n converges ..... on [0, 1] .

2. The sequence $< f_n >$ defined by $f_n(x) = \frac{x}{1+nx}$, $0 \le x < \infty$ is uniformly convergent to ..... on $[0, \infty[$.

3. $\sum_{n=1}^{\infty} \frac{1}{n^2+x^2}, 0 \le x < \infty$ is ..... convergent on $[0, \infty[$.

4. The series $\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + ...$
converges ..... in $0 < a \le x \le b < 2\pi$.

5. The series $\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$ is ..... convergent in $[1, \infty[$.

6. Uniform convergence is only a ..... but not a ..... condition for the continuity of the sum function.

True or False

Write 'T' for true and 'F' for false statement.

1. The sequence $< f_n >$ is said to converge uniformly to the function f on an interval I if for every $\epsilon > 0$, these exists a positive integer m such that $|f_n(x) - f(x)| > \epsilon$ for all $n \ge m$ and for all $x \in I$.

2. 0 is a point of uniform convergence of the sequence $< f_n >$ where $f_n(x) = 1 - (1-x^2)^n$.

3. The series $\sum_{n=0}^{\infty} xe^{-nx}$ is non uniformly convergent in $[0, 1]$.

4. 0 is a point of non uniform convergence of the sequence $< f_n(x) >$ where

$$f_n(x) = nxe^{-nx^2}, \quad x \in R.$$

5. $\sum_{n=1}^{\infty} \frac{x}{n(n+1)}$ is uniformly convergent on $]0, \infty[$.

6. $\int_0^1 \left(\sum_1^{\infty} \frac{x^n}{n^2} \right) dx = \sum_1^{\infty} \frac{1}{n^2}$.

7. The series, for which $f_n(x) = \frac{nx}{1+n^2x^2}, 0 \le x \le 1$.

can not be differentiated term by term at $x = 0$.

8. If $f(x) = \sum_1^{\infty} \frac{1}{n^3 + n^4x^2}$, $f'(x) = -2x \sum_1^{\infty} \frac{1}{n^2(1+nx^2)^2}$ for all x.

9. The series, the sum of whose first n terms is $n^2x(1-x)^n, 0 \le x \le 1$, can be integrated term by term.

Answers

Multiple Choice Questions

1. (a) 2. (a) 3. (a) 4. (b) 5. (b)

Fill in the Blank(s)

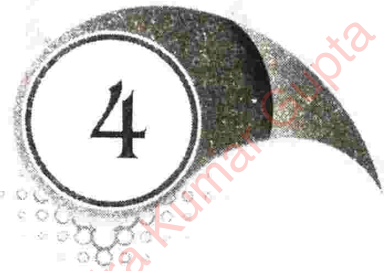
1. uniformly 2. 0 3. uniformly
4. uniformly 5. uniformly
6. sufficient, necessary

True or False

1. *F* 2. *F* 3. *T* 4. *T* 5. *F*
6. *F* 7. *T* 8. *T* 9. *F*

□

Chapter



4

More About Continuity and Darboux Theorem

In the present chapter we shall study the sequential continuity, boundedness and intermediate value properties of continuous functions, uniform continuity, Meaning of sign of derivative and Darboux theorem.

4.1 Criteria for Continuity or Equivalent Definitions of Continuity

Theorem 1: (Heine's definition of continuity). Sequential Continuity: *The necessary and sufficient condition for a function f defined on* $I \subset \mathbb{R}$ *to be continuous at* $a \in I$ *is that for each sequence* $< a_n >$ *in* I *which converges to* a*, we have*

$$\lim_{n \to \infty} f(a_n) = f(a) .$$

Proof: The condition is necessary: Let f be continuous at a and let $< a_n >$ be a sequence in I such that $\lim_{n \to \infty} a_n = a$.

Let ε be any positive number. Since f is continuous at a, therefore, for a given $\varepsilon > 0$, we can find a number $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \text{ whenever } |x - a| < \delta. \quad \ldots(1)$$

Also, since $\lim_{n \to \infty} a_n = a$, therefore, there exists a positive integer m such that

$$|a_n - a| < \delta \text{ whenever } n > m. \quad \ldots(2)$$

Setting $x = a_n$ in (1), we get

$$|f(a_n) - f(a)| < \varepsilon \text{ whenever } |a_n - a| < \delta. \quad \ldots(3)$$

From (2) and (3), we get

$$|f(a_n) - f(a)| < \varepsilon \text{ whenever } n > m.$$

$$\therefore \quad \lim_{n \to \infty} f(a_n) = f(a) .$$

Hence the condition is necessary.

The condition is sufficient

Suppose for every sequence $< a_n >$ in I converging to a, we have

$$\lim_{n \to \infty} f(a_n) = f(a) .$$

Then we have to show that f is continuous at a.

Let us suppose that f is not continuous at a. Then there exists an $\varepsilon > 0$ such that for every $\delta > 0$ there is an x such that $|x - a| < \delta$ but $|f(x) - f(a)| \geq \varepsilon$.

If we take $\delta = 1/n$, we find that for each positive integer n, there is an a_n such that

$$|a_n - a| < 1/n \text{ but } |f(a_n) - f(a)| \geq \varepsilon.$$

Then $\lim_{n \to \infty} a_n = a$, but $f(a_n)$ does not converge to $f(a)$ *i.e.*,

$$\lim_{n \to \infty} f(a_n) \neq f(a).$$

But this is a contradiction. Hence f must be continuous at $x = a$.

Illustrative Examples

Example 1: *A function f defined on $[0, 1]$ is given by*

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1 - x, & \text{if } x \text{ is irrational} . \end{cases}$$

Show that f takes every value between 0 and 1 (both inclusive), but it is continuous only at the point $x = \frac{1}{2}$.

Solution: Let $c \in [0, 1]$.

If c is rational, then $f(c) = c$.

If c is irrational, then $1 - c$ is also irrational

and $0 < 1 - c < 1$ *i.e.*, $1 - c \in [0, 1]$.

We have $f(1 - c) = 1 - (1 - c) = c$.

Thus f takes every value c in $[0, 1]$.

Now to show that f is continuous only at the point $x = \frac{1}{2}$.

Let x_0 be any point of $[0, 1]$. For each positive integer n we select a rational number a_n and an irrational number b_n, both in $[0, 1]$, such that

$$|a_n - x_0| < 1/n, |b_n - x_0| < 1/n.$$

$$\therefore \quad \lim_{n \to \infty} a_n = x_0 = \lim_{n \to \infty} b_n.$$

If f is to be continuous at x_0, then we must have

$$\lim_{n \to \infty} f(a_n) = f(x_0) = \lim_{n \to \infty} f(b_n).$$

Now $f(a_n) = a_n$ for all n and $f(b_n) = 1 - b_n$ for all n.

$$\therefore \quad \lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} a_n = x_0$$

and $$\lim_{n \to \infty} f(b_n) = \lim_{n \to \infty} (1 - b_n) = 1 - x_0.$$

So for f to be continuous at x_0, we must have

$$x_0 = f(x_0) = 1 - x_0 \quad i.e., \quad x_0 = \frac{1}{2}.$$

Thus $x = \frac{1}{2}$ is the only possible point of $[0, 1]$ where f can be continuous.

Now we shall show that f is actually continuous at the point $x = 1/2$.

We have $f(1/2) = \frac{1}{2}$.

Let $\varepsilon > 0$ be given.

Take a positive real number $\delta = \frac{1}{2}\varepsilon$. Then if x is rational, we have

$$\left|x - \frac{1}{2}\right| < \delta \Rightarrow \left|f(x) - f\left(\frac{1}{2}\right)\right| = \left|x - \frac{1}{2}\right| < \delta = \frac{1}{2}\varepsilon < \varepsilon$$

and if x is irrational, we have

$$\left|x - \frac{1}{2}\right| < \delta \Rightarrow \left|f(x) - f\left(\frac{1}{2}\right)\right| = \left|(1 - x) - \frac{1}{2}\right|$$

$$= \left| x - \frac{1}{2} \right| < \delta = \frac{1}{2}\varepsilon < \varepsilon.$$

Thus, we have

$$\left| x - \frac{1}{2} \right| < \delta \Rightarrow \left| f(x) - f\left(\frac{1}{2}\right) \right| < \varepsilon,$$

so that f is continuous at $x = \frac{1}{2}$.

Hence f is continuous only at the point $x = 1/2$.

Example 2: *Let f be a function defined on $]0, 1[$ by setting $f(x) = 0$ when x is irrational and $f(x) = 1/q$ when x is a rational number of the form p/q where p and q are positive integers having no factor in common. Show that f is continuous at each irrational point and discontinuous at each rational point.*

Solution: First, let $a = p/q$ be any rational number in $]0, 1[$, where p and q are positive integers having no factor in common. For each positive integer n, we choose a positive irrational number x_n such that $|x_n - a| < 1/n$. Then the sequence $\langle x_n \rangle$ converges to a. Also it is given that $f(x_n) = 0$ for each n, so that $\lim_{n \to \infty} f(x_n) = 0 \neq f(a)$, since $f(a) = f(p/q) = 1/q \neq 0$.

It follows from the preceding theorem that f is not continuous at a rational point.

Next, let b be an irrational number in $]0, 1[$ so that $f(b) = 0$. Let $\varepsilon > 0$ be given. Choose a positive integer n such that $1/n < \varepsilon$. Evidently there can be only a finite number of rational numbers p/q in $]0, 1[$ such that $q < n$. Hence, we can find a number $\delta > 0$ such that no rational number in $]b - \delta, b + \delta[\subset]0, 1[$ has its denominator less than n.

Thus we have shown that

$$|x - b| < \delta \Rightarrow |f(x) - f(b)| = |f(x)| = 0 < \varepsilon, \quad \text{if } x \text{ is irrational} \qquad \ldots(1)$$

and

$$|x - b| < \delta \Rightarrow |f(x) - f(b)| = |f(x)| \leq 1/n < \varepsilon, \quad \text{if } x \text{ is rational.} \qquad \ldots(2)$$

From (1) and (2), it follows that $|f(x) - f(b)| < \varepsilon$ whenever $|x - b| < \delta$.

Hence f is continuous at b.

Theorem 2: *A function $f : \mathbf{R} \to \mathbf{R}$ is continuous iff for every open set G in $\mathbf{R}$, the inverse image $f^{-1}(G)$ is an open set in $\mathbf{R}$.*

Proof: The 'only if' part: Let f be continuous and let G be any open set in $\mathbf{R}$. If $f^{-1}(G)$ is empty, then it is open. If $f^{-1}(G)$ is not empty, let $a \in f^{-1}(G)$. Then $f(a) \in G$. Since G is an open set containing $f(a)$, therefore, there exists an $\varepsilon > 0$ such that

$$]f(a) - \varepsilon, f(a) + \varepsilon[\subset G.$$

Now f is continuous at a, so we can find a number $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

i.e., $x \in]a - \delta, a + \delta[\Rightarrow f(x) \in]f(a) - \varepsilon, f(a) + \varepsilon[\subset G$

or $x \in]a - \delta, a + \delta[\Rightarrow f(x) \in G$

$$\Rightarrow x \in f^{-1}(G)$$

Hence $]a - \delta, a + \delta[\subset f^{-1}(G)$. Thus $f^{-1}(G)$ is a neighbourhood of a. Since a is any point of $f^{-1}(G)$, therefore, it follows that $f^{-1}(G)$ is open.

The 'if' part: Let the inverse image $f^{-1}(G)$ of every open set G be open. To show that f is continuous. Consider any point $a \in \mathbf{R}$. Let $\varepsilon > 0$ be given. Then $]f(a) - \varepsilon, f(a) + \varepsilon[$ is an open set containing $f(a)$ and hence by hypothesis $f^{-1}(]f(a) - \varepsilon, f(a) + \varepsilon[)$ is an open set containing a. Consequently, we can find a number $\delta > 0$ such that

$$]a - \delta, a + \delta[\subset f^{-1}(]f(a) - \varepsilon, f(a) + \varepsilon[), \text{ so that}$$

$$f(]a - \delta, a + \delta[) \subset]f(a) - \varepsilon, f(a) + \varepsilon[.$$

Thus, for any given $\varepsilon > 0$ we have found a number $\delta > 0$ such that

$$x \in]a - \delta, a + \delta[\Rightarrow f(x) \in]f(a) - \varepsilon, f(a) + \varepsilon[\text{ i.e.,}$$

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

This shows that f is continuous at a. Since a is any point of $\mathbf{R}$, hence f is continuous on $\mathbf{R}$.

Theorem 3: *A function $f : \mathbf{R} \to \mathbf{R}$ is continuous on $\mathbf{R}$ iff for every closed set H in $\mathbf{R}$, $f^{-1}(H)$ is closed in $\mathbf{R}$.*

Proof: First, let f be continuous and let H be any closed set in $\mathbf{R}$. Then $\mathbf{R} \sim H$ is an open set in $\mathbf{R}$, so that by the preceding theorem, $f^{-1}(\mathbf{R} \sim H)$ is an open set in $\mathbf{R}$. Since

$$f^{-1}(\mathbf{R} \sim H) = \mathbf{R} \sim f^{-1}(H),$$

hence it follows that $\mathbf{R} \sim f^{-1}(H)$ is an open set in $\mathbf{R}$ and consequently $f^{-1}(H)$ is a closed set in $\mathbf{R}$.

Conversely, let $f^{-1}(H)$ be closed for every closed set H. To show that f is continuous. Let G be any open set in $\mathbf{R}$. Then $\mathbf{R} \sim G$ is a closed set in $\mathbf{R}$ and hence by hypothesis $f^{-1}(\mathbf{R} \sim G)$ is a closed set in $\mathbf{R}$.

Since $f^{-1}(\mathbf{R} \sim G) = \mathbf{R} \sim f^{-1}(G)$, therefore, this means that $\mathbf{R} \sim f^{-1}(G)$ is a closed set in $\mathbf{R}$ and hence $f^{-1}(G)$ is an open set in $\mathbf{R}$. Thus $f^{-1}(G)$ is open in $\mathbf{R}$ whenever G is open in $\mathbf{R}$. Therefore f is continuous by the preceding theorem.

4.2 Algebra of Continuous Functions

Theorem 1: *Let f and g be defined on an interval I. If f and g are continuous at $a \in I$, then $f + g$ is also continuous at a.*

Proof: Let $< a_n >$ be any sequence in I converging to a. Since f and g are both continuous at $a \in I$, therefore

$$\lim_{n \to \infty} f(a_n) = f(a) \text{ and } \lim_{n \to \infty} g(a_n) = g(a).$$

We have, $$\lim_{n \to \infty} (f + g)(a_n) = \lim_{n \to \infty} \{f(a_n) + g(a_n)\}$$

$$= \lim_{n \to \infty} f(a_n) + \lim_{n \to \infty} g(a_n)$$

$$= f(a) + g(a) = (f + g)(a),$$

which shows that $f + g$ is continuous at a.

Aliter: Since both f and g are continuous at a, therefore

$$f(a) = \lim_{x \to a} f(x) \text{ and } g(a) = \lim_{x \to a} g(x)$$

Now $$\lim_{x \to a} (f + g)(x) = \lim_{x \to a} \{f(x) + g(x)\}$$

[$\because$ by def. of the function $f + g$, we have $(f + g)(x) = f(x) + g(x)$]

$$= \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

[$\because$ limit of a sum = sum of the limits provided both the limits exist]

$$= f(a) + g(a) = (f + g)(a).$$

Since $\lim_{x \to a} (f + g)(x) = (f + g)(a)$, therefore $f + g$ is continuous at a.

Theorem 2: *Let f and g be defined on an interval I. If f and g are continuous at $a \in I$, then fg is continuous at a.*

Proof: Let $< a_n >$ be any sequence in I converging to a. Since f and g are both continuous at $a \in I$, therefore,

$$\lim_{n \to \infty} f(a_n) = f(a) \text{ and } \lim_{n \to \infty} g(a_n) = g(a).$$

We have, $$\lim_{n \to \infty} (fg)(a_n) = \lim_{n \to \infty} \{f(a_n)\, g(a_n)\}$$

$$= \lim_{n \to \infty} f(a_n) \cdot \lim_{n \to \infty} g(a_n)$$

$$= f(a) \cdot g(a) = (fg)(a),$$

which shows that $f\,g$ is continuous at a.

Alternative Proof: Since both f and g are continuous at a, therefore

$$f(a) = \lim_{x \to a} f(x) \text{ and } g(a) = \lim_{x \to a} g(x).$$

Now $$\lim_{x \to a} (fg)(x) = \lim_{x \to a} f(x) \cdot g(x)$$

[$\because$ by def. of the function fg, we have $(fg)(x) = f(x)\, g(x)$]

$$= \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

[$\because$ limit of a product = product of the limits provided both the limits exist]

$$= f(a)\, g(a) = (fg)(a).$$

Since $\lim_{x \to a} (f\,g)(x) = (f\,g)(a)$, therefore fg is continuous at a.

Theorem 3: *If f is continuous at a point a and $c \in \mathrm{R}$, then cf is continuous at a.*

Proof: Let $< a_n >$ be any sequence in I converging to a. Since f is continuous at a, therefore

$$\lim_{n \to \infty} f(a_n) = f(a).$$

We have, $$\lim_{n \to \infty} (c\,f)(a_n) = \lim_{n \to \infty} c\,f(a_n) = c \lim_{n \to \infty} f(a_n)$$

$$= c\,f(a) = (c\,f)(a),$$

which shows that cf is continuous at a.

Alternative Proof: Since f is continuous at a, therefore $f(a) = \lim_{x \to a} f(x)$.

Now $$\lim_{x \to a} (c\,f)(x) = \lim_{x \to a} c\,f(x)$$

[$\because$ by def. of the function cf, we have $(c\,f)(x) = c\,f(x)$]

$$= c \lim_{x \to a} f(x) = c\,f(a) = (c\,f)(a).$$

Since $\lim_{n \to \infty} (c\,f)(x) = (c\,f)(a)$, therefore $c\,f$ is continuous at a.

Theorem 4: *Let f and g be defined on an interval I, and let $g(a) \neq 0$. If f and g are continuous at $a \in I$, then f / g is continuous at a.*

Proof: Let $< a_n >$ be any sequence in I converging to a.

Since f and g are continuous at a, therefore

$$\lim_{n \to \infty} f(a_n) = f(a) \text{ and } \lim_{n \to \infty} g(a_n) = g(a).$$

Also, since $g(a) \neq 0$, therefore, there exists a positive integer m such that $g(a_n) \neq 0$, whenever $n > m$.

We have, $$\lim_{n \to \infty} (f / g)(a_n) = \lim_{n \to \infty} \{f(a_n) / g(a_n)\}$$

$$= \lim_{n \to \infty} f(a_n) / \lim_{n \to \infty} g(a_n)$$

$$= f(a) / g(a) = (f / g)(a)$$

which shows that f / g is continuous at a.

Aliter: Since both f and g are continuous at a, therefore

$$f(a) = \lim_{x \to a} f(x) \text{ and } g(a) = \lim_{x \to a} g(x)$$

Since $g(a) \neq 0$ and $g(x)$ is continuous at $x = a$, therefore there exists a neighbourhood of a at each point of which $g(x) \neq 0$.

Now $$\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \lim_{x \to a} \frac{f(x)}{g(x)}$$

[$\because$ by def. of the function f / g we have $(f / g)(x) = f(x) / g(x)$]

$$= \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

[$\because$ limit of a quotient = quotientof the limits provided both the limits exist and the limit of the denominator is not zero]

$$= \frac{f(a)}{g(a)} = \left(\frac{f}{g}\right)(a).$$

Since $\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \left(\frac{f}{g}\right)(a)$, therefore $\frac{f}{g}$ is continuous at a.

Theorem 5: *If f is continuous at a then $|f|$ is also continuous at a.*

Proof: Let $< a_n >$ be any sequence in I converging to a. Since f is continuous at a, therefore

$$\lim_{n \to \infty} f(a_n) = f(a)$$

Now $$\lim_{n \to \infty} |f|(a_n) = \lim_{n \to \infty} |f(a_n)| = \left| \lim_{n \to \infty} f(a_n) \right|$$

$$= |f(a)| = |f|(a), \text{ showing that } |f| \text{ is continuous at } a.$$

Aliter: It is given that f is continuous at a. To prove that $|f|$ is also continuous at a.

Take any given $\varepsilon > 0$.

Since f is continuous at a, therefore there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon, \text{ whenever } |x - a| < \delta. \qquad ...(1)$$

Now if x and y are any real numbers, we know that

$$||x| - |y|| \leq |x - y|.$$

$$\therefore \quad ||f(x)| - |f(a)|| \leq |f(x) - f(a)| \qquad ...(2)$$

From (1) and (2), we have

$$||f(x)| - |f(a)|| < \varepsilon, \text{ whenever } |x - a| < \delta$$

$$\Rightarrow \quad ||f|(x) - |f|(a)| < \varepsilon, \text{ whenever } |x - a| < \delta.$$

[$\because$ by def. of the function $|f|$, we have $|f|(x) = |f(x)|$]

Thus for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$||f|(x) - |f|(a)| < \varepsilon, \text{ whenever } |x - a| < \delta.$$

$\therefore$ by definition of continuity of a function at a point, the function $|f|$ is continuous at a.

Note: The converse is not true. For example, if

$f(x) = -1$, for $x < a$ and $f(x) = 1$ for $x \geq a$ then

$\lim_{x \to a} |f(x)| = 1 = |f(a)|$, but $\lim_{x \to a} f(x)$ does not exist.

Thus $|f|$ is continuous at a while f is not continuous at a.

Theorem 6: *Let f and g be defined on an interval I. If both f and g are continuous at $a \in I$, then the functions max. $\{f, g\}$ and min. $\{f, g\}$ are both continuous at a.*

Proof: We have $\max\{f, g\} = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$,

$$\min.\{f, g\} = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|.$$

The theorem now follows from theorems 1, 3, 5, of article 4.2.

Theorem 7: *Let f and g be defined on intervals I and J respectively and let $f(I) \subset J$. If f is continuous at $a \in I$ and g is continuous at $f(a)$, then the composite map $g \circ f$ is continuous at a.*

Proof: Let $< a_n >$ be any sequence in I converging to a. Then $< f(a_n) >$ converges to $f(a)$, f being continuous at a. Again, since $f(I) \subset J$, hence $< f(a_n) >$ is a sequence in J.

Now g being continuous at $f(a)$ and $< f(a_n) >$ is a sequence in J converging to $f(a)$, therefore $< g(f(a_n)) >$ converges to $g(f(a))$ *i.e.*, $< (g \circ f)(a_n) >$ converges to $(g \circ f)(a)$.

Thus $< a_n > \to a \Rightarrow < (g \circ f)(a_n) > \to (g \circ f)(a)$.

Hence $g \circ f$ is continuous at a.

4.3 Boundedness and Intermediate Value Properties of Continuous Functions

Theorem 1: (Borel's Theorem) *If f is a continuous function on the closed interval $[a, b]$, then the interval can always be divided up into a finite number of subintervals such that, given $\varepsilon > 0$, $|f(x_1) - f(x_2)| < \varepsilon$, where x_1 and x_2 are any two points in the same sub-interval.*

Proof: Let us assume that the theorem is false. Then for any mode of sub-division, there must be at least one of the subintervals for which the theorem is false. Let us divide $[a, b]$ into two equal parts. Let c be the point of division. Then the theorem must be false in at least one of the two parts. Suppose it is false in $[c, b]$. Renaming the interval $[c, b]$ as $[a_1, b_1]$, we divide $[a_1, b_1]$ into two equal parts. Again, the theorem must be false in at least one of these two parts. Continuing this process of repeated bisection indefinitely, we get a sequence of closed intervals

$$[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n], \ldots \text{ such that}$$

$$a \le a_1 \le a_2 \le \ldots \le a_n \le \ldots \le b_n \le \ldots \le b_2 \le b_1 \le b, \qquad \ldots(1)$$

$$b_1 - a_1 = \frac{1}{2}(b - a), b_2 - a_2 = \frac{1}{2}(b_1 - a_1) = \frac{1}{2^2}(b - a), \ldots\ldots,$$

$$b_n - a_n = \frac{1}{2^n}(b - a), \ldots \qquad \ldots(2)$$

and the theorem is false for each interval $[a_n, b_n]$. $\ldots(3)$

From (1), we find that the sequence $< a_n >$ is increasing and bounded above by b and $< b_n >$ is decreasing and bounded below by a. Hence both $< a_n >$ and $< b_n >$ are convergent. So there exist $l_1, l_2 \in \mathbb{R}$ such that $\lim a_n = l_1$ and $\lim b_n = l_2$.

From (2), $\lim (b_n - a_n) = \lim \frac{1}{2^n}(b - a) = (b - a) \lim \frac{1}{2^n} = (b - a) \cdot 0 = 0$

i.e., $\lim b_n - \lim a_n = 0$ *i.e.*, $l_2 - l_1 = 0$ *i.e.*, $l_2 = l_1$.

Now $l_1 = \sup < a_n >$,

$< a_n >$ being an increasing and bounded above sequence

$= \inf < b_n >$,

$< b_n >$ being a decreasing and bounded below sequence.

$\therefore$ $a_n \le l_1$ and $l_1 \le b_n$ *i.e.*, $a_n \le l_1 \le b_n$ $\forall n \in \mathbb{N}$

$\Rightarrow$ $a \le l_1 \le b$ *i.e.*, either $a < l_1 < b$ or $a = l_1$ or $b = l_1$.

Case I: $a < l_1 < b$. Since f is continuous in $[a, b]$, it is also continuous at $x = l_1$.

Then for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(l_1)| < \frac{1}{2}\varepsilon, \text{ whenever } |x - l_1| < \delta.$$

Now we can choose n so large that the interval $[a_n, b_n]$ lies completely inside the interval $]l_1 - \delta, l_1 + \delta[$.

Hence if x_1 and x_2 are any two points in $[a_n, b_n]$, then

$$|f(x_1) - f(l_1)| < \varepsilon/2, |f(x_2) - f(l_1)| < \varepsilon/2.$$

Now $$|f(x_1) - f(x_2)| = |f(x_1) - f(l_1) + f(l_1) - f(x_2)|$$
$$\leq |f(x_1) - f(l_1)| + |f(l_1) - f(x_2)|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \Rightarrow \text{the result is true in } [a_n, b_n].$$

This is a contradiction to the fact (3) and so our initial assumption that the theorem is not true is wrong. Hence the theorem must be true.

Similarly we can prove the cases for $l_1 = a$ or $l_1 = b$.

Note: The possibility $l_1 = a$ or $l_1 = b$ shows that the interval considered in the theorem must be closed so that f is continuous at l_1.

Theorem 2: (Boundedness Theorem) *If a function $f(x)$ is continuous in a closed interval $[a, b]$, then it is bounded in that interval.*

Proof: By the above theorem, for a given $\varepsilon > 0$, we can sub-divide the interval $[a, b]$ into a finite number of sub-intervals say $[a = a_0, a_1], [a_1, a_2], \ldots, [a_{n-1}, a_n = b]$ such that

$$|f(x_1) - f(x_2)| < \varepsilon \qquad \ldots(1)$$

for any two points x_1, x_2 in the same sub-interval. Let x be any point in the first sub-interval $[a, a_1]$. Then by (1), we have, $\forall\, x \in [a, a_1]$

$$|f(x) - f(a)| < \varepsilon \text{ i.e., } f(a) - \varepsilon < f(x) < f(a) + \varepsilon. \qquad \ldots(2)$$

In particular, for $x = a_1$, $|f(a_1) - f(a)| < \varepsilon$. $\qquad \ldots(3)$

Again $\forall\, x \in [a_1, a_2], |f(x) - f(a_1)| < \varepsilon$. $\qquad \ldots(4)$

$\therefore$ $\forall\, x \in [a_1, a_2]$, we have

$$|f(x) - f(a)| = |f(x) - f(a_1) + f(a_1) - f(a)|$$
$$\leq |f(x) - f(a_1)| + |f(a_1) - f(a)|$$
$$< \varepsilon + \varepsilon, \text{ from (3) and (4)}$$
$$= 2\varepsilon.$$

Thus $\forall\, x \in [a_1, a_2]$, we have $|f(x) - f(a)| < 2\varepsilon$

i.e., $$f(a) - 2\varepsilon < f(x) < f(a) + 2\varepsilon. \qquad \ldots(5)$$

From (2) and (5), we see that all the values of $f(x)$ in the first two sub-intervals lie between

$$f(a) - 2\varepsilon \text{ and } f(a) + 2\varepsilon.$$

Proceeding in the same way, we can show that $\forall\, x \in [a_{n-1}, a_n = b]$, we have

$$f(a) - n\varepsilon < f(x) < f(a) + n\varepsilon.$$

Hence all the values of $f(x)$ in the interval $[a, b]$ will lie between $f(a) - n\varepsilon$ and $f(a) + n\varepsilon$.

Thus $f(x)$ is bounded in $[a, b]$.

Note: A bounded function in $[a, b]$ need not be continuous in $[a, b]$. For example, the function

$$f(x) = \sin(1/x) \text{ for } x \neq 0,\ f(0) = 0$$

is bounded in $[0, 1]$ but not continuous in $[0, 1]$ since it is discontinuous at $x = 0$.

Theorem 3: (The Mostest Theorem) *If a function $f(x)$ is continuous in $[a, b]$, then it attains its supremum and infimum at least once in $[a, b]$.*

Proof: Since $f(x)$ is continuous in $[a, b]$, $f(x)$ is bounded in $[a, b]$. Let M and m be the supremum and infimum of f in $[a, b]$ respectively. We shall show that f attains its supremum M at least once in this interval. Let, if possible, $f(x)$ does not attain M, then

$$f(x) \neq M \ \forall x \in [a, b] \quad i.e., \quad M - f(x) \neq 0 \ \forall x \in [a, b].$$

Now, M, being a constant, is always a continuous function and f is given to be continuous in $[a, b]$.

$\therefore$ $M - f(x)$ is also continuous in $[a, b]$

$\Rightarrow$ $\dfrac{1}{M - f(x)}$ is also continuous in $[a, b]$.

$[\because M - f(x) \neq 0 \ \forall x \in [a, b]]$

Consequently $1/\{M - f(x)\}$ is bounded in $[a, b]$. Let $G \in R_+$ be its upper bound in $[a, b]$ so that $\forall x \in [a, b]$

$$\frac{1}{M - f(x)} \leq G \ i.e., \ M - f(x) \geq \frac{1}{G} \quad i.e., \quad f(x) \leq M - (1/G) < M$$

e., $M - (1/G) < M$ is also an upper bound of f.

ıt this contradicts the fact that M is the supremum of f in $[a, b]$. Hence $f(x) = M$ ɔr at least one value of x in $[a, b]$. Similarly we can show that f attains its infimum at least once in $[a, b]$.

Theorem 4: *If f is continuous at $x = x_0$ where $f(x_0) \neq 0$, then a positive number δ can be found such that $f(x)$ has the same sign as $f(x_0)$ for every value of x in $]x_0 - \delta, x_0 + \delta[$.*

Proof: Since f is continuous at $x = x_0$, hence for a given $\varepsilon > 0$, we can find a number $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$, whenever $|x - x_0| < \delta$

i.e., $f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$, whenever $x_0 - \delta < x < x_0 + \delta$.

Now $f(x_0) \neq 0 \Rightarrow |f(x_0)| > 0$.

If we choose ε such that $0 < \varepsilon < |f(x_0)|$, we see that $f(x_0) - \varepsilon$ and $f(x_0) + \varepsilon$ have the same sign as $f(x_0)$. It implies that $f(x)$ has the same sign as $f(x_0)$ for all x in the interval $]x_0 - \delta, x_0 + \delta[$.

Theorem 5: ((Bolzano's Theorem)) *If f is continuous in $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then there is at least one value of x for which $f(x)$ vanishes.*

Proof: For definiteness, let $f(a) < 0$ and $f(b) > 0$. Consider a subset S of $[a, b]$ defined as follows :

$$S = \{x : a \leq x \leq b \text{ and } f(x) < 0\}.$$

Then $S \neq \varnothing$, since $a \in S$, $f(a)$ being < 0. By definition, b is an upper bound of S. It follows that S has a supremum, say u, by the completeness property of real numbers. Obviously $a \leq u \leq b$. Now, we shall show that $f(u) = 0$.

Step I: First we shall show that $u \neq a$. Since $f(a) < 0$, we can find a positive number δ such that $f(x) < 0$ whenever $a \leq x < a + \delta$.

It follows that $[a, a + \delta[\ S$ and hence the supremum of S must be greater than or equal to $a + \delta$.

$\therefore \quad u \geq a + \delta$ *i.e.*, $u \neq a$.

Step II: We shall show that $u \neq b$. In fact, since $f(b) > 0$, therefore, there exists a positive number δ_1 such that $f(x) > 0$ whenever $b - \delta_1 < x \leq b$.

It gives that $b - \delta_1$ is an upper bound of S, and hence

$$u = \sup S \leq b - \delta_1 < b \quad \textit{i.e.}, \; u < b.$$

Step III: We shall show that $f(u) \not> 0$. Since $a < u < b$, therefore, if $f(u) > 0$, then we can find a positive number δ_2 such that $f(x) > 0$ whenever

$$u - \delta_2 < x < u + \delta_2 .$$

Also, since $u = \sup S$, there exists $x_0 \in S$ such that $u - \delta_2 < x_0 \leq u$. This means that $f(x_0) > 0$.

Again, $x_0 \in S \Rightarrow f(x_0) < 0$, by definition of S. This contradiction implies that $f(u) \not> 0$.

Step IV: We shall show that $f(u) \not< 0$. Since, if $f(u) < 0$, then we can find a positive number δ_3 such that $u + \delta_3 < b$ and $f(x) < 0$ whenever $u - \delta_3 < x < u + \delta_3$.

If x_1 is any point such that $u < x_1 < u + \delta_3$, then $f(x_1) < 0$. But this is a contradiction to the fact that u is the supremum of all those points of $[a, b]$ for which f is negative. Consequently $f(u) \not< 0$.

It follows from steps III and IV that $f(u) = 0$.

Theorem 6: (The Intermediate Value Theorem) *If a function f is continuous in the closed interval $[a, b]$, then $f(x)$ must take at least once all values between $f(a)$ and $f(b)$.*

Proof: Let $f(a) < d < f(b)$. Let us define a function F such that

$$F(x) = f(x) - d.$$

Since f is continuous in the closed interval $[a, b]$, F must also be continuous in $[a, b]$. Also $F(a) < 0$ and $F(b) > 0$ *i.e.*, $F(a)$ and $F(b)$ are of opposite signs. It follows that there exists x_0 in $]a, b[$ such that $F(x_0) = 0$ *i.e.*, $f(x_0) - d = 0$ or $f(x_0) = d$.

Since d is any value between $f(a)$ and $f(b)$ it follows that f takes all values between $f(a)$ and $f(b)$ at least once.

The converse of the above theorem is not true. For example, let f be the function defined as $f(x) = \sin(1/x), x \neq 0$ and $f(0) = 0$.

In the interval $[-2/\pi, 2/\pi]$ this function takes all values between $f(-2/\pi)$ and $f(2/\pi)$ *i.e.*, between -1 and 1 an infinite number of times as x varies from $-2/\pi$ to $2/\pi$, but this function is not continuous in $[-2/\pi, 2/\pi]$ as it is discontinuous at $x = 0$.

Corollary 1: *Let f be continuous on $[a, b]$ and let $k \in [m, M]$ where $m = \inf f$ and $M = \sup f$ on $[a, b]$. Then there exists $c \in [a, b]$ such that $f(c) = k$.*

Proof: Since f is continuous on $[a, b]$ and every function defined and continuous on a closed interval attains its supremum and infimum, therefore, there exist $x_1, x_2 \in [a, b]$ such that

$$m = f(x_1) \text{ and } M = f(x_2).$$

If $x_1 = x_2$, then f is a constant function on $[a, b]$ and the result follows.

Let $x_1 < x_2$. Then $[x_1, x_2] \subset [a, b]$ and f is continuous on $[x_1, x_2]$.

Hence by the above theorem there exists $c \in [x_1, x_2] \subset [a, b]$ such that $f(c) = k$.

Similarly we can prove the result if $x_1 > x_2$.

Corollary 2: *Let f be continuous on $[a, b]$. Then $f([a, b]) = [m, M]$, where $m = \inf f$ and $M = \sup f$ on $[a, b]$ and thus $f([a, b])$ is a closed set.*

Proof: By Corollary 1, f takes all values between m and M and hence $[m, M] \subset f([a, b])$. Since every value of f on $[a, b]$ lies between m and M, hence, $f([a, b]) \subset [m, M]$. Thus $f([a, b]) = [m, M]$ which is a closed set because every closed interval is a closed set.

4.4 Uniform Continuity

Let a function f be continuous for every value of x in $[a, b]$. It means that if $x_0 \in [a, b]$, then, given $\varepsilon > 0$, there exists $\delta > 0$, such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$.

The number δ will depend upon x_0 as well as ε and so we may write it symbolically as $\delta(\varepsilon, x_0)$. For some functions it may happen that given $\varepsilon > 0$ the same δ serves for all $x_0 \in [a, b]$ in the condition of continuity. Such functions are called uniformly continuous on $[a, b]$.

Definition: *A function f defined on an interval I is said to be uniformly continuous on I if given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$, where x, y are in I.* **(Meerut 2001)**

It should be noted carefully that uniform continuity is a property associated with an interval and not with a single point. The concept of continuity is local in character whereas the concept of uniform continuity is global in character.

Note: A function f is not uniformly continuous on I, if there exists some $\varepsilon > 0$ for which no $\delta > 0$ serves *i.e.*, for any $\delta > 0$, there exist $x, y \in I$ such that $|f(x) - f(y)| \geq \varepsilon$ and $|x - y| < \delta$.

The following two theorems express the relation between continuity and uniform continuity. The first one gives that uniform continuity always implies continuity. The second one gives a sufficient condition under which continuity implies uniform continuity.

Theorem 1: *If f is uniformly continuous on an interval I, then it is continuous on I.*

Proof: Let x_0 be any point of I and let $\varepsilon > 0$ be given. Since f is uniformly continuous on I, therefore, given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta, \forall x, y \text{ in } I.$$

Taking $y = x_0$, we have in particular,

$$|f(x) - f(x_0)| < \varepsilon \text{ whenever } |x - x_0| < \delta \ \forall x \text{ in } I.$$

This means that f is continuous at x_0.

Since x_0 is an arbitrary point of I, therefore f is continuous at every point of I.

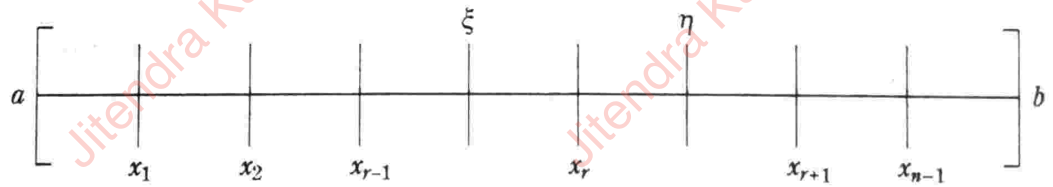
Hence f is continuous on I.

Theorem 2: *A function which is continuous in a closed and bounded interval $I = [a, b]$ is uniformly continuous in $[a, b]$.*

Proof: Take any given $\varepsilon > 0$. By theorem 1 of article 4.3, the interval $[a, b]$ can be divided up into sub-intervals $[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b]$ such that for any two points α, β in the same sub-interval, we have $|f(\alpha) - f(\beta)| < \frac{1}{2}\varepsilon$. ...(1)

Let δ be a positive number which does not exceed the least of the numbers

$$x_1 - a, x_2 - x_1, \ldots, b - x_{n-1}.$$



Let ξ, η be any two points in $[a, b]$ such that $|\xi - \eta| \geq \delta$.

If these two points are in the same sub-interval, then by (1), we have

$$|f(\xi) - f(\eta)| < \frac{1}{2}\varepsilon.$$

If ξ, η do not lie in the same sub-interval, then surely they lie one in each of the two consecutive intervals. Let x_r be the point of division such that $x_{r-1} < \xi < x_r < \eta < x_{r+1}$. Then we have

$$\begin{aligned} |f(\xi) - f(\eta)| &= |f(\xi) - f(x_r) + f(x_r) - f(\eta)| \\ &\leq |f(\xi) - f(x_r)| + |f(x_r) - f(\eta)| \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, \text{ by (1).} \end{aligned}$$

Thus we have shown that, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(\xi) - f(\eta)| < \varepsilon \text{ whenever } |\xi - \eta| < \delta, \quad \forall\, \xi, \eta \text{ in } [a, b].$$

Hence f is uniformly continuous in $[a, b]$.

Illustrative Examples

Example 3: *Give an example to show that a function continuous in an open interval may fail to be uniformly continuous in the interval.*

Solution: Consider the function f defined on the open interval $]0, 1[$ as follows :

$$f(x) = 1/x, \forall\, x \in]0, 1[\,.$$

First we shall show that f is continuous in $]0, 1[$ *i.e.*, f is continuous at each point c in $]0, 1[$.

We have $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} \dfrac{1}{x} = \dfrac{1}{c} = f(c)$.

So $f(x)$ is continuous at each point c in $]0, 1[$ and hence $f(x)$ is continuous in $]0, 1[$.

Now we shall show that $f(x)$ is not uniformly continuous in $]0, 1[$.

For any $\delta > 0$, we can find a positive integer m such that $1/m < \delta$.

Let $x_1 = 1/m$ and $x_2 = 1/2m$. Then $0 < x_1 < 1$ and $0 < x_2 < 1$ so that $x_1, x_2 \in]0, 1[$.

We have $$|x_1 - x_2| = \left|\frac{1}{m} - \frac{1}{2m}\right| = \left|\frac{1}{2m}\right| = \frac{1}{2m} < \frac{1}{m} < \delta$$

and $$|f(x_1) - f(x_2)| = \left|\frac{1}{x_1} - \frac{1}{x_2}\right| = |m - 2m| = |-m| = m > \frac{1}{2}.$$

[$\because$ m is a + ive integer]

Thus if we take $\varepsilon = \dfrac{1}{2} > 0$, then what ever $\delta > 0$ we try there exist $x_1, x_2 \in]0, 1[$ such that

$$|x_1 - x_2| < \delta \text{ but } |f(x_1) - f(x_2)| > \varepsilon = \frac{1}{2}.$$

In this way for $\varepsilon = \dfrac{1}{2} > 0$, there exists no $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < \varepsilon \text{ whenever } |x_1 - x_2| < \delta, x_1, x_2 \in]0, 1[\,.$$

Hence $f(x) = 1/x$ is not uniformly continuous in $]0, 1[$.

Example 4: *Prove that the function f defined on R^+ as $f(x) = \sin \dfrac{1}{x}$, $\forall\, x > 0$ is continuous but not uniformly continuous on R^+.*

Solution: Let a be any positive real number.

We have $$f(a - 0) = \lim_{h \to 0} f(a - h) = \lim_{h \to 0} \sin \frac{1}{a - h} = \sin \frac{1}{a},$$

$$f(a+0) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} \sin \frac{1}{a+h} = \sin \frac{1}{a}.$$

Also $f(a) = \sin \frac{1}{a}$.

Since $f(a+0) = f(a-0) = f(a)$, f is continuous at a.

But a is an arbitrary point of R^+, so f is continuous on R^+.

It remains to show that f is not uniformly continuous on R^+.

We shall show that no δ works for $\varepsilon = \frac{1}{2}$.

Let δ be any positive number. Take $x_1 = \frac{1}{n\pi}$, $x_2 = \frac{1}{n\pi + (\pi/2)} = \frac{2}{(2n+1)\pi}$,

where n is a positive integer such that $x_1 - x_2 = \frac{1}{n\pi} - \frac{2}{(2n+1)\pi} < \delta$.

(We can always choose such a positive integer n for each positive δ.)

Now $|x_1 - x_2| < \delta$ but $|f(x_1) - f(x_2)| = |\sin n\pi - \sin \frac{1}{2}(2n+1)\pi| = 1 > \varepsilon$.

This shows that for this choice of ε, we are unable to find $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < \varepsilon \text{ whenever } |x_1 - x_2| < \delta, \quad \forall x_1, x_2 \in R^+.$$

Hence f is not uniformly continuous on R^+.

Example 5: *Show that the function f defined by $f(x) = x^3$, is uniformly continuous in $[-2, 2]$.*

Solution: Let $x_1, x_2 \in [-2, 2]$. We have

$$\begin{aligned} |f(x_2) - f(x_1)| = |x_2^3 - x_1^3| &= |(x_2 - x_1)(x_2^2 + x_1^2 + x_1 x_2)| \\ &\le |x_2 - x_1|\{|x_2|^2 + |x_1|^2 + |x_1||x_2|\} \\ &\le 12|x_2 - x_1|. \end{aligned}$$

$[\because x_1, x_2 \in [-2, 2] \Rightarrow |x_1| \le 2 \text{ and } |x_2| \le 2]$

$\therefore$ $|f(x_2) - f(x_1)| < \varepsilon$ whenever $|x_2 - x_1| < \varepsilon / 12$.

Thus given $\varepsilon > 0$, there exists $\delta = \varepsilon / 12$ such that

$$|f(x_2) - f(x_1)| < \varepsilon \text{ whenvever } |x_2 - x_1| < \delta \ \forall\ x_1, x_2 \in [-2, 2].$$

Hence $f(x)$ is uniformly continuous in $[-2, 2]$.

Example 6: *Let $f : R \to R$ given by $f(x) = x^2$. Show that f is not uniformly continuous on* **R**.

Solution: Let $\varepsilon > 0$ be given. The function $f(x)$ will be uniformly continuous on **R** if we are able to find $\delta > 0$ such that

$$x_1, x_2 \in R, |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon. \qquad \ldots(1)$$

The function $f(x)$ will not be uniformly continuous on R if we produce some $\varepsilon > 0$ for which no δ works *i.e.*, for which for every $\delta > 0$, there exist $x_1, x_2 \in R$ such that $|x_1 - x_2| < \delta$ but $|f(x_1) - f(x_2)| \ge \varepsilon$.

So here we shall show that for some given $\varepsilon > 0$, there exists no $\delta > 0$ which satisfies the condition (1).

By the axiom of Archimedes, for any $\delta > 0$, there exists a positive integer n such that

$$n\delta^2 > \varepsilon. \quad \ldots(2)$$

If we take $x_1 = n\delta$ and $x_2 = n\delta + \frac{1}{2}\delta$, then $|x_1 - x_2| = \frac{1}{2}\delta < \delta$,

but $|f(x_1) - f(x_2)| = |x_1^2 - x_2^2| = |x_1 - x_2||x_1 + x_2|$

$$= \frac{1}{2}\delta\left(2n\delta + \frac{1}{2}\delta\right) = n\delta^2 + \frac{1}{4}\delta^2 > \varepsilon, \text{ by (2)}.$$

Hence for these two points x_1, x_2, we would always have $|f(x_1) - f(x_2)| > \varepsilon$, whatever $\delta > 0$ we take. This contradicts (1). Hence f is not uniformly continuous on $\mathbf{R}$.

Example 7: *In the closed interval* $[-1, 1]$ *let* f *be defined by*

$$f(x) = x^2 \sin(1/x^2) \text{ for } x \neq 0 \text{ and } f(0) = 0.$$

In the given interval (i) Is the function bounded ? (ii) Is it continuous ? (iii) Is it uniformly continuous ?

Solution: (i) If $x \in [-1, 1]$ and $x \neq 0$, we have

$$|f(x)| = |x^2 \sin(1/x^2)| = |x^2| \cdot |\sin(1/x^2)|$$

$$= |x|^2 \cdot |\sin(1/x^2)| \leq 1 \cdot 1 = 1.$$

$$[\because |\sin(1/x^2)| \leq 1 \text{ and } -1 \leq x \leq 1 \Rightarrow |x| \leq 1]$$

Also $f(0) = 0 \Rightarrow |f(0)| = 0 < 1$.

Thus $|f(x)| \leq 1, \forall\ x \in [-1, 1]$ and so f is bounded in $[-1, 1]$.

(ii) Let $c \in [-1, 1]$ and $c \neq 0$.

We have $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} x^2 \sin\frac{1}{x^2} = c^2 \sin\frac{1}{c^2} = f(c)$.

$\therefore$ $f(x)$ is continuous at every point c of $[-1, 1]$ if $c \neq 0$.

Now to check the continuity of $f(x)$ at $x = 0$.

We have $f(0 - 0) = \lim\limits_{h \to 0} f(0 - h) = \lim\limits_{h \to 0} f(-h), h > 0$

$$= \lim_{h \to 0} (-h)^2 \sin\left\{\frac{1}{(-h)^2}\right\} = \lim_{h \to 0} h^2 \sin\frac{1}{h^2} = 0.$$

$$\left[\because \lim_{h \to 0} h^2 = 0 \text{ and } \left|\sin\frac{1}{h^2}\right| \leq 1 \text{ if } h \neq 0\right]$$

Again $f(0 + 0) = \lim\limits_{h \to 0} f(0 + h) = \lim\limits_{h \to 0} f(h)$

$$= \lim_{h \to 0} h^2 \sin \frac{1}{h^2} = 0.$$

Also $f(0) = 0$.

Since $f(0-0) = f(0) = f(0+0)$, therefore $f(x)$ is continuous at $x = 0$.

Thus $f(x)$ is continuous at each point of $[-1, 1]$ and so it is continuous in $[-1, 1]$.

(iii) Since f is continuous in the closed interval $[-1, 1]$, therefore it is also uniformly continuous in $[-1, 1]$.

Example 8: *Prove that if f and g are bounded and uniformly continuous on an interval I, then the product function fg is also uniformly continuous on I. Is boundedness of each function necessary for the uniform continuity of the product ? If not so, give a counter example.*

Solution: It is given that the functions f and g are bounded and uniformly continuous on I.

To prove that fg is also uniformly continuous on I.

Since f is bounded on I, therefore there exists $k_1 > 0$ such that

$$|f(x)| \leq k_1, \forall\, x \in I. \qquad \ldots(1)$$

Again g is also bounded on I and so there exists $k_2 > 0$ such that

$$|g(x)| \leq k_2, \forall\, x \in I. \qquad \ldots(2)$$

Now take any given $\varepsilon > 0$.

Since f is uniformly continuous on I, therefore there exists $\delta_1 > 0$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{2k_2},$$

whenever $|x - y| < \delta_1$, where $x, y \in I$. $\qquad \ldots(3)$

Again g is also uniformly continuous on I and so there exists $\delta_2 > 0$ such that

$$|g(x) - g(y)| < \frac{\varepsilon}{2k_1},$$

whenever $|x - y| < \delta_2$, where $x, y \in I$. $\qquad \ldots(4)$

Take $\delta = \min(\delta_1, \delta_2)$. Then from (3) and (4), we have

$$|f(x) - f(y)| < \frac{\varepsilon}{2k_2}$$

and $$|g(x) - g(y)| < \frac{\varepsilon}{2k_1}, \qquad \ldots(5)$$

whenever $|x - y| < \delta$, where $x, y \in I$.

Now $\delta > 0$ is such that if $x, y \in I$ and $|x - y| < \delta$, then

$$|(f\,g)(x) - (f\,g)(y)| = |f(x)\,g(x) - f(y)\,g(y)|$$

$$= |f(x)\,g(x) - f(y)\,g(x) + f(y)\,g(x) - f(y)\,g(y)|$$

$$= |\{f(x) - f(y)\}\,g(x) + \{g(x) - g(y)\}\,f(y)|$$

$$\leq |f(x) - f(y)| . |g(x)| + |g(x) - g(y)| . |f(y)|$$
$$< \frac{\varepsilon}{2k_2} \cdot k_2 + \frac{\varepsilon}{2k_1} \cdot k_1, \text{ using (1), (2) and (5)}$$
$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$|(fg)(x) - (fg)(y)| < \varepsilon$, whenever $|x - y| < \delta$ where $x, y \in I$.

Hence $f\ g$ is uniformly continuous on I.

Second part: Boundedness of each function is not necessary for the uniform continuity of the product as is obvious from the following examples.

Example Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = x, \forall\, x \in \mathbf{R} \text{ and } g(x) = 1, \forall\, x \in \mathbf{R}.$$

Here both the functions f and g are uniformly continuous on R. The function f is not bounded. But the product function $fg = f$ is uniformly continuous on R.

Example Consider the functions f and g defined on $[0, \infty[$ by

$$f(x) = g(x) = \sqrt{x}, \forall\, x \in [0, \infty[.$$

Both the functions f and g are not bounded. The product function fg is given by

$$(fg)(x) = x, \forall\, x \in [0, \infty[$$

which is obviously uniformly continuous on $[0, \infty[$.

Example 9: *Find the points of discontinuity of the function* $f : \mathbf{R} \to \mathbf{R}$ *defined by*

$$f(x) = \lim_{m \to \infty} \left\{ \lim_{n \to \infty} (\cos m!\, \pi x)^{2n} \right\}.$$

Solution: Let x be a rational number, say p/q, where p, q are integers prime to each other. Choosing m sufficiently large, $m!\, \pi x$ can be made an integral multiple of π so that $\cos(m!\, \pi x) = \pm 1$.

$$\therefore \quad \lim_{n \to \infty} (\cos m!\, \pi x)^{2n} = \lim_{n \to \infty} (\pm 1)^{2n} = 1.$$

Hence $f(x) = 1$, when x is rational.

If x is irrational, then for any integral value of m, $\cos m!\, \pi x$ will always lie between -1 and $+1$.

$$\therefore \quad (\cos m!\, \pi x)^{2n} = (r_m)^{2n} \text{ where } |r_m| < 1, \text{ for a fixed value of } m.$$

Hence $f(x) = \lim_{m \to \infty} \lim_{n \to \infty} (r_m)^{2n} = 0$, when x is irrational.

Since $f(x)$ is 1 for rational values of x and 0 for irrational values of x, f is totally discontinuous *i.e.*, discontinuous for every value of x. This is so because at any point a (rational or irrational) the limits $f(a + 0)$ as well as $f(a - 0)$ do not exist. Note that there are infinite number of rational and infinite number of irrational points

in any neighbourhood $]a-h, a+h[$ of a, however small h may be and at these points the functional values differ widely.

Here $\overline{f(a+0)} = 1$, $\underline{f(a+0)} = 0$, $\overline{f(a-0)} = 1$, $\underline{f(a-0)} = 0$.

Hence there is a discontinuity of second kind at every point.

Example 10: *Show that the function* $f : \mathrm{R} \to \mathrm{R}$ *defined by*

$$f(x) = \lim_{n \to \infty} \left[\lim_{t \to 0} \frac{\sin^2 (n ! \pi x)}{\sin^2 (n ! \pi x) + t^2} \right]$$

is equal to 0 *when* x *is rational and to* 1 *when* x *is irrational. Hence show that the function is totally discontinuous.*

Solution: Let x be a rational number say p/q, where p, q are integers prime to each other. By taking n sufficiently large, $n ! \pi x$ can be made an integral multiple of π so that $\sin (n ! \pi x) = 0$.

Hence $f(x) = \lim_{t \to 0} \frac{0}{0 + t^2} = 0$, when x is rational.

If x is irrational, then $0 < \sin^2 (n ! \pi x) < 1$.

Hence
$$f(x) = \lim_{n \to \infty} \lim_{t \to 0} \frac{1}{1 + t^2 / \sin^2 (n ! \pi x)}$$
$$= \frac{1}{1+0} = 1, \text{ when } x \text{ is irrational.}$$

Thus $f(x) = 0$ when x is rational and 1 when x is irrational. Hence f is totally discontinuous.

Example 11: *Show that the function* $f : \mathrm{R} \to \mathrm{R}$ *defined by*

$$f(x) = \lim_{t \to \infty} \frac{(1 + \sin \pi x)^t - 1}{(1 + \sin \pi x)^t + 1}$$

is discontinuous at the points $x = 0, 1, 2, \ldots, n, \ldots$

Solution: For $x = 0, 1, 2, 3, \ldots, n, \ldots$ we have $\sin \pi x = 0$, so that at these values of x, $f(x) = \lim_{n \to \infty} \frac{(1+0)^t - 1}{(1+0)^t + 1} = 0$.

Now if $2m < x < 2m + 1$ (m being an integer), then $\sin \pi x$ is positive. Hence for such values of x, we have

$$f(x) = \lim_{t \to \infty} \frac{1 - \frac{1}{(1 + \sin \pi x)^t}}{1 + \frac{1}{(1 + \sin \pi x)^t}} = \frac{1 - \frac{1}{\infty}}{1 + \frac{1}{\infty}} = 1. \qquad \left[\because \frac{1}{\infty} = 0 \right]$$

Again if $2m + 1 < x < 2m + 2$, $\sin \pi x$ is negative and so

$$\lim_{t \to \infty} (1 + \sin \pi x)^t = 0.$$

Hence for such values of x, $f(x) = \dfrac{0-1}{0+1} = -1$.

From the values of $f(x)$ mentioned above, we observe that

(i) if x is an even integer, then

$$f(x) = 0,\ f(x+0) = 1 \text{ and } f(x-0) = -1$$

and (ii) if x is an odd integer, then

$$f(x) = 0,\ f(x+0) = -1 \text{ and } f(x-0) = 1.$$

Hence f has discontinuities of the first kind at

$$x = 0, 1, 2, \ldots, n, \ldots$$

Example 12: *Let a function $f : \mathbb{R} \to \mathbb{R}$ satisfy the equation*

$f(x+y) = f(x) + f(y),\ \forall\ x, y \in \mathbb{R}$. *Show that*

(i) If f is continuous at the point $x = a$, then it is continuous for all $x \in \mathbb{R}$.

(ii) If f is continuous then $f(x) = kx$, for some constant k.

Solution: **(i)** Let f be continuous at the point $x = a$.

We have $f(a+0) = \lim\limits_{h \to 0} f(a+h) = \lim\limits_{h \to 0} [f(a) + f(h)]$,

by definition of f

$$= \lim_{h \to 0} f(a) + \lim_{h \to 0} f(h) = f(a) + \lim_{h \to 0} f(h).$$

Since f is continuous at a, we have $f(a) = f(a+0)$

$$\Rightarrow \quad f(a) = f(a) + \lim_{h \to 0} f(h) \Rightarrow \lim_{h \to 0} f(h) = 0. \qquad \ldots(1)$$

Similarly $f(a) = f(a-0) \Rightarrow \lim\limits_{h \to 0} f(-h) = 0.$ $\qquad \ldots(2)$

Now let α be any real number. We shall show that f is continuous at α.

We have $f(\alpha+0) = \lim\limits_{h \to 0} f(\alpha+h) = \lim\limits_{h \to 0} [f(\alpha) + f(h)]$

$$= \lim_{h \to 0} f(\alpha) + \lim_{h \to 0} f(h) = f(\alpha), \text{ using } (1).$$

Similarly, using (2) we have $f(\alpha - 0) = f(\alpha)$.

Thus $f(\alpha+0) = f(\alpha-0) = f(\alpha)$.

Hence f is continuous at $x = \alpha$. Since α is arbitrary so f is continuous for all $x \in \mathbb{R}$.

(ii) Here we consider the following cases :

Case I: Let $x = 0$. Since $f(x+x) = f(x) + f(x)$, we have

$$f(0) = f(0+0) = f(0) + f(0) \text{ and so } f(0) = 0.$$

Hence $f(x) = kx$ for every constant k in this case.

Case II: Let x be any positive integer. Then

$$f(x) = f(1 + 1 + \ldots x \text{ times}) = f(1) + f(1) + \ldots x \text{ times}$$
$$= xf(1) = kx, \text{ where } k = f(1) \textit{ i.e., a constant.}$$

Case III: Let x be any negative integer. Put $x = -y$ so that y is a positive integer.

Now $0 = f(0)$, by Case I
$$= f(y - y) = f(y) + f(-y).$$
$\therefore$ $f(-y) = -f(y)$.

Hence $f(x) = -f(y) = -yf(1)$, by Case II
$$= kx, \text{ where } k = f(1) \textit{ i.e., a constant.}$$

Case IV: Let x be any rational number. Put $x = p/q$ where q is a positive integer and p is any integer, positive, negative or zero.

Now $f\left(q \cdot \frac{p}{q}\right) = f\left(\frac{p}{q} + \frac{p}{q} + \ldots q \text{ times}\right)$

$$= f\left(\frac{p}{q}\right) + f\left(\frac{p}{q}\right) + \ldots q \text{ times.}$$

$\therefore$ $f(p) = qf\left(\frac{p}{q}\right)$. But $f(p) = kp$, by previous cases.

Hence $qf\left(\frac{p}{q}\right) = kp$ or $f\left(\frac{p}{q}\right) = k\frac{p}{q}$

or $f(x) = kx$, in this case also.

Case V: Finally let x be any real number. Let $< x_n >$ be a sequence of rational numbers converging to x. Since f is continuous at x, the sequence $< f(x_n) >$ converges to $f(x)$. Thus, we have

$$\lim_{n \to \infty} x_n = x \quad \text{and} \quad \lim_{n \to \infty} f(x_n) = f(x)$$

Since x_n is a rational number, we have by case IV,

$$f(x_n) = kx_n.$$
$\therefore$ $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} kx_n = k \lim_{n \to \infty} x_n = kx$

or $f(x) = kx$.

Hence $f(x) = kx$, $\forall\ x \in R$.

4.5 Meaning of the Sign of Derivative

Let $f'(c) > 0$ where c is an interior point of the domain of the fuction f; then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) > 0.$$

If $\varepsilon > 0$ be any number $< f'(c)$, there exists $\delta > 0$ such that

$$|x - c| \le \delta \Rightarrow \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \varepsilon$$

i.e., $x \in [c - \delta, c + \delta], x \neq c$

$$\Rightarrow \quad \frac{f(x) - f(c)}{x - c} \in]\, f'(c) - \varepsilon, f'(c) + \varepsilon\, [\qquad \ldots(1)$$

Since ε was chosen smaller than $f'(c)$ we conclude from (1) that

$$\frac{f(x) - f(c)}{x - c} > 0 \text{ when } x \in [c - \delta, c + \delta], x \neq c.$$

We then have, $f(x) - f(c) > 0$ when $c < x \le c + \delta$

and $\quad f(x) - f(c) < 0$ when $c - \delta \le x < c$.

Thus we have shown that if $f'(c) > 0$, there exists a neighbourhood $[c - \delta, c + \delta]$ of c such that

$$f(x) > f(c)\ \forall\, x \in]\, c, c + \delta\, [\text{ and } f(x) < f(c)\ \forall\, x \in [c - \delta, c]\,.$$

If $f'(c) < 0$, it can be similarly shown that there exists a neighbourhood $]\, c - \delta, c + \delta\, [$ of c such that

$$f(x) > f(c)\ \forall\, x \in [c - \delta, c\, [\text{ and } f(x) < f(c)\ \forall\, x \in]\, c, c + \delta\, [\,.$$

Similarly, it can be shown for the end points a and b that there exist intervals, $]\, a, a + \delta]$ and $[\, b - \delta, b\, [$ such that

$$f'(a) > 0 \Rightarrow f(x) > f(a)\ \forall\, x \in]\, a, a + \delta\,]$$

$$f'(a) < 0 \Rightarrow f(x) < f(a)\ \forall\, x \in]\, a, a + \delta\,]$$

$$f'(b) > 0 \Rightarrow f(x) < f(b)\ \forall\, x \in [\, b - \delta, b\, [$$

and $\quad f'(b) < 0 \Rightarrow f(x) > f(b)\ \forall\, x \in |\, b - \delta, b\, [$

4.6 Intermediate value Theorem for Derivatives or Darboux Theorem

Theorem: *If f is finitely differentiable in a closed interval $[a, b]$ and $f'(a)$, $f'(b)$ are of opposite signs, then there exists at least one point $c \in]\, a, b\, [$ such that $f'(c) = 0$.*

Proof : For definiteness, suppose that $f'(a) > 0$ and $f'(b) < 0$. Then there exist intervals $]\, a, a + h\,]$ and $[b - h, b[$, h being positive such that

$$f(x) > f(a)\ \forall\, x \in]\, a, a + h\,]$$

$$f(x) > f(b)\ \forall\, x \in [\, b - h, b\,]\,.$$

Further f being finitely differentiable, is continuous in $[a, b]$ and hence it is bounded on $[a, b]$ and attains its supremum and infimum at least once in $[a, b]$.

Thus if M be the least upper bound of f in $[a, b]$, then there exists $c \in [a, b]$ such that $f(c) = M$. It is clear from (1) and (2) that the upper bound is not attained at the end points a asnd b so that $c \in]a, b[$. We shall prove that $f'(c) = 0$.

If $f'(c) > 0$, then there exists an interval $]c, c + h]$, $h > 0$, such that

$$f(x) > f(c) = M \ \forall x \in]c, c + h],$$

which is not possible since M is the least upper bound of the function $f(x)$ in $[a, b]$.

If $f'(c) < 0$, then there exists an interval $[c - h, c[$, $h > 0$ such that

$$f(x) > f(c) = M \ \forall \ x \in [c - h, c[,$$

which is again not possible for the same reason as mentioned earlier.

Hence we conclude that $f'(c) = 0$.

Corollary 1: *If f is finitely differentiable on $[a, b]$ and $f'(a) \neq f'(b)$ and k is any number lying between $f'(a)$ and $f'(b)$, then there exists at least one point $c \in]a, b[$ such that $f'(c) = k$. In other words $f'(x)$ takes every value intermediate between $f'(a)$ and $f'(b)$.*

Proof : Let k be any real number lying between $f'(a)$ and $f'(b)$. Define a function ϕ by $\phi(x) = f(x) - kx$. Since f and kx both are finitely differentiable in $[a, b]$, ϕ is also finitely differentiable in $[a, b]$.

We have $\phi'(x) = f'(x) - k \ \forall x \in [a, b]$.

Hence $\phi'(a) = f'(a) - k$ and $\phi'(b) = f'(b) - k$.

Since k lies between $f'(a)$ and $f'(b)$, $\phi'(a)$ and $\phi'(b)$ are of opposite signs. Hence by the above theorem, there exists at least one point c of $]a, b[$ such that $\phi'(c) = 0$

or $\qquad f'(c) - k = 0$ or $f'(c) = k$

Corollary 2: *If f is finitely differentiable on $[a, b]$ and $f'(x) \neq 0$ for any $x \in]a, b[$, then $f'(x)$ retains the same sign, positive or negative in $]a, b[$ i.e., $f'(x)$ is either positive or negative for all values of $x \in]a, b[$.*

Proof : If possible, let x_1 and x_2 be two distinct elements of $]a, b[$ and let $f'(x_1)$ and $f'(x_2)$ be of opposite signs. Then by the above theorem, there exists $c \in]a, b[$ such that $f'(c) = 0$ which contradicts the fact that $f'(x) \neq 0 \ \forall x \in]a, b[$. Hence $f'(x)$ must retain the same sign in $]a, b[$.

Corollary 3: *If f is finitely differentiable on $I = [a, b]$, then the range $f'(I)$ of f' on I is either an interval or a singleton.*

Proof: Let $f'(I) = J$ and let p_1, p_2 be two distinct elements of J. Then there exist two distinct elements x_1, x_2 of I such that $f'(x_1) = p_1$ and $f'(x_2) = p_2$. Suppose that $x_1 < x_2$. Then

$$[x_1, x_2] \subset [a, b].$$

Let p be any real number lying between p_1 and p_2. Then, by the above corollary 1, there exists $c \in]x_1, x_2[\subset [a, b]$ such that $f'(c) = p$. Hence $p \in J$. This shows that every number lying between p_1 and p_2 belongs to J. Hence J is an interval.

If J does not contain at least two distinct elements, then obviously it is a singleton.

Comprehensive Exercise 1

1. Let f be the function defined on $[-1, 1]$ by

   $f(x) = x$, if x is irrational, $f(x) = 0$, if x is rational.

   Show that f is continuous only at $x = 0$.

2. Let $f : R \to R$ be such that

   $f(x) = x$ when x is irrational

   $= -x$ when x is rational.

   Show that $f(x)$ is continuous only at $x = 0$.

3. Show that the function f defined on R by

   $f(x) = 1$ when x is rational, $f(x) = -1$ when x is irrational

   is discontinuous at every point of R.

4. Show that $f : R \to R$ given by $f(x) = x^2$ is continuous but not uniformly continuous on R.

5. Explain fully uniform continuity and discuss the uniform continuity of the function $f(x) = x^2, \forall x \in R$ in $]0, 1[$.

6. Let $f :]-1, 1[\to [0, 1]$ be a function defined by $f(x) = x^2$. Using definition show that f is uniformly continuous on its domain.

7. Show that the function $f(x) = 1/x, x > 0$ is continuous in $(0, 1)$ but not uniformly continuous.

8. Define uniform continuity and show that the function $f(x) = x^2 + 3x, x \in [-1, 1]$ is uniformly continuous in $[-1, 1]$.

9. Show that a function which is continuous in a closed interval is bounded in that interval. Verify the theorem for

   $$f(x) = \cos x \text{ in } [-\frac{1}{2}\pi, \frac{1}{2}\pi] .$$

10. Give an example to show that a function continuous on an open interval need not be bounded on that interval.

11. If the function f is continuous in the closed interval $[a, b]$, prove that it attains its least upper bound and greatest lower bound in $[a, b]$. Verify the theorem for the function $f(x) = \sin x$ in $[0, 2\pi]$.

12. If f is continuous in $[a, b]$ and $f(a) . f(b) < 0$, show that $f(c) = 0$ for at least one $c \in [a, b]$.

13. Let f be continuous on $[a, b]$ and suppose that $f(x) = 0$ for every rational x in $[a, b]$. Prove that $f(x) = 0$ for all x in $[a, b]$.

14. If a function is continuous on a closed interval $[a, b]$, then it attains its bounds at least once in $[a, b]$. Give an example of a function which is continuous and bounded, and attains its supremum but does not attain its infimum.

15. Prove that the identity mapping of any interval I is uniformly continuous on I.

16. Show that the function $f : \mathrm{R} \to \mathrm{R}$ defined by

$$f(x) = \frac{1}{1 + e^{1/\sin(n!\,\pi x)}}$$

can be made discontinuous at any rational point in the interval $[0, 1]$ by a proper choice of n.

17. If $f : \mathrm{R} \to \mathrm{R}$ is a continuous function and satisfies the relation $f(x + y) = f(x)\, f(y), \forall\, x, y \in \mathrm{R}$, then either $f(x) = 0 \quad \forall\, x \in \mathrm{R}$ or there exists an $a > 0$ such that

$$f(x) = a^x, \forall\, x \in \mathrm{R}.$$

18. Show that a function f is continuous at a iff for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x_1, x_2 \in]a - \delta, a + \delta[\Rightarrow |f(x_1) - f(x_2)| < \varepsilon.$$

19. Discuss the nature of discontinuity of the function f, defined by

$$f(x) = \lim_{n \to \infty} \frac{\log(2 + x) - x^{2n} \sin x}{1 + x^{2n}}$$

at $x = 1$. Show that $f(0)$ and $f(\pi/2)$ differ in sign.

20. Let f and g be continuous on $[a, b]$ and let $f(a) < g(a)$ but $f(b) > g(b)$. Prove that $f(c) = g(c)$ for some $c \in]a, b[$.

21. Let $f : \mathrm{R} \to \mathrm{R}$ be continuous and let f be zero on a dense set (*i.e.*, a set whose intersection with every interval is non-empty). Then f is identically zero.

22. Determine the discontinuities of the function $f : \mathrm{R} \to \mathrm{R}$ defined by

$$f(x) = \lim_{n \to \infty} \frac{\{1 + \sin(\pi/x)\}^n - 1}{\{1 + \sin(\pi/x)\}^n + 1}, 0 < x \le 1.$$

23. Find the type of discontinuity at $x = 1$ for the function

$$f(x) = \lim_{n \to \infty} \frac{x^n}{1 + x^n e^x}.$$

24. Show that the function

$$\phi(x) = \lim_{n \to \infty} \frac{x^{2n+2} - \cos x}{x^{2n} + 1}$$

does not vanish anywhere in the interval $[0, 2]$ though $\phi(0)$ and $\phi(2)$ differ in sign. Discuss the continuity of the function at $x = 1$.

25. Examine for continuity the function f defined by

$$f(x) = \lim_{n \to \infty} \frac{e^x - x^n \sin x}{1 + x^n} \left(0 \le x \le \frac{\pi}{2}\right)$$

at $x = 1$. Explain why the function f does not vanish anywhere in $[0, \pi/2]$ although $f(0) \cdot f(\pi/2) < 0$.

Answers 1

5. Uniformly continuous in $]0, 1[$

19. Discontinuity of the first kind at $x = 1$

22. f has a discontinuity of second kind at $x = 0$ and ordinary discontinuous at $x = 1, \frac{1}{2}, \frac{1}{3}, \ldots$

23. Discontinuous of the first kind

25. Discontinuous at $x = 1$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The necessary and sufficient condition for a function f defined on $I \subset R$ to be continuous at $a \in I$ is that for each sequence $< a_n >$ in I which converges to a, we have $\lim_{n \to \infty} f(a_n) =$

(a) $f'(a)$ (b) $f^2(a)$

(c) $f(a)$ (d) none of these

2. The function f defined on $[-1, 1]$ by

$f(x) = x$, if x is irrational

$f(x) = 0$, if x is rational.

is continuous at $x =$

(a) 0 (b) 1

(c) $-1, 1$ (d) none of these

3. At $x = 1$, the function $f(x) = \lim_{n \to \infty} \frac{x^n}{1 + x^n e^x}$

(a) is continous
(b) has discontinuity of first kind
(c) has discontinuity of second kind
(d) is uniformly continuous.

4. For all real values of x, the function $f(x) = x^2$ is

(a) continuous (b) discontinuous
(c) uniformly continuous (d) not uniformly continuous.

5. In the interval $[-1, 1]$ the function f defined by $f(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $f(0) = 0$ is

(a) bounded (b) uniformly continuous
(c) unbounded (d) none of these

Fill In The Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. If f and g are continuous at $a \in I$ then $f + g$ is …… at a.
2. If a function $f(x)$ is continuous in a closed interval $[a, b]$, it is …… in that interval.
3. If f is continuous in a closed and bounded interval I, it is …… on I.
4. The function $f : R \to R$ defined by

$$f(x) = \lim_{t \to \infty} \frac{(1 + \sin \pi x)^t - 1}{(1 + \sin \pi x)^t + 1}$$

is …… at the points $x = 0, 1, 2, \ldots$

5. The function $f(x) = x^3$ is …… continuous in $[-2, 2]$.
6. The function f defined on R^+ as $f(x) = \sin \frac{1}{x}$, $\forall\ x > 0$ is …… but not …… on R^+.
7. The function $f(x) = \lim_{n \to \infty} \frac{e^x - x^n \sin x}{1 + x^n} \left(0 \leq x \leq \frac{\pi}{2}\right)$ is …… at $x = 1$.

True or False

Write 'T' for true and 'F' for false statement.

1. If a function $f(x)$ is continuous in a closed interval $[a, b]$, then it is bounded in $[a, b]$.
2. If a function $f(x)$ is uniformly continuous in an interval I, then it is also continuous in I.

3. If a function $f(x)$ is continuous in an open interval I, then it is also uniformly continuous in I.
4. If a function $f(x)$ is continuous in a closed interval $[a, b]$, then it is also uniformly continuous in $[a, b]$.
5. If a function f is continuous at a, then $|f|$ is also continuous at a.
6. A function continuous on an open interval is bounded on that interval.
7. The function $f(x) = x^2 + 3x$, $x \in [-1, 1]$ is uniformly continuous in this interval.
8. The function $f(x) = x^2$ is uniformly continuous on $]0, 1[$.
9. A function defined on $[0, 1]$ and given by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1 - x, & \text{if } x \text{ is irrational} \end{cases},$$

is discontinuous at $x = \frac{1}{2}$.

Answers

Multiple Choice Questions

1. (c) 2. (a) 3. (b) 4. (a), (d) 5. (a), (b)

Fill in the Blank(s)

1. continuous
2. bounded
3. uniformly continuous
4. discontinuous
5. uniformly
6. continuous, uniformly continuous
7. discontinuous

True or False

1. *T* 2. *T* 3. *F* 4. *T* 5. *T*
6. *F* 7. *T* 8. *T* 9. *F*

□

Chapter

5

Limit and Continuity of Functions of Two Variables

5.1 Functions of Two Variables

Let R denote the set of real numbers.

We have $R^2 = R \times R = \{(x, y) : x \in R, y \in R\}$.

Let $A \subseteq R^2$.

If $f : A \to R$, then f is called a real valued function of two real variables. The set A is called the domain of this function and the range of this function is some subset of R.

Definition: *We say that $z = f(x, y)$ is a real valued function of two independent real variables x and y, if for each pair of values of x and y of a certain set $\{(x, y)\}$ over which the point (x, y) ranges, there exists a unique real value of z.*

For example, $z = x^2 + y^2 - xy$, $z = x \sin y + y \cos x$ etc., are real valued functions of two real independent variables x and y.

5.2 Neighbourhood of a Point

Rectangular neighbourhood of a point:

Let (a, b) be any point of R^2.

The set $N(a, b) = \{(x, y) : x \in R, y \in R, a - h < x < a + h, b - k < y < b + k\}$ is called a rectangular neighbourhood of the point (a, b), where h and k are arbitrarily small positive real numbers.

In particular,

$$N(a, b) = \{(x, y) : x \in R, y \in R, a - \delta < x < a + \delta, b - \delta < y < b + \delta\}$$

is a square neighbourhood of the point (a, b), where δ is an arbitrarily small positive real number.

A neighbourhood of a point (a, b) is denoted by $N(a, b)$.

We shall often write the word neighbourhood in the abbreviated form as nhd or nbd.

Circular neighbourhood of a point:

The set $N(a, b) = \{(x, y) : x \in R, y \in R \text{ and } \sqrt{(x - a)^2 + (y - b)^2} < \delta\}$ is called a circular neighbourhood of the point (a, b), where δ is an arbitrarily small positive real number.

It can be seen that every circular neighbourhood of a point contains a rectangular neighbourhood and vice versa.

Deleted neighbourhood of a point: If from the neighbourhood $N(a, b)$ of the point (a, b), we delete the point (a, b), then the remaining set is called a deleted nbd of (a, b).

5.3 Simultaneous and Iterated Limits

For a function $f(x, y)$ of two variables x and y, we can define several kinds of limits. If (a, b) is the limiting point of a set of values in two dimensional space, then we have the limits

$$\lim_{(x, y) \to (a, b)} f(x, y), \quad \lim_{x \to a} \lim_{y \to b} f(x, y), \quad \lim_{y \to b} \lim_{x \to a} f(x, y).$$

The first type of limit is known as **'simultaneous limit'** or **'double limit'** and the last two types are known as **'iterated limits'** or **'repeated limits'**.

The simultaneous limit, $\lim_{(x, y) \to (a, b)} f(x, y)$ is also written as $\lim_{\substack{x \to a \\ y \to b}} f(x, y)$.

The notion of an iterated limit is nothing but a limit of a limit and it can be obtained as in the case of a function of a single variable. However, the notion of

simultaneous limit is quite different from that of the limit of a function of single variable.

Simultaneous limit

Definition: *We say that the simultaneous limit of* $f(x, y)$ *exists and is equal to* A *as* $(x, y) \to (a, b)$, *if for every* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that*

$$|f(x, y) - A| < \varepsilon$$

for all values of x *and* y *in the nbd of* (a, b) *defined by*

$$|x - a| < \delta, |y - b| < \delta.$$

Non-existence of simultaneous limit:

For the existence of simultaneous limit, not only must we have same limiting value if the variable point (x, y) approaches the limiting point (a, b) through any set of values dense at the point, but we must also have the same limiting value as the variable point approaches its limiting position along any curve whatsoever.

Thus if we can find two methods of approach to the limiting point which give different limiting values then we can conclude that the simultaneous limit does not exist.

Illustrative Examples

Example 1: *Show that the simultaneous limit* $\lim\limits_{(x, y) \to (0, 0)} \dfrac{xy^3}{x^2 + y^6}$ *does not exist.*

Solution: First let (x, y) approach $(0, 0)$ along the line $y = x$ which is a line through the origin. For this, we first put $y = x$ in the function and then allow x to approach 0. We thus get

$$\lim_{x \to 0} \frac{x^4}{x^2 + x^6} = \lim_{x \to 0} \frac{x^2}{1 + x^4} = 0.$$

Again, let (x, y) approach $(0, 0)$ along the curve $x = y^3$. For this, we put $x = y^3$ in the function and then allow y to approach zero. In this case, we get

$$\lim_{y \to 0} \frac{y^6}{y^6 + y^6} = \frac{1}{2}.$$

Since two methods of approach to the limiting point give different limiting values, the simultaneous limit does not exist.

Note 1: It can be seen that if we approach the origin along any line $y = mx$, the limit comes out to be zero. But we cannot conclude that the simultaneous limit exists and is 0 as we have seen that the limit along $x = y^3$ comes out to be $\frac{1}{2}$.

Note 2: The existence of the simultaneous limit, $\lim\limits_{(x, y) \to (a, b)} f(x, y)$ implies

that the single limits $\lim_{x \to a} f(x, b)$, $\lim_{y \to b} f(a, y)$ also exist. However, it does not follow that the single limits $\lim_{x \to a} f(x, y)$, $\lim_{y \to b} f(x, y)$ exist for $y \neq b$, $x \neq a$ respectively.

It is shown by the following example :

Example 2: *Show that the simultaneous limit* $\lim_{(x, y) \to (0, 0)} y \sin \frac{1}{x}$ *exists and is equal to* 0 *but the single limit* $\lim_{x \to 0} y_1 \sin \frac{1}{x}$ $(y_1 \neq 0)$ *does not exist.*

Solution: First we shall show that the simultaneous limit $\lim_{(x, y) \to (0, 0)} y \sin \frac{1}{x}$ exists.

Let $\varepsilon > 0$ be given. Taking $\delta = \varepsilon$, we see that for all x, y satisfying the inequalities $0 < |x| < \delta, 0 < |y| < \delta$, we have

$$\left| y \sin \frac{1}{x} - 0 \right| = \left| y \sin \frac{1}{x} \right| = |y| \left| \sin \frac{1}{x} \right|$$

$$\leq |y| \qquad \left[\because \left| \sin \frac{1}{x} \right| \leq 1 \right]$$

$$< \delta = \varepsilon.$$

$\therefore \lim_{(x, y) \to (0, 0)} y \sin \frac{1}{x} = 0$ *i.e.*, the simultaneous limit exists.

But, for any constant value of $y = y_1 \neq 0$, we get

$\lim_{x \to 0} y_1 \sin \frac{1}{x} = y_1 \lim_{x \to 0} \sin \frac{1}{x}$, which does not exist.

Repeated limits or Iterated limits: If a function f is defined in some deleted neighbourhood of (a, b), then the limit $\lim_{y \to b} f(x, y)$, if it exists, is a function of x, say $\phi(x)$. If then the limit, $\lim_{x \to a} \phi(x)$ exists and is equal to λ, we write

$$\lim_{x \to a} \lim_{y \to b} f(x, y) = \lambda$$

and say that λ is a *repeated limit* of f as $y \to b, x \to a$.

If we change the order of taking the limits, we get the other repeated limit

$$\lim_{y \to b} \lim_{x \to a} f(x, y) = \mu \text{ (say)}$$

when first $x \to a$ and then $y \to b$.

These two limits may or may not be equal.

Note: In case the simultaneous limit exists, these two repeated limits if they exist are necessarily equal but the converse is not true. However, *if the repeated limits are not equal, the smultaneous limit cannot exist.*

Example 3: *Let* $f(x, y) = \dfrac{y - x}{y + x} \cdot \dfrac{1 + x}{1 + y}$, $(x, y) \neq (0, 0)$. *Show that the two repeated limits exist at the origin but are unequal.*

Solution: We have $\lim\limits_{x \to 0} \lim\limits_{y \to 0} f(x, y) = \lim\limits_{x \to 0} -\left(\dfrac{1 + x}{1}\right) = -1.$

Again, $\lim\limits_{y \to 0} \lim\limits_{x \to 0} f(x, y) = \lim\limits_{y \to 0} \left(\dfrac{1}{1 + y}\right) = 1.$

Thus the two repeated limits exist at the origin but are not equal.

Now we give a theorem which is only a **sufficient** but not a **necessary** condition for the interchange of the order of iterated limits.

Theorem 1: *Let the simultaneous limit* $\lim\limits_{(x, y) \to (a, b)} f(x, y)$ *exist and be equal to* A *and let the limit* $\lim\limits_{x \to a} f(x, y)$ *exist for each constant value of* y *in the nhd of* b *and likewise let the limit* $\lim\limits_{y \to b} f(x, y)$ *exist for each constant value of* x *in the nhd of* a. *Then*

$$\lim_{x \to a} \lim_{y \to b} f(x, y) = \lim_{y \to b} \lim_{x \to a} f(x, y) = A.$$

Proof: Since the limit $\lim\limits_{x \to a} f(x, y)$ exists for each value of y in the nhd of b, we get a set of these limiting values which defines a function of y, say $F(a, y)$. Thus, we can write

$$\lim_{x \to a} f(x, y) = F(a, y), \qquad \ldots(1)$$

where $F(a, y)$ may or may not be identical with $f(a, y)$.

Let $\varepsilon > 0$ be given. By virtue of (1), there exists $\delta_1 > 0$ such that for each value of y in a nhd of b, say in the nhd defined by $|y - b| < \delta_0$, we get

$$|F(a, y) - f(x, y)| < \varepsilon / 2 \qquad \ldots(2)$$

for all x satisfying the inequality $|x - a| < \delta_1$.

Again, since the simultaneous limit of $f(x, y)$ at (a, b) is A, there exists $\delta_2 > 0$ such that

$$|f(x, y) - A| < \varepsilon / 2 \qquad \ldots(3)$$

for all x, y satisfying $|x - a| < \delta_2$, $|y - b| < \delta_2$.

Taking $\delta = \min(\delta_0, \delta_2)$, we get for all y satisfying $|y - b| < \delta$,

$$|F(a, y) - A| = |F(a, y) - f(x, y) + f(x, y) - A|$$

$$\leq |F(a, y) - f(x, y)| + |f(x, y) - A|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ by (2) and (3).}$$

It gives that $\lim_{y \to b} F(a, y) = A$

or $\lim_{y \to b} \lim_{x \to a} f(x, y) = A$, using (1).

Similarly, we can prove that

$$\lim_{x \to a} \lim_{y \to b} f(x, y) = A.$$

Thus $\lim_{(x, y) \to (a, b)} f(x, y) = \lim_{x \to a} \lim_{y \to b} f(x, y) = \lim_{y \to b} \lim_{x \to a} f(x, y)$.

Example 4: *Give an example to show that the order of iterated limits can be interchanged although the simultaneous limit does not exist.*

Solution: We consider the function $f(x, y) = \frac{xy}{x^2 + y^2}$ for simultaneous and iterated limits at $(0, 0)$. First let (x, y) approach $(0, 0)$ along the line $y = x$. Putting $y = x$ and then making x approach 0, we get

$$\lim_{x \to 0} \frac{x^2}{x^2 + x^2} = \frac{1}{2}.$$

Again, let (x, y) approach $(0, 0)$ along y-axis. Putting $x = 0$, we get

$$\lim_{y \to 0} \frac{0}{0 + y^2} = 0.$$

Since two methods of approach give different results, the simultaneous limit does not exist.

For iterated limits, we get

$$\lim_{x \to 0} \lim_{y \to 0} \frac{xy}{x^2 + y^2} = \lim_{x \to 0} \frac{x \cdot 0}{x^2 + 0} = 0$$

and $$\lim_{y \to 0} \lim_{x \to 0} \frac{xy}{x^2 + y^2} = \lim_{y \to 0} \frac{0 \cdot y}{0 + y^2} = 0.$$

Thus the order of iterated limits can be interchanged.

Example 5: *Show that* $\lim_{(x, y) \to (0, 0)} xy \frac{x^2 - y^2}{x^2 + y^2} = 0.$

Solution: Let $\varepsilon > 0$ be given.

For all $(x, y) \neq (0, 0)$, we have

$$\left| xy\,\frac{(x^2 - y^2)}{x^2 + y^2} - 0 \right| = \left| xy\,\frac{(x^2 - y^2)}{x^2 + y^2} \right| = |x^2 - y^2| \cdot \frac{|xy|}{x^2 + y^2} \quad \ldots(1)$$

Now for all x and y, we have $(|x| - |y|)^2 \geq 0$

$\Rightarrow$ $|x|^2 + |y|^2 - 2|x| \cdot |y| \geq 0$

$\Rightarrow$ $x^2 + y^2 \geq 2|xy|$

$\Rightarrow$ $\dfrac{|xy|}{x^2 + y^2} \leq \dfrac{1}{2}$, if $(x, y) \neq (0, 0)$. $\quad \ldots(2)$

$\therefore$ from (1) and (2), for all $(x, y) \neq (0, 0)$, we have

$$\left| xy\,\frac{(x^2 - y^2)}{x^2 + y^2} - 0 \right| \leq \frac{1}{2}|x^2 - y^2| \leq \frac{1}{2}|x^2 + y^2|$$

$$= \frac{1}{2}(x^2 + y^2) < \varepsilon,$$

if $x^2 < \varepsilon$ and $y^2 < \varepsilon$ *i.e.*, if $|x| < \sqrt{\varepsilon}$ and $|y| < \sqrt{\varepsilon}$.

Now if we take $\delta = \sqrt{\varepsilon}$, then we see that for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{xy\,(x^2 - y^2)}{x^2 + y^2} - 0 \right| < \varepsilon, \text{ whenever } |x| < \delta \text{ and } |y| < \delta.$$

$\therefore$ $\lim\limits_{(x, y) \to (0, 0)} xy\,\dfrac{x^2 - y^2}{x^2 + y^2} = 0.$

Alternative Solution: Take any given $\varepsilon > 0$. Putting $x = r\cos\theta$, $y = r\sin\theta$, we have

$$\left| xy\,\frac{x^2 - y^2}{x^2 + y^2} - 0 \right| = \left| xy\,\frac{x^2 - y^2}{x^2 + y^2} \right| = |r^2 \sin\theta\cos\theta\cos 2\theta|$$

$$= \left| \frac{r^2}{4}\sin 4\theta \right| = \frac{r^2}{4}|\sin 4\theta|$$

$$\leq \frac{r^2}{4} \qquad [\because |\sin 4\theta| \leq 1]$$

$$= \frac{x^2 + y^2}{4}$$

$$< \varepsilon, \text{ if } \frac{x^2}{4} < \frac{\varepsilon}{2} \text{ and } \frac{y^2}{4} < \frac{\varepsilon}{2} \text{ i.e.,}$$

if $|x| < \sqrt{2\varepsilon}$ and $|y| < \sqrt{2\varepsilon}$.

Now if we take $\delta = \sqrt{2\varepsilon}$, then we see that for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| xy \frac{x^2 - y^2}{x^2 + y^2} - 0 \right| < \varepsilon, \text{ whenever } |x| < \delta \text{ and } |y| < \delta.$$

Hence $\lim_{(x, y) \to (0, 0)} xy \frac{x^2 - y^2}{x^2 + y^2} = 0.$

Example 6: *Show that* $\lim_{(x, y) \to (0, 0)} \frac{3x - 2y}{2x - 3y}$ *does not exist.*

Solution: When $(x, y) \to (0, 0)$ along the st. line $y = x$, we have

$$\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} \frac{3x - 2x}{2x - 3x} = \lim_{x \to 0} \frac{x}{-x} = \lim_{x \to 0} -1 = -1.$$

Again when $(x, y) \to (0, 0)$ along the st. line $y = 0$, we have

$$\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} \frac{3x - 0}{2x - 0} = \lim_{x \to 0} \frac{3x}{2x} = \lim_{x \to 0} \frac{3}{2} = \frac{3}{2}.$$

Since the two methods of approach to the limiting point give different limiting values, the simultaneous limit does not exist.

Example 7: *Show that* $\lim_{(x, y) \to (0, 0)} \frac{2xy^2}{x^2 + y^4}$ *does not exist.*

Solution: Let (x, y) approach $(0, 0)$ along the curve $x = my^2$. For this, we put $x = my^2$ in the function and then allow y to approach zero. Thus in this case we have

$$\lim_{(x, y) \to (0, 0)} \frac{2xy^2}{x^2 + y^4} = \lim_{y \to 0} \frac{2my^4}{(m^2 + 1) y^4}$$

$$= \lim_{y \to 0} \frac{2m}{1 + m^2}$$

$$= \frac{2m}{1 + m^2}$$

which is different for different values of m.

For example, if $m = 1$, then this limit $= \frac{2 \cdot 1}{1 + 1^2} = 1$

and if $m = 2$, then this limit $= \frac{2 \cdot 2}{1 + 2^2} = \frac{4}{5}$.

Since two methods of approach to the limiting point give different limiting values, therefore the simultaneous limit

$$\lim_{(x, y) \to (0, 0)} \frac{2xy^2}{x^2 + y^4} \text{ does not exist.}$$

Example 8: *Show that the simultaneous limit exists at the origin but the limits do not, where*

$$f(x, y) = \begin{cases} x \sin \frac{1}{y} + y \sin \frac{1}{x}, & xy \neq 0 \\ 0, & xy = 0. \end{cases}$$

Solution: Here $\lim_{y \to 0} f(x, y)$, $\lim_{x \to 0} f(x, y)$ do not exist and therefore both the repeated limits $\lim_{x \to 0} \lim_{y \to 0} f(x, y)$ and $\lim_{y \to 0} \lim_{x \to 0} f(x, y)$ do not exist.

However, the simultaneous limit $\lim_{(x, y) \to (0, 0)} f(x, y)$ exists and is equal to 0 as shown below. Take any given $\varepsilon > 0$.

For all $(x, y) \neq (0, 0)$ such that $xy = 0$, we have

$$|f(x, y) - 0| = |0 - 0| = 0 < \varepsilon.$$

Again for all $(x, y) \neq (0, 0)$ such that $xy \neq 0$, we have

$$|f(x, y) - 0| = |f(x, y)| = \left| x \sin \frac{1}{y} + y \sin \frac{1}{x} \right|$$

$$\leq \left| x \sin \frac{1}{y} \right| + \left| y \sin \frac{1}{x} \right| \qquad [\because |a + b| \leq |a| + |b|]$$

$$= |x| \left| \sin \frac{1}{y} \right| + |y| \left| \sin \frac{1}{x} \right| \qquad [\because |ab| = |a||b|]$$

$$\leq |x| + |y| \qquad \left[\because \left| \sin \frac{1}{y} \right| \leq 1 \text{ and } \left| \sin \frac{1}{x} \right| \leq 1\right]$$

$$\leq 2\sqrt{x^2 + y^2}$$

$$[\because |x| \leq \sqrt{x^2 + y^2} \text{ and } |y| \leq \sqrt{x^2 + y^2}]$$

$$< \varepsilon, \text{ if } x^2 < \frac{\varepsilon^2}{8} \text{ and } y^2 < \frac{\varepsilon^2}{8} \quad i.e.,$$

$$\text{if } |x| < \frac{\varepsilon}{2\sqrt{2}} \text{ and } |y| < \frac{\varepsilon}{2\sqrt{2}}.$$

Now if we take $\delta = \frac{\varepsilon}{2\sqrt{2}}$, then we see that for any given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x, y) - 0| < \varepsilon$, whenever $|x| < \delta$ and $|y| < \delta$.

Hence, $\lim_{(x, y) \to (0, 0)} f(x, y) = 0$.

Comprehensive Exercise 1

1. Show that $\lim\limits_{(x, y) \to (0, 0)} \dfrac{2xy^2}{x^2 + y^4}$ does not exist.

2. Show that $\lim\limits_{(x, y) \to (0, 0)} (x + y) = 0$.

3. Show that $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{2xy}{x^2 + y^2}$ does not exist.

4. Show that $\lim\limits_{(x, y) \to (0, 0)} \dfrac{\sqrt{x^2 y^2 + 1} - 1}{x^2 + y^2} = 0$.

5. Show that $\lim\limits_{(x, y) \to (0, 0)} \dfrac{x^3 + y^3}{x - y}$ does not exist.

6. Let $f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & \text{if } x^4 + y^2 \neq 0 \\ 0, & \text{if } x = y = 0. \end{cases}$

   Show that the straight line approach gives the limit zero at the origin but the function possesses no limit at the origin.

7. Show that $\lim\limits_{(x, y) \to (0, 0)} \dfrac{xy^2}{x^2 + y^2} = 0$.

8. Show that the limit, when $(x, y) \to (0, 0)$, exists in each case :

   (i) $\lim \dfrac{xy}{\sqrt{x^2 + y^2}}$ (ii) $\lim \dfrac{x^3 y^3}{x^2 + y^2}$

   (iii) $\lim \dfrac{x^4 + y^4}{x^2 + y^2}$ (iv) $\lim \dfrac{x^3 - y^3}{x^2 + y^2}$

9. Show that the repeated limits exist at the origin and are equal but the simultaneous limit does not exist, where $f(x, y) = \begin{cases} 1, & \text{if } xy \neq 0 \\ 0, & \text{if } xy = 0. \end{cases}$

10. Show that the limit and the repeated limits exist when $(x, y) \to (0, 0)$:

$$f(x, y) = \begin{cases} xy \dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

11. Show that $\lim_{(x, y) \to (0, 0)} f(x, y)$ and $\lim_{y \to 0} \lim_{x \to 0} f(x, y)$ exist, but $\lim_{x \to 0} \lim_{y \to 0} f(x, y)$ does not exist, where

$$f(x, y) = \begin{cases} y + x \sin\left(\frac{1}{y}\right), & \text{if } y \neq 0 \\ 0, & \text{if } y = 0. \end{cases}$$

12. Show that if $f(x, y) = \frac{x - y}{x + y}$, then the repeated limits exist but the double limit does not exist when $(x, y) \to (0, 0)$.

5.4 Continuity of Functions of two Variables

Definition: *The function $f(x, y)$ is said to be continuous at (a, b) if for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|x - a| < \delta, |y - b| < \delta \Rightarrow |f(x, y) - f(a, b)| < \varepsilon.$$

Equivalently, $f(x, y)$ is said to be continuous at (a, b) if the simultaneous limit

$$\lim_{(x, y) \to (a, b)} f(x, y)$$

exists and is equal to its functional value $f(a, b)$ at (a, b).

If f is not continuous at $(a, b) \in D \subseteq R^2$, then f is said to be discontinuous at (a, b).

f is said to be continuous on the domain D if f is continuous at each point of D.

Note 1: Let $D \subseteq R^2$ and $f : D \to R$ be a function continuous at $(a, b) \in D$.

Let $f_1(x) = f(x, b)$. Then f_1 is a function of a single variable x.

f is continuous at $(a, b) \Rightarrow$ for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$|f(x, y) - f(a, b)| < \varepsilon$ for $(x, y) \in D$ and $|x - a| < \delta, |y - b| < \delta$

$\Rightarrow$ $|f(x, b) - f(a, b)| < \varepsilon$ for $(x, b) \in D$ and $|x - a| < \delta$

$\Rightarrow$ $|f_1(x) - f_1(a)| < \varepsilon$ for $(x, b) \in D$ and $|x - a| < \delta$

$\Rightarrow$ f_1 is continuous at a.

Similarly, we can show that $f_2(y) = f(a, y)$ is continuous at b.

Hence, if $f(x, y)$ is continuous at (a, b), then

(i) $f(x, b)$ is continuous at $x = a$ and (ii) $f(a, y)$ is continuous at $y = b$.

The converse of the above result is not true. Thus, if $f(x, b)$ is continuous at $x = a$ and $f(a, y)$ is continuous at $y = b$ then $f(x, y)$ need not be continuous at (a, b).

Note 2: For $f(x, y)$ to be continuous in both the variables together, it must have the same limiting value by all possible approaches to the critical point. Thus the necessary and sufficient condition is that the function is not only continuous in

each direction but the continuity is uniform for all directions.

In the definition of continuity, if we put $x = a + r\cos\theta$, $y = b + r\sin\theta$, we get

$$|f(a + r\cos\theta, b + r\sin\theta) - f(a, b)| < \varepsilon,$$

which must hold for all values of r less than some number r_0 which is independent of θ. In other words we say that the transformed function must be uniformly continuous in r for all values of $|\theta| \le 2\pi$.

Illustrative Examples

Example 9: *Examine the continuity at* $(1, 2)$ *of the function*

$$f(x, y) = \begin{cases} x^2 + 4y & \text{when } (x, y) \neq (1, 2) \\ 0 & \text{when } (x, y) = (1, 2). \end{cases}$$

Solution: We have $\lim\limits_{(x, y) \to (1, 2)} x^2 + 4y = 1^2 + 4 \times 2 = 9$, so that the limit exists and is equal to 9.

Since $f(1, 2) = 0$ and $\lim\limits_{(x, y) \to (1, 2)} f(x, y) = 9$, we have

$$\lim_{(x, y) \to (1, 2)} f(x, y) \neq f(1, 2).$$

Hence, the function is not continuous at $(1, 2)$.

Example 10: *Show that the function* $f : \mathbf{R}^2 \to \mathbf{R}$ *defined by* $f(x, y) = \dfrac{(y^2 - x^2)\,yx}{x^2 + y^2}$ *for* $(x, y) \neq (0, 0)$ *and* $f(0, 0) = 0$ *is continuous at the point* $(0, 0)$.

Solution: First we shall show that $\lim\limits_{(x, y) \to (0, 0)} f(x, y) = 0$.

Let $\varepsilon > 0$ be given.

For all $(x, y) \neq (0, 0)$, we have

$$|f(x, y) - 0| = \left|\frac{(y^2 - x^2)\,yx}{x^2 + y^2}\right| = |y^2 - x^2| \cdot \frac{|xy|}{x^2 + y^2} \quad \ldots(1)$$

Now for all x and y, we have $(|x| - |y|)^2 \ge 0$

$$\Rightarrow \quad |x|^2 + |y|^2 - 2|x|.|y| \ge 0 \quad \Rightarrow \quad x^2 + y^2 \ge 2|xy|$$

$$\Rightarrow \quad \frac{|xy|}{x^2 + y^2} \le \frac{1}{2}, \text{ if } (x, y) \neq (0, 0). \quad \ldots(2)$$

$\therefore$ from (1) and (2), for all $(x, y) \neq (0, 0)$, we have

$$|f(x, y) - 0| \le \frac{1}{2}|y^2 - x^2| \le \frac{1}{2}|y^2 + x^2| = \frac{1}{2}(x^2 + y^2)$$

$$< \varepsilon, \text{ if } x^2 < \varepsilon \text{ and } y^2 < \varepsilon \text{ i.e.,}$$

$$\text{if } |x| < \sqrt{\varepsilon} \text{ and } |y| < \sqrt{\varepsilon}.$$

Now if we take $\delta = \sqrt{\varepsilon}$, then we see that for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x, y) - 0| < \varepsilon, \text{ whenever } |x| < \delta \text{ and } |y| < \delta$$

$\therefore \quad \lim_{(x, y) \to (0, 0)} f(x, y) = 0.$

But $\quad f(0, 0) = 0.$ [Given]

$\therefore \quad \lim_{(x, y) \to (0, 0)} f(x, y) = f(0, 0).$

Hence $f(x, y)$ is continuous at the origin.

Example 11: *Discuss the continuity of the function*

$$f(x, y) = \frac{2xy^2}{x^3 + 3y^3}, (x, y) \neq (0, 0) \text{ and } f(0, 0) = 0,$$

with respect to both variables.

Solution: First we observe that $\lim_{(x, y) \to (0, 0)} f(x, y)$ does not exist.

When $(x, y) \to (0, 0)$ along the st. line $y = x$, we have

$$\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} \frac{2x \cdot x^2}{x^3 + 3x^3} = \lim_{x \to 0} \frac{2x^3}{4x^3} = \lim_{x \to 0} \frac{1}{2} = \frac{1}{2}.$$

Again when $(x, y) \to (0, 0)$ along the st. line $y = 0$, we have

$$\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} \frac{2x \cdot 0^2}{x^3 + 3 \cdot 0^3} = \lim_{x \to 0} 0 = 0.$$

Since the two methods of approach to the limiting point give different limiting values, the simultaneous limit does not exist. Consequently the function is not continuous at $(0, 0)$ in (x, y) together.

However, the function is continuous in x alone and in y alone at the origin. Because if we put either variable zero and then let the other variable approach zero, we find the limiting value zero, which is the value of the function at $(0, 0)$.

Example 12: *Show that the function* $f(x, y) = \frac{xy^3}{x^2 + y^6}, (x, y) \neq (0, 0)$ *and* $f(0, 0) = 0$ *is not continuous at* $(0, 0)$ *in* x, y *together but that the function is continuous in* x *alone and in* y *alone at the origin.* **(Purvanchal 2009)**

Solution: Simultaneous limit of $f(x, y)$ at $(0, 0)$ does not exist as shown below:

First let (x, y) approach $(0, 0)$ through any line $y = mx$. We have

$$\lim_{x \to 0} \frac{x \cdot m^3 x^3}{x^2 + m^6 x^6} = \lim_{x \to 0} \frac{m^3 x^2}{1 + m^6 x^4} = \frac{0}{1+0} = 0.$$

Now let (x, y) approach $(0, 0)$ through the curve $x = y^3$.

Then $$\lim_{y \to 0} \frac{y^3 \cdot y^3}{y^6 + y^6} = \frac{1}{2}.$$

Since the limits obtained by two different approaches are different, the simultaneous limit does not exist. Consequently the function is not continuous at $(0, 0)$ in (x, y) together.

The function is however continuous in x alone and in y alone at the origin. For putting either variable zero and then letting the other variable approach zero, we get the limiting value zero, which is the value of $f(0, 0)$.

Example 13: *Show that the function* $f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{(x^2 + y^2)}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

is continuous at the origin in x-y together. **(Rohilkhand 2008; Garhwal 10)**

Solution: First we shall show that $\lim_{(x, y) \to (0, 0)} f(x, y) = 0$.

Let $\varepsilon > 0$ be given. For all $(x, y) \neq (0, 0)$, we have

$$|f(x, y) - 0| = \left| \frac{xy}{\sqrt{(x^2 + y^2)}} - 0 \right| = \left| \frac{xy}{\sqrt{(x^2 + y^2)}} \right| = \left| \frac{r \cos\theta \, r \sin\theta}{r} \right|$$

[Putting $x = r\cos\theta$, $y = r\sin\theta$]

$$= r\,|\cos\theta||\sin\theta|$$

$$\leq r \qquad [\because |\cos\theta| \leq 1, |\sin\theta| \leq 1]$$

$$= \sqrt{(x^2 + y^2)}$$

$$< \varepsilon, \text{ if } x^2 < \frac{\varepsilon^2}{2} \text{ and } y^2 < \frac{\varepsilon^2}{2} \text{ i.e.,}$$

$$\text{if } |x| < \frac{\varepsilon}{\sqrt{2}} \text{ and } |y| < \frac{\varepsilon}{\sqrt{2}}.$$

Now if we take $\delta = \dfrac{\varepsilon}{\sqrt{2}}$, then we see that for any given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x, y) - 0| < \varepsilon$, whenever $|x| < \delta$ and $|y| < \delta$.

Hence, $\lim_{(x, y) \to (0, 0)} f(x, y) = 0$.

Since $f(0, 0) = 0$, therefore $\lim_{(x, y) \to (0, 0)} f(x, y) = f(0, 0)$.

Hence, the given function f is continuous at $(0, 0)$ in x-y together.

Important Note: If $\lim_{r \to 0} f(a + r\cos\theta, b + r\sin\theta) = f(a, b)$ for every value of θ, then it is not necessary that the function is continuous at (a, b).

For example, we have seen in Example 12 of article 5.4 that the simultaneous limit of $f(x, y) = \dfrac{xy^3}{x^2 + y^6}$, $x \neq 0$, $y \neq 0$ and $f(0, 0) = 0$ does not exist as $(x, y) \to (0, 0)$. Therefore, this function is discontinuous at the origin. However, if we put $x = r\cos\theta$, $y = r\sin\theta$, we get

$$\lim_{r \to 0} \frac{r^4 \cos\theta \sin^3\theta}{r^2\cos^2\theta + r^6 \sin^6\theta} = \lim_{r \to 0} r^2 \frac{\cos\theta \sin^3\theta}{\cos^2\theta + r^4 \sin^6\theta}$$

$$= 0 = f(0, 0)$$

for each constant value of θ.

5.5 Taylor's Theorem for Functions of two Variables

To expand $f(x + h, y + k)$ in powers of h and k in case $f(x, y)$ and all its partial derivatives are continuous in a certain domain of the point (x, y).

Regarding $f(x + h, y + k)$ as a function of one variable only, say that of x (that is, supposing that only x varies while y remains constant), and expanding by Taylor's theorem for one variable only, we have

$$f(x + h, y + k) = f(x, y + k) + h\frac{\partial f(x, y + k)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f(x, y + k)}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 f(x, y + k)}{\partial x^3} + \ldots \quad \ldots(1)$$

Now expanding each term on the right hand side of (1) by Taylor's theorem regarding y as variable and x as constant, we have

$$f(x + h, y + k) = f(x, y) + k\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \frac{k^3}{3!}\frac{\partial^3 f(x, y)}{\partial y^3} + \ldots$$

$$+ h\frac{\partial}{\partial x}\left\{f(x, y) + k\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \ldots\right\}$$

$$+ \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left\{f(x, y) + k\frac{\partial f(x, y)}{\partial y} + \ldots\right\}$$

$$+ \frac{h^3}{3!}\frac{\partial^3}{\partial x^3}\{f(x, y) + \ldots\} + \ldots$$

Hence,

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\cdot\frac{\partial^2 f}{\partial x\,\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)$$

$$+ \frac{1}{3!}\left(h^3\frac{\partial^3 f}{\partial x^3} + 3h^2 k\frac{\partial^3 f}{\partial x^2\,\partial y} + 3hk^2\frac{\partial^3 f}{\partial x\,\partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}\right) + \ldots$$

Symbolically we may put this result as

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f$$

$$+ \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f + \ldots,$$

$$f(x+h, y+k) = f(x, y) + (hf_x + kf_y) + \frac{1}{2!}(h^2 f_{xx} + 2hk\, f_{xy} + k^2 f_{yy})$$

$$+ \frac{1}{3!}(h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy}) + \ldots\ldots \quad \ldots(2)$$

Corollary 1: Putting $x = a$ and $y = b$ in equation (2), we get

$$f(a+h, b+k) = f(a, b) + \{hf_x(a, b) + kf_y(a, b)\} + \frac{1}{2!}\{h^2 f_{xx}(a, b)$$

$$+ 2hk\, f_{xy}(a, b) + k^2 f_{yy}(a, b)\} + \frac{1}{3!}\{h^3 f_{xxx}(a, b) + 3h^2 k\, f_{xxy}(a, b)$$

$$+ 3hk^2 f_{xyy}(a, b) + k^3 f_{yyy}(a, b)\} + \ldots$$

Corollary 2: Putting $a = 0, b = 0, h = x, k = y$, in equation (2), we get

$$f(x, y) = f(0, 0) + x f_x(0, 0) + y f_y(0, 0)$$

$$+ \frac{x^2}{2!} f_{xx}(0, 0) + xy f_{xy}(0, 0) + \frac{y^2}{2!} f_{yy}(0, 0) + \ldots,$$

where $f_x(0, 0) = \left(\frac{\partial f}{\partial x}\right)_{(0, 0)}$, $f_y(0, 0) = \left(\frac{\partial f}{\partial y}\right)_{(0, 0)}$, $f_{xx}(0, 0) = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(0, 0)}$,

$f_{xy}(0, 0) = \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)_{(0, 0)}$, $f_{yy}(0, 0) = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(0, 0)}$ $= \ldots$

which is **Maclaurin's theorem** for two variables.

Illustrative Examples

Example 14: *Find the first six terms of the expansion of the function* $e^x \log(1+y)$ *in a Taylor series in the neighbourhood of the point* $(0, 0)$.

Solution: Let $F(x, y) = e^x \log(1+y)$. Then

$$F(x, y) = e^x \log(1+y) \Rightarrow F(0,0) = 0$$

$$F_x(x, y) = e^x \log(1+y) \Rightarrow F_x(0,0) = 0$$

$$F_{xx}(x, y) = e^x \log(1+y) \Rightarrow F_{xx}(0,0) = 0$$

$$F_y(x, y) = \frac{e^x}{(1+y)} \Rightarrow F_y(0,0) = 1$$

$$F_{yy}(x, y) = -\frac{e^x}{(1+y)^2} \Rightarrow F_{yy}(0,0) = -1$$

$$F_{xy}(x, y) = \frac{e^x}{(1+y)} \Rightarrow F_{xy}(0,0) = 1.$$

Using Taylor's theorem, we get

$$F(x, y) = F(0,0) + xF_x(0,0) + yF_y(0,0) + \frac{x^2}{2!}F_{xx}(0,0) + xy\,F_{xy}(0,0) + \frac{y^2}{2!}F_{yy}(0,0) + \ldots$$

i.e., $$e^x \log(1+y) = 0 + x(0) + y(1) + \frac{x^2}{2!}(0) + xy(1) + \frac{y^2}{2!}(-1) + \ldots$$

i.e., $$e^x \log(x+y) = y + xy - \frac{y^2}{2} + \ldots$$

Example 15: *Find the expansion for* $\cos x \cos y$ *in powers of* x, y *upto fourth order terms.*

Solution: Let $f(x, y) = \cos x \cos y$. Then

$$f(x, y) = \cos x \cos y \Rightarrow f(0,0) = 1$$

$$f_x(x, y) = -\sin x \cos y \Rightarrow f_x(0,0) = 0$$

$$f_y(x, y) = -\cos x \sin y \Rightarrow f_y(0,0) = 0$$

$$f_{x^2}(x, y) = -\cos x \cos y \Rightarrow f_{x^2}(0,0) = -1$$

$$f_{xy}(x, y) = \sin x \sin y \Rightarrow f_{xy}(0,0) = 0$$

$$f_{y^2}(x, y) = -\cos x \cos y \Rightarrow f_{y^2}(0,0) = -1$$

$$f_{x^3}(x, y) = \sin x \cos y \Rightarrow f_{x^3}(0,0) = 0$$

$$f_{x^2y}(x, y) = \cos x \sin y \Rightarrow f_{x^2y}(0,0) = 0$$

$$f_{xy^2}(x, y) = \sin x \cos y \Rightarrow f_{xy^2}(0,0) = 0$$

$$f_{y^3}(x, y) = \cos x \sin y \Rightarrow f_{y^3}(0,0) = 0$$

$$f_{x^4}(x, y) = \cos x \cos y \Rightarrow f_{x^4}(0,0) = 1$$

$$f_{x^3y}(x, y) = -\sin x \sin y \quad \Rightarrow \quad f_{x^3y}(0,0) = 0$$
$$f_{x^2y^2}(x, y) = \cos x \cos y \quad \Rightarrow \quad f_{x^2y^2}(0,0) = 1$$
$$f_{xy^3}(x, y) = -\sin x \sin y \quad \Rightarrow \quad f_{xy^3}(0,0) = 0$$
$$f_{y^4}(x, y) = \cos x \cos y \quad \Rightarrow \quad f_{y^4}(0,0) = 1.$$

Using Taylor's theorem, we get

$$f(x, y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0) +$$
$$+ \frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(0,0) + \frac{1}{4!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^4 f(0,0) + \ldots$$
$$= f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{x^2}{2} f_{xx}(0,0)$$
$$+ xyf_{xy}(0,0) + \frac{y^2}{2} f_{yy}(0,0)$$
$$+ \frac{1}{6} x^3 f_{xxx}(0,0) + \frac{1}{2} x^2 y f_{xxy}(0,0) + \frac{1}{2} xy^2 f_{xyy}(0,0)$$
$$+ \frac{1}{6} y^3 f_{yyy}(0,0)$$
$$+ \frac{1}{24} x^4 f_{x^4}(0,0) + \frac{1}{6} x^3 y f_{x^3y}(0,0) + \frac{1}{4} x^2 y^2 f_{x^2y^2}(0,0)$$
$$+ \frac{1}{6} xy^3 f_{xy^3}(0,0) + \frac{1}{24} y^4 f_{y^4}(0,0) + \ldots\ldots$$

i.e., $$\cos x \cos y = 1 + 0 + 0 + \frac{x^2}{2}(-1) + xy(0) + \frac{y^2}{2}(-1) + \frac{x^3}{6}(0)$$
$$+ \frac{x^2 y}{2}(0) + \frac{xy^2}{2}(0) + \frac{y^3}{6}(0) + \frac{x^4}{24}(1) + \frac{x^3 y}{6}(0) + \frac{x^2 y^2}{4}(1)$$
$$+ \frac{xy^3}{6}(0) + \frac{y^4}{24}(1) + \ldots\ldots$$

i.e., $$\cos x \cos y = 1 - \frac{x^2}{2} - \frac{y^2}{2} + \frac{x^4}{24} + \frac{x^2 y^2}{4} + \frac{y^4}{24} + \ldots\ldots$$

Example 16: *If* $F(x, y) = \tan^{-1}(xy)$, *compute an approximate value of* $F(0 \cdot 9, -1 \cdot 2)$.

Solution: Let us expand $F(x, y)$ near the point $(1, -1)$.

By Taylor's theorem, we get

$$F(0 \cdot 9, -1 \cdot 2) = F(1 - 0 \cdot 1, -1 - 0 \cdot 2)$$
$$= F(1, 1) + \{(1 - 0 \cdot 9) F_x(1, 1) + (1 - 0 \cdot 2) F_y(1, 1)\}$$
$$+ \frac{1}{2!}\{(1 - 0 \cdot 9)^2 F_{xx}(1, 1) + 2(1 - 0 \cdot 9)(1 - 0 \cdot 2) F_{xy}(1, 1)$$
$$+ (1 - 0 \cdot 2)^2 F_{yy}(1, 1)\} + \ldots \quad \ldots(1)$$

Here $a = 1, b = -1$.

$$F(x, y) = \tan^{-1} xy \quad \Rightarrow \quad F(1, -1) = \tan^{-1}(-1) = -\frac{\pi}{4}.$$

$$F_x = \frac{y}{1 + x^2 y^2} \quad \Rightarrow \quad F_x(1, -1) = \frac{-1}{1+1} = -\frac{1}{2}.$$

$$F_y = \frac{x}{1 + x^2 y^2} \quad \Rightarrow \quad F_y(1, -1) = \frac{1}{1+1} = \frac{1}{2}.$$

$$F_{x^2} = -\frac{(2x)\, y}{(1 + x^2 y^2)^2} \quad \Rightarrow \quad F_{x^2}(1, -1) = \frac{-(2)(-1)}{(1+1)^2} = \frac{1}{2}.$$

$$F_{y^2} = \frac{-x(2x^2 y)}{(1 + x^2 y^2)^2} \quad \Rightarrow \quad F_{y^2}(1, -1) = \frac{2}{(1+1)^2} = \frac{1}{2}.$$

$$F_{xy} = \frac{1 + x^2 y^2 - x(2xy^2)}{(1 + x^2 y^2)^2}$$

$$= \frac{1 - x^2 y^2}{(1 + x^2 y^2)^2} \quad \Rightarrow \quad F_{xy}(1, -1) = \frac{1-1}{(1+1)^2} = 0$$

Putting all these values in (1), we get

$$F(0 \cdot 9, -1 \cdot 2) = -\frac{\pi}{4} + (-0 \cdot 1)\left(-\frac{1}{2}\right) + (-0 \cdot 2)\frac{1}{2} + \frac{1}{2}\left[(-0 \cdot 1)^2 \cdot \left(\frac{1}{2}\right) + 2(-0 \cdot 1)(-0 \cdot 2)0 + (-0 \cdot 2)^2 \left(\frac{1}{2}\right)\right] + \ldots.$$

$$= -\frac{\pi}{4} + 0 \cdot 05 - 0 \cdot 1 + 0 \cdot 0125 = -0 \cdot 823.$$

Example 17: *Expand $e^x \cdot \cos y$ near the point $\left(1, \frac{\pi}{4}\right)$ by Taylor's Theorem.*

Solution: We have $F(x + h, y + k) = F(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) F$

$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 F + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 F + \ldots \qquad \ldots(1)$$

Again $e^x \cos y = F(x, y) = F\left[1 + (x - 1), \frac{\pi}{4} + \left(y - \frac{\pi}{4}\right)\right] = F\left(1 + h, \frac{\pi}{4} + k\right)$

where $h = x - 1, k = y - \frac{\pi}{4}$.

$$F(x, y) = e^x \cos y \quad \Rightarrow \quad F\left(1, \frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}$$

$$\frac{\partial F}{\partial x} = e^x \cos y \quad \Rightarrow \quad \frac{\partial F}{\partial x}\left(1, \frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}$$

$$\frac{\partial F}{\partial y} = -e^x \sin y \quad \Rightarrow \quad \frac{\partial F}{\partial y}\left(1, \frac{\pi}{4}\right) = \frac{-e}{\sqrt{2}}$$

$$\frac{\partial^2 F}{\partial x^2} = e^x \cos y \quad \Rightarrow \quad \frac{\partial^2 F}{\partial x^2}\left(1, \frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}$$

$$\frac{\partial F}{\partial y^2} = -e^x \cos y \quad \Rightarrow \quad \frac{\partial^2 F}{\partial y^2}\left(1, \frac{\pi}{4}\right) = \frac{-e}{\sqrt{2}}$$

$$\frac{\partial^2 F}{\partial x\, dy} = -e^x \sin y \quad \Rightarrow \quad \frac{\partial^2 F}{\partial y\, \partial y}\left(1, \frac{\pi}{4}\right) = \frac{-e}{\sqrt{2}}$$

Substituting these values in Taylor's theorem, we get

$$e^x \cos y = \frac{e}{\sqrt{2}} + \left[(x-1)\frac{e}{\sqrt{2}} + \left(y - \frac{\pi}{4}\right)\frac{-e}{\sqrt{2}}\right]$$

$$+ \frac{1}{2!}\left[(x-1)^2 \frac{e}{\sqrt{2}} + 2(x-1)\left(y - \frac{\pi}{4}\right)\left(\frac{-e}{\sqrt{2}}\right) + \left(y - \frac{\pi}{4}\right)^2 \left(\frac{-e}{\sqrt{2}}\right)\right] + \ldots$$

Comprehensive Exercise 2

1. Investigate for continuity at $(1, 2)$, the function

$$f(x, y) = \begin{cases} x^2 + 2y, & (x, y) \neq (1, 2) \\ 0, & (x, y) = (1, 2). \end{cases}$$

2. Show that the function f defined as follows has a removable discontinuity at $(2, 3)$.

$$f(x, y) = \begin{cases} 3xy, & (x, y) \neq (2, 3) \\ 6, & (x, y) = (2, 3). \end{cases}$$

Suitably redefine the function to make it continuous at $(2, 3)$.

3. Show that the function f defined by

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is not continuous at the origin.

4. Prove that $f(x, y) = xy\dfrac{x^2 - y^2}{x^2 + y^2}$ when $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$ is continuous at the origin.

5. A function f is defined as follows :

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & \text{when } (x, y) \neq (0, 0) \\ 0, & \text{when } (x, y) = (0, 0). \end{cases}$$

Show that this function is continuous at the origin in x alone and y alone but is discontinuous at the origin with regard to both the variables x and y.

6. Show that the function f defined by $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$, $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$ is discontinuous in $x - y$ at the origin.
Show also that this function is continuous along the radius vector $\theta = \pi / 2$.

7. Show that the function $f(x, y) = \dfrac{xy}{x^3 + y^2}$, when $x \neq 0$, $y \neq 0$ and $f(0, 0) = 0$ is discontinuous in (x, y) at the origin.

8. Investigate the continuity at $(0, 0)$ of the function

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^3 + y^3}, & x^3 + y^3 \neq 0 \\ 0, & x^3 + y^3 = 0. \end{cases}$$

9. Show that the function f is continuous at the origin, where

$$f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

10. Show that the following functions are discontinuous at the origin :

(i) $f(x, y) = \begin{cases} \dfrac{1}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$

(ii) $f(x, y) = \begin{cases} \dfrac{x^2 y^2}{x^4 + y^4}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$

(iii) $f(x, y) = \begin{cases} \dfrac{x^4 - y^4}{x^4 + y^4}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$

(iv) $f(x, y) = \begin{cases} \dfrac{x^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$

11. Show that the following functions are continuous at the origin :

(i) $f(x, y) = \begin{cases} \dfrac{x^3 y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$

(ii) $f(x, y) = \begin{cases} \dfrac{x^2 y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$

12. Show that the following function is discontinuous at (0, 0) :

$$f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y. \end{cases}$$

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13. Expand $e^x \cdot \sin y$ in powers of x and y as far as terms of third degree.

14. Expand : $F(x, y) = x^2 y + 3y - 2$ in powers of $(x - 1)$ and $(y + 2)$ by Taylor's theorem.

15. Obtain Taylor's expansion of $\tan^{-1} \dfrac{y}{x}$ about (1, 1) upto and including the second degree terms. Hence compute $F(1 \cdot 1, 0 \cdot 9)$.

6. Expand x^y in powers of $(x - 1)$ and $(y - 1)$ upto the third degree terms.

Answers 2

1. Discontinuous

2. $f(x, y) = \begin{cases} 3xy, & (x, y) \neq (2, 3) \\ 18, & (x, y) = (2, 3) \end{cases}$

8. Discontinuous

13. $e^x \sin y = y + xy + \dfrac{x^2 y}{2} - \dfrac{y^3}{6} + \ldots$

14. $x^2 y + 3y - 2 = -10 - 4(x - 1) + 4(y + 2) - 2(x - 1)^2 + 2(x - 1)(y + 2) + (x - 1)^2 (y + 2)$

15. $\tan^{-1} \dfrac{y}{x} = \dfrac{\pi}{4} - \dfrac{1}{2}(x - 1) + \dfrac{1}{2}(y - 1) + \dfrac{1}{4}(x - 1)^2 - \dfrac{1}{4}(y - 1)^2 + \ldots$; $0 \cdot 7862$

16. $e^{xy} = 1 + (x - 1) + (x - 1)(y - 1) + \dfrac{1}{2}(x - 1)^2 (y - 1)$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. $\lim_{(x, y) \to (0,0)} \frac{3x - 2y}{2x - 3y} =$

   (a) $-\frac{3}{2}$ (b) $\frac{2}{3}$

   (c) 1 (d) does not exist.

2. $\lim_{(x, y) \to (0,0)} \frac{2xy^2}{x^2 + y^4} =$

   (a) 0 (b) does not exist

   (c) 1 (d) 2.

Fill in the Blank(s)

Fill in the blanks "....." so that the following statements are complete and correct.

1. We say that the simultaneous limit of $f(x, y)$ exists and is equal to A as $(x, y) \to (a, b)$, if for every given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x, y) - A| < \varepsilon$ for all values of x and y in the nhd of (a, b) defined by $|x - a| < \delta$, ...

2. $\lim_{(x, y) \to (0,0)} xy \frac{y^2 - x^2}{x^2 + y^2} =$ .....

3. The function $f(x, y)$ is said to be continuous at (a, b) if

   $$\lim_{(x, y) \to (a, b)} f(x, y) = \ldots.$$

True or False

Write 'T' for true and 'F' for false statement.

1. $\lim_{(x, y) \to (0,0)} \frac{xy}{x^2 + y^2} = 0.$

2. The function

   $$f(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

   is discontinuous at the origin.

3. The function

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{(x^2 + y^2)}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at the origin.

4. The function $f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

is discontinuous at the origin.

Answers

Multiple Choice Questions

1. (d) 2. (b)

Fill in the Blank(s)

1. $|y - b| < \delta$ 2. 0 3. $f(a, b)$

True or False

1. *F* 2. *T* 3. *T* 4. *F*

□

Chapter

6

Maxima and Minima

(Of Functions of Several Independent Variables)

6.1 Definition

Let $f(x, y, z,...)$ be any function of several independent variables $x, y, z,...$ supposed to be continuous for all values of these variables in the neighbourhood of their values $a, b, c,...$ respectively. Then $f(a, b, c,...)$ is said to be a **maximum** or a **minimum** value of $f(x, y, z,...)$ according as $f(a + h, b + k, c + l,...)$ is **less** or **greater** than $f(a, b, c,...)$ for all sufficiently small independent values of $h, k, l,...$, positive or negative, provided they are not all zero.

6.2 Necessary Conditions for the Existence of Maxima or Minima

From the definition it is obvious that we shall have a maximum or a minimum of $f(x, y, z,...)$ for those values of $x, y, z,...$ for which the expression $f(x + h, y + k, z + l,...) - f(x, y, z,...)$ is of **invariable sign** for all sufficiently

small independent values of $h, k, l, \ldots$ provided they are not all equal to zero. There will be a maximum or a minimum according as this sign is negative or positive. Expanding by Taylor's theorem for several variables, we have

$$f(x+h, y+k, z+l, \ldots)$$

$$= \left[1 + \frac{1}{1!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z} + \ldots\right) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z} + \ldots\right)^2 + \ldots\right] f(x, y, z, \ldots).$$

$$\therefore \quad f(x+h, y+k, z+l, \ldots) - f(x, y, z, \ldots)$$

$$= \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z} + \ldots\right) + \text{terms of the second and higher orders in } h, k, l, \ldots. \quad \ldots(1)$$

Now by taking $h, k, l, \ldots$ sufficiently small, the first degree terms in $h, k, l, \ldots$ can be made to govern the sign of the right hand side and therefore of the left hand side of (1). But if $h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z} + \ldots$, is not equal to zero, the sign of this expression will change by changing the sign of each of $h, k, l, \ldots,$. Hence as a necessary condition for the occurrence of a maximum or a minimum of $f(x, y, z, \ldots)$, we must have

$$h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z} + \ldots = 0. \quad \ldots(2)$$

Since (2) is true whatever be the values of $h, k, l, \ldots$ independent of each other, we must have as a necessary consequence

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0, \ldots$$

If there are n independent vairables, we have then obtained n simultaneous equations to give us the values $a, b, c, \ldots$ of the n variables $x, y, z, \ldots$ for which $f(x, y, z, \ldots)$ may have a maximum or a minimum value.

The conditions $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0, \ldots$ *are necessary but not sufficient for the existence of maxima and minima.*

6.3 Stationary and Extreme Points

A point $(a_1, a_2, \ldots, a_n)$ is called a **stationary point**, if all the first order partial derivatives of the function $f(x_1, x_2, \ldots, x_n)$ vanish at that point. Also then the

value of the function $f(x_1, x_2, \ldots, x_n)$ is said to be stationary at that point. A stationary point which is either a maximum or a minimum is called an **extreme point** and the value of the function at that point is called an **extreme value**. A stationary point is not necessarily an extreme point. Thus a stationary value may be a maximum or a minimum or neither of these two. To decide whether a stationary point is really an extreme point, a further investigation is required.

6.4 Lagrange's Necessary and Sufficient Conditions for the Maxima or Minima of a Function of three Independent Variables

Necessary Conditions: Let $f(x, y, z)$ be a function of three independent variables x, y and z. Then as derived in article 6.2, for $f(x, y, z)$ to be a maximum or a minimum at any point (a, b, c), it is **necessary** that $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$ **and** $\frac{\partial f}{\partial z} = 0$ at that point.

Hence the points where the value of the function $f(x, y, z)$ is stationary (*i.e.*, may be a maximum or a minimum) are obtained by solving the simultaneous equations

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0.$$

Sufficient Conditions: Before deriving the sufficient conditions for the existence of a maximum or a minimum of a function of three independent variables, we obtain the following two algebraic lemmas regarding the signs of quadratic expressions.

Lemma 1: Let $I_2 = ax^2 + 2hxy + by^2$ be a quadratic expression in two variables x and y. We can write

$$I_2 = \frac{1}{a}[a^2x^2 + 2ahxy + aby^2], \text{ if } a \neq 0$$
$$= \frac{1}{a}[(ax + hy)^2 + (ab - h^2)\, y^2].$$

The expression within the square brackets will be positive if $ab - h^2$ is positive and in that case the sign of the expression I_2 will be the same as that of a.

In case $ab - h^2$ is not positive, we can say nothing about the sign of the expression within the square brackets and hence nothing about the sign of the given quadratic expression I_2.

Lemma 2. In three variables x, y and z,

$$I_3 \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

$$= \frac{1}{a}[a^2x^2 + aby^2 + acz^2 + 2fayz + 2gazx + 2haxy], \text{ if } a \neq 0$$

$$= \frac{1}{a}[a^2x^2 + 2ax(gz + hy) + aby^2 + acz^2 + 2fayz]$$

$$= \frac{1}{a}[(ax + hy + gz)^2 + aby^2 + acz^2 + 2fayz - (gz + hy)^2]$$

$$= \frac{1}{a}[(ax + hy + gz)^2 + (ab - h^2)y^2 + 2yz(fa - gh) + (ac - g^2)z^2].$$

Now I_3 will be of the same sign as a provided the expression within the square brackets is positive which will of course be so if

$$ab - h^2 \text{ and } \{(ab - h^2)(ac - g^2) - (fa - gh)^2\}$$

are both positive *i.e.*, if

$$ab - h^2 \text{ and } a(abc + 2fgh - af^2 - bg^2 - ch^2)$$

are both positive.

Thus I_3 will be positive if

$$a, \begin{vmatrix} a & h \\ h & b \end{vmatrix}, \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

be all positive and will be negative if these three expressions are alternately negative and positive.

Now we are in a position to derive Lagrange's sufficient conditions for the existence of a maximum or a minimum of a function of three independent variables at a stationary point.

Let a set of the values of x, y, z obtained by solving the equations

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0 \text{ be } a, b, c .$$

Let the values of the six second order partial derivatives

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial z^2}, \frac{\partial^2 f}{\partial y\,\partial z}, \frac{\partial^2 f}{\partial z\,\partial x} \text{ and } \frac{\partial^2 f}{\partial x\,\partial y}$$

at the point (a, b, c) be denoted by A, B, C, F, G and H respectively.

Then, we have

$$f(a + h, b + k, c + l) - f(a, b, c)$$

$$= \frac{1}{2!}(Ah^2 + Bk^2 + Cl^2 + 2Fkl + 2Glh + 2Hhk) + R_3 , \quad \ldots(1)$$

where R_3 consists of terms of third and higher orders of small quantities h, k and l. By taking h, k and l sufficiently small, the second degree terms in h, k and l can be made to govern the sign of the right hand side and therefore of the left hand side of

(1). If this group of terms forms an expression of invariable sign for all such values of h, k and l, we shall have a maximum or a minimum value of $f(x, y, z)$ at (a, b, c) according as that sign is negative or positive.

Hence by our lemma 2, if the expressions

$$A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

be all positive, we shall have a minimum of $f(x, y, z)$ at (a, b, c) and if these expressions be alternately negative and positive, we shall have a maximum of $f(x, y, z)$ at (a, b, c), whilst if these conditions are not satisfied, we shall in general have neither a maximum nor a minimum of $f(x, y, z)$ at (a, b, c).

6.5 Working Rule for Finding the Maxima and Minima of a Function of Three Independent Variables

Suppose $f(x, y, z)$ is a given function of three independent variables x, y and z. Find $\partial f / \partial x$, $\partial f / \partial y$ and $\partial f / \partial z$ and solve the simultaneous equations $\partial f / \partial x = 0$, $\partial f / \partial y = 0$ and $\partial f / \partial z = 0$. All the triads (a, b, c) of the values of x, y and z obtained on solving these equations will give the stationary values of $f(x, y, z)$ *i.e.*, will give the points at which the function $f(x, y, z)$ may be a maximum or a minimum.

To discuss the maximum or minimum of $f(x, y, z)$ at any point (a, b, c) obtained on solving the equations $\partial f / \partial x = 0$, $\partial f / \partial y = 0$ and $\partial f / \partial z = 0$, we find the values at this point of the six partial derivatives of second order of $f(x, y, z)$ symbolically denoted as follows :

$$A = \frac{\partial^2 f}{\partial x^2}, B = \frac{\partial^2 f}{\partial y^2}, C = \frac{\partial^2 f}{\partial z^2}, F = \frac{\partial^2 f}{\partial y\, \partial z}, G = \frac{\partial^2 f}{\partial z\, \partial x}$$

and $$H = \frac{\partial^2 f}{\partial x\, \partial y}.$$

If the expressions

$$A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

be all positive, we shall have a minimum of $f(x, y, z)$ at (a, b, c) and if these expressions be alternately negative and positive, we shall have a maximum of $f(x, y, z)$ at (a, b, c), whilst if these conditions are not satisfied, we shall in general have neither a maximum nor a minimum of $f(x, y, z)$ at (a, b, c).

Illustrative Examples

Example 1: *Discuss the maximum or minimum values of u where*

$$u = x^2 + y^2 + z^2 + x - 2z - xy .$$

Solution: For a maximum or a minimum of u, we must have

$$\frac{\partial u}{\partial x} = 2x - y + 1 = 0 ,$$

$$\frac{\partial u}{\partial y} = - x + 2y = 0 ,$$

and $$\frac{\partial u}{\partial z} = 2z - 2 = 0 .$$

These equations give $x = -2/3,\ y = -1/3 ,\ z = 1$.

$\therefore$ $(-2/3, -1/3, 1)$ is the only point at which u is stationary *i.e.*, at which u may have a maximum or a minimum.

Now $$\frac{\partial^2 u}{\partial x^2} = 2 , \frac{\partial^2 u}{\partial y^2} = 2 , \frac{\partial^2 u}{\partial z^2} = 2 , \frac{\partial^2 u}{\partial y\ \partial z} = 0 , \frac{\partial^2 u}{\partial z\ \partial x} = 0$$

and $$\frac{\partial^2 u}{\partial x\ \partial y} = -1 .$$

If A, B, C, F, G and H denote the respective values of these six partial derivatives of second order at the point $(-2/3, -1/3, 1)$, then

$$A = 2 , B = 2 , C = 2 , F = 0 , G = 0 , H = -1 .$$

Now we have

$$A = 2 , \begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

and $$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 6 .$$

Since these three expressions are all positive, we have a minimum of u when $x = -2/3 , y = -1/3 , z = 1$.

Example 2: *Show that the point such that the sum of the squares of its distances from n given points shall be minimum, is the centre of the mean position of the given points.*

Solution: Let the n given points be $(a_1, b_1, c_1) , (a_2 , b_2 , c_2) , \ldots\ldots , (a_n , b_n , c_n)$ and let (x, y, z) be the coordinates of the required point.

If u denotes the sum of the squares of the distances of (x, y, z) from the n given points, then

$$u = \Sigma [(x - a_1)^2 + (y - b_1)^2 + (z - c_1)^2]$$

$$= \Sigma (x - a_1)^2 + \Sigma (y - b_1)^2 + \Sigma (z - c_1)^2 .$$

For a maximum or a minimum of u, we must have

$$\frac{\partial u}{\partial x} = 2\Sigma (x - a_1) = 2nx - 2\Sigma a_1 = 0,$$

$$\frac{\partial u}{\partial y} = 2\Sigma (y - b_1) = 2ny - 2\Sigma b_1 = 0,$$

and $$\frac{\partial u}{\partial z} = 2\Sigma (z - c_1) = 2nz - 2\Sigma c_1 = 0.$$

Solving these equations, we get

$$x = \frac{\Sigma a_1}{n}, y = \frac{\Sigma b_1}{n}, z = \frac{\Sigma c_1}{n}.$$

Now $$A = \frac{\partial^2 u}{\partial x^2} = 2n, B = \frac{\partial^2 u}{\partial y^2} = 2n, C = \frac{\partial^2 u}{\partial z^2} = 2n,$$

$$F = \frac{\partial^2 u}{\partial y\, \partial z} = 0, G = \frac{\partial^2 u}{\partial z\, \partial x} = 0, H = \frac{\partial^2 u}{\partial x\, \partial y} = 0.$$

We have $$A = 2n, \begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 2n & 0 \\ 0 & 2n \end{vmatrix} = 4n^2,$$

and $$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 2n & 0 & 0 \\ 0 & 2n & 0 \\ 0 & 0 & 2n \end{vmatrix} = 8n^3.$$

Since these three expressions are all positive, u is minimum when

$$x = \frac{\Sigma a_1}{n}, y = \frac{\Sigma b_1}{n} \text{ and } z = \frac{\Sigma c_1}{n}$$

i.e., u is minimum when the point (x, y, z) is the centre of the mean position of the n given points.

Example 3: *Find the maximum value of u where*

$$u = \frac{xyz}{(a + x)(x + y)(y + z)(z + b)}.$$

Solution: We have

$$\log u = \log x + \log y + \log z - \log (a + x) - \log (x + y) - \log (y + z) - \log (z + b).$$

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{x} - \frac{1}{a + x} - \frac{1}{x + y} = \frac{ay - x^2}{x(a + x)(x + y)}$$

or $$\frac{\partial u}{\partial x} = \frac{(ay - x^2)\, u}{x(a + x)(x + y)}.$$

Similarly $$\frac{\partial u}{\partial y} = \frac{(xz - y^2)\, u}{y(x + y)(y + z)}$$

and $$\frac{\partial u}{\partial z} = \frac{(by - z^2)\, u}{z\,(y + z)\,(z + b)}.$$

Now for a maximum or a minimum of u, we must have

$$\frac{\partial u}{\partial x} = 0 \quad i.e., \quad ay - x^2 = 0$$

$$\frac{\partial u}{\partial y} = 0 \quad i.e., \quad xz - y^2 = 0$$

and $$\frac{\partial u}{\partial z} = 0 \quad i.e., \quad by - z^2 = 0.$$

From the above equations, it follows that a, x, y, z and b are in geometrical progression. Let r be the common ratio of this geometrical progression. Then

$$ar^4 = b \quad \text{or} \quad r = (b/a)^{1/4}.$$

Also $x = ar$, $y = ar^2$, $z = ar^3$.

Substituting these values, we get

$$u = \frac{ar \,.\, ar^2 \,.\, ar^3}{a\,(1+r) \,.\, ar\,(1+r) \,.\, ar^2\,(1+r) \,.\, ar^3\,(1+r)}$$

$$= \frac{1}{a\,(1+r)^4} = \frac{1}{a\,[1 + (b/a)^{1/4}]^4} = \frac{1}{(a^{1/4} + b^{1/4})^4}.$$

This gives a stationary value of u. To decide whether this value of u is a maximum or a minimum we proceed to find the second order partial derivatives of u.

We have

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2xu}{x\,(a+x)\,(x+y)} + (ay - x^2)\,\frac{\partial}{\partial x}\left[\frac{u}{x\,(a+x)\,(x+y)}\right].$$

$\therefore$ when $x = ar$, $y = ar^2$, $z = ar^3$, we have

$$A = \frac{\partial^2 u}{\partial x^2} = \frac{-2 \,.\, ar \,.\, u}{ar \,.\, a\,(1+r) \,.\, ar\,(1+r)}$$

$$= \frac{-2u}{a^2 r\,(1+r)^2}, \text{ which is negative.}$$

Hence the above stationary value of u is a maximum.

Ans. Maximum value of $u = \dfrac{1}{(a^{1/4} + b^{1/4})^4}$.

Note: In the complicated problems in order to find whether a stationary value of u is a maximum or a minimum, it is sufficient to find the value of a second partial differential coefficient of u with respect to any of the independent variables. The value of u will be maximum or minimum according as the value of this second partial derivative at the stationary point under consideration is –ive or +ive.

Comprehensive Exercise 1

1. Show that $u = (x + y + z)^3 - 3(x + y + z) - 24xyz + a^3$ has minimum at (1, 1, 1) and maximum at $(-1, -1, -1)$.
2. Find the maximum or minimum values of u where
$$u = axy^2z^3 - x^2y^2z^3 - xy^3z^3 - xy^2z^4.$$
3. Find the maximum value of
$$(ax + by + cz)\, e^{-(\alpha^2x^2 + \beta^2y^2 + \gamma^2z^2)}.$$

Answers 1

2. u is maximum when $x = a/7$, $y = 2a/7$, $z = 3a/7$ and the maximum value is $108a^2/7$.
3. $\sqrt{\left\{\frac{1}{2e}\left(\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2}\right)\right\}}$

6.6 Lagrange's Method of Undetermined Multipliers

Let $u = f(x_1, x_2, \ldots, x_n)$

be a function of n variables $x_1, x_2, \ldots, x_n$. Let these variables be connected by m equations

$$\phi_1(x_1, x_2, \ldots, x_n) = 0,\ \phi_2(x_1, x_2, \ldots, x_n) = 0, \ldots,$$
$$\phi_m(x_1, x_2, \ldots, x_n) = 0$$

so that only $n - m$ of the n variables are independent.

For a maximum or a minimum of u, we have

$$du = \frac{\partial u}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2 + \frac{\partial u}{\partial x_3}dx_3 + \ldots + \frac{\partial u}{\partial x_n}dx_n = 0.$$

Also differentiating the m given equations connecting the variables, we get

$$d\phi_1 = \frac{\partial \phi_1}{\partial x_1}dx_1 + \frac{\partial \phi_1}{\partial x_2}dx_2 + \frac{\partial \phi_1}{\partial x_3}dx_3 + \ldots + \frac{\partial \phi_1}{\partial x_n}dx_n = 0,$$

$$d\phi_2 = \frac{\partial \phi_2}{\partial x_1}dx_1 + \frac{\partial \phi_2}{\partial x_2}dx_2 + \frac{\partial \phi_2}{\partial x_3}dx_3 + \ldots + \frac{\partial \phi_2}{\partial x_n}dx_n = 0,$$

... ...
... ...

$$d\phi_m = \frac{\partial \phi_m}{\partial x_1} dx_1 + \frac{\partial \phi_m}{\partial x_2} dx_2 + \frac{\partial \phi_m}{\partial x_3} dx_3 + \ldots + \frac{\partial \phi_m}{\partial x_n} dx_n = 0 .$$

Multiplying the above $m+1$ equations obtained on differentiation by $1, \lambda_1, \lambda_2, \ldots, \lambda_m$ respectively and adding, we get an equation which may be written as

$$P_1\, dx_1 + P_2\, dx_2 + P_3\, dx_3 + \ldots + P_n\, dx_n = 0 , \qquad \ldots(1)$$

where $$P_r = \frac{\partial u}{\partial x_r} + \lambda_1 \frac{\partial \phi_1}{\partial x_r} + \lambda_2 \frac{\partial \phi_2}{\partial x_r} + \ldots + \lambda_m \frac{\partial \phi_m}{\partial x_r} .$$

Now the m multipliers $\lambda_1, \lambda_2, \ldots, \lambda_m$ are at our choice. We choose them such that they satisfy the m linear equations

$$P_1 = 0 , P_2 = 0 , \ldots, P_m = 0 .$$

Then the equation (1) reduces to

$$P_{m+1}\, dx_{m+1} + P_{m+2}\, dx_{m+2} + \ldots + P_n\, dx_n = 0 . \qquad \ldots(2)$$

It is immaterial which of the $n - m$ of the n variables $x_1, x_2, \ldots, x_n$ are regarded as independent. Let us regard the $n - m$ variables $x_{m+1}, x_{m+2}, \ldots, x_n$ as independent. Then since the $n - m$ quantities $dx_{m+1}, dx_{m+2}, \ldots, dx_n$ are all independent of one another, their coefficients must be separately zero in the relation (2). Hence we must have

$$P_{m+1} = 0 , P_{m+2} = 0 , \ldots, P_n = 0 .$$

Thus we get $m + n$ equations

$$P_1 = 0 , P_2 = 0 , \ldots, P_n = 0$$

and $$\phi_1 = 0 , \phi_2 = 0 , \ldots, \phi_m = 0 ,$$

which together with the relation $u = f(x_1, x_2, \ldots, x_n)$ determine the m multipliers $\lambda_1, \lambda_2, \ldots, \lambda_m$, the values of $x_1, x_2, \ldots, x_n$ and u at the stationary point. This method is known as **Lagrange's method of undetermined multipliers.** It is very convenient to apply and it often gives us the maximum or minimum values of u without actually determining the values of the multipliers $\lambda_1, \ldots, \lambda_m$. The only drawback of this method is that it does not determine the nature of the stationary point.

Illustrative Examples

Example 4: *If* $u = x^2 + y^2 + z^2$, *where* $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$, *find the maximum or minimum values of* u.

Solution: We have $u = x^2 + y^2 + z^2$, ...(1)

where the variables x, y, z are connected by the relation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1 . \qquad \ldots(2)$$

For a maximum or a minimum of u, we have $du = 0$

$\Rightarrow \quad 2x\,dx + 2y\,dy + 2z\,dz = 0$

$\Rightarrow \quad x\,dx + y\,dy + z\,dz = 0$ . ...(3)

Also differentiating the given relation (2), we get

$$2ax\,dx + 2by\,dy + 2cz\,dz + 2fy\,dz + 2fz\,dy + 2gz\,dx + 2gx\,dz + 2hx\,dy + 2hy\,dx = 0$$

or $(ax + hy + gz)\,dx + (hx + by + fz)\,dy + (gx + fy + cz)\,dz = 0$ . ...(4)

Multiplying (3) by 1, (4) by λ and adding, and then equating the coefficients of dx, dy, dz to zero, we have

$$x + \lambda\,(ax + hy + gz) = 0\,, \quad ...(5)$$

$$y + \lambda\,(hx + by + fz) = 0\,, \quad ...(6)$$

and $z + \lambda\,(gx + fy + cz) = 0$ . ...(7)

Multiplying (5) by x, (6) by y, (7) by z and adding, we get

$$x^2 + y^2 + z^2 + \lambda\,(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = 0$$

or $u + \lambda \cdot 1 = 0$, using (1) and (2).

$\therefore \quad \lambda = -u$.

Hence from (5), we have

$$x - u\,(ax + hy + gz) = 0$$

or $x\,(1 - au) - huy - guz = 0$

or $\left(a - \frac{1}{u}\right)x + hy + gz = 0$ . ...(8)

Similarly from (6) and (7), we have

$$hx + (b - 1/u)\,y + fz = 0\,, \quad ...(9)$$

and $gx + fy + (c - 1/u)\,z = 0$ . ...(10)

Eliminating x, y, z from (8), (9), (10), we get

$$\begin{vmatrix} a - (1/u) & h & g \\ h & b - (1/u) & f \\ g & f & c - (1/u) \end{vmatrix} = 0\,. \quad ...(11)$$

Hence the required maximum or minimum values of u are the roots of the equation (11).

Example 5: *Find the stationary values of* $x^2 + y^2 + z^2$ *subject to the conditions*

$$ax^2 + by^2 + cz^2 = 1 \text{ and } lx + my + nz = 0\,.$$

Interpret the result geometrically.

Solution: Let $u = x^2 + y^2 + z^2$, ...(1)

where the variables x, y and z are connected by the relations

$$ax^2 + by^2 + cz^2 = 1\,, \quad ...(2)$$

and $lx + my + nz = 0$ . ...(3)

For a stationary value of u , we have

$$du = 0$$

$$\Rightarrow \quad 2x\,dx + 2y\,dy + 2z\,dz = 0$$

$$\Rightarrow \quad x\,dx + y\,dy + z\,dz = 0 . \qquad ...(4)$$

Also differentiating the given relations (2) and (3), we get

$$2ax\,dx + 2by\,dy + 2cz\,dz = 0$$

i.e., $ax\,dx + by\,dy + cz\,dz = 0$ ...(5)

and $l\,dx + m\,dy + n\,dz = 0$ . ...(6)

Multiplying (4) by 1, (5) by λ and (6) by μ and adding, and then equating the coefficients of dx , dy , dz to zero, we get

$$x + \lambda ax + \mu l = 0 , \qquad ...(7)$$

$$y + \lambda by + \mu m = 0 , \qquad ...(8)$$

and $z + \lambda cz + \mu n = 0$ . ...(9)

Multiplying the equations (7), (8) and (9) by x, y and z respectively and adding, we get

$$x^2 + y^2 + z^2 + \lambda\,(ax^2 + by^2 + cz^2) + \mu\,(lx + my + nz) = 0 ,$$

or $u + \lambda .1 + \mu .0 = 0$, using (1), (2) and (3)

or $\lambda = -u$.

Substituting for λ in the equations (7), (8) and (9), we get

$$x = \frac{\mu l}{au - 1} , \; y = \frac{\mu m}{bu - 1} , \; z = \frac{\mu n}{cu - 1} .$$

Substituting these values of x, y, z in (3), we get

$$\frac{\mu l^2}{au - 1} + \frac{\mu m^2}{bu - 1} + \frac{\mu n^2}{cu - 1} = 0$$

or $$\frac{l^2}{au - 1} + \frac{m^2}{bu - 1} + \frac{n^2}{cu - 1} = 0 . \qquad ...(10)$$

Hence the stationary (*i.e.*, maximum or minimum) values of u are given by the equation (10). The equation (10) is a quadratic in u and so it gives two stationary values of u .

Geometrical interpretation: The surface $ax^2 + by^2 + cz^2 = 1$ represents an ellipsoid (or a hyperboloid) whose centre is origin, and $lx + my + nz = 0$ is a plane passing through the origin. Therefore the point (x, y, z) satisfying both the conditions (2) and (3) lies on the conic in which (2) and (3) intersect. Also $x^2 + y^2 + z^2$ gives the square of the distance of (x, y, z) from the origin which is also the centre of the conic of intersection. The maximum and minimum values of

this distance are the major and minor semi-axes of the conic. So the equation (10) gives the squares of the lengths of the semi-axes of the conic of intersection.

Example 6: *Find the maximum and minimum values of*

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4},$$

when $lx + my + nz = 0$ *and* $x^2 / a^2 + y^2 / b^2 + z^2 / c^2 = 1$.

Interpret the result geometrically.

Solution: Let $u = x^2 / a^4 + y^2 / b^4 + z^2 / c^4$. Then for a maximum or a minimum of u, we have

$$du = 0 \quad \Rightarrow \quad \frac{x}{a^4}\,dx + \frac{y}{b^4}\,dy + \frac{z}{c^4}\,dz = 0 \qquad \ldots(1)$$

Also differentiating the two given equations connecting the variables x, y and z, we get

$$l\,dx + m\,dy + n\,dz = 0, \qquad \ldots(2)$$

and

$$\frac{x}{a^2}\,dx + \frac{y}{b^2}\,dy + \frac{z}{c^2}\,dz = 0. \qquad \ldots(3)$$

Multiplying (1), (2) and (3) by l, λ and μ respectively and adding, and then equating to zero the coefficients of dx, dy and dz, we get

$$\frac{x}{a^4} + \lambda l + \mu \frac{x}{a^2} = 0, \qquad \ldots(4)$$

$$\frac{y}{b^4} + \lambda m + \mu \frac{y}{b^2} = 0, \qquad \ldots(5)$$

and

$$\frac{z}{c^4} + \lambda n + \mu \frac{z}{c^2} = 0. \qquad \ldots(6)$$

Multiplying the equations (4), (5) and (6) by x, y and z respectively and adding, we get

$$u + \lambda .0 + \mu .1 = 0 \quad \text{or} \quad \mu = -u.$$

Putting $\mu = -u$ in (4), we get

$$\frac{x}{a^4} + \lambda l - \frac{xu}{a^2} = 0, \quad \text{or} \quad \frac{x}{a^2}\left\{u - \frac{1}{a^2}\right\} = \lambda l,$$

or

$$x = \frac{\lambda l a^4}{a^2 u - 1}.$$

Similarly from (5) and (6), we get

$$y = \frac{\lambda m b^4}{b^2 u - 1} \quad \text{and} \quad z = \frac{\lambda n c^4}{c^2 u - 1}.$$

Substituting these values of x, y, z in $lx + my + nz = 0$, we get

$$\frac{l^2 a^4}{a^2 u - 1} + \frac{m^2 b^4}{b^2 u - 1} + \frac{n^2 c^4}{c^2 u - 1} = 0. \qquad \ldots(7)$$

The equation (7) gives the required maximum or minimum values of u.

Geometrical interpretation: The equation of the tangent plane to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at any point (x, y, z) on it is

$$\frac{Xx}{a^2} + \frac{Yy}{b^2} + \frac{Zz}{c^2} = 1. \qquad ...(8)$$

If p be the length of the perpendicular from origin which is also the centre of the ellipsoid to the tangent plane (1), then

$$p^2 = \frac{1}{x^2/a^4 + y^2/b^4 + z^2/c^4}.$$

If the point (x, y, z) on the ellipsoid also lies on the given plane $lx + my + nz = 0$, the problem consists of finding out the maximum or minimum values of the perpendicular distance from the origin to the tangent planes to the ellipsoid at the points common to the plane $lx + my + nz = 0$ and the ellipsoid.

Example 7: *Prove that of all rectangular parallelopipeds of the same volume, the cube has the least surface.*

Solution: Let x, y, z be the dimensions of the rectangular parallelopiped, V be its volume and S be its surface. Then

$$S = 2xy + 2yz + 2zx \qquad ...(1)$$

and $$xyz = V = \text{some constant}. \qquad ...(2)$$

For a maximum or a minimum of S, we have

$$dS = 2(y + z)\,dx + 2(z + x)\,dy + 2(x + y)\,dz = 0$$

i.e., $$(y + z)\,dx + (z + x)\,dy + (x + y)\,dz = 0. \qquad ...(3)$$

Also differentiating (2) and observing that V is constant, we have

$$yz\,dx + zx\,dy + xy\,dz = 0. \qquad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dx, dy and dz, we get

$$(y + z) + \lambda yz = 0, \qquad ...(5)$$

$$(z + x) + \lambda zx = 0, \qquad ...(6)$$

and $$(x + y) + \lambda xy = 0. \qquad ...(7)$$

These give $-\lambda = \frac{1}{y} + \frac{1}{z} = \frac{1}{z} + \frac{1}{x} = \frac{1}{x} + \frac{1}{y}$.

$\therefore$ $$\frac{1}{y} - \frac{1}{x} = 0 \quad \text{or} \quad x = y.$$

Similarly $y = z$.

Hence for a stationary value of S, we have

$$x = y = z = V^{1/3}, \text{ from (2).}$$

Thus S is stationary when the rectangular parallelopiped is a cube.

Let us now find the nature of this stationary value of S.

Here S is a function of three variables x, y, z which are connected by the relation (2) so that only two variables are independent. Let us regard x and y as independent variables and z to be dependent on them.

Then from (1), $\frac{\partial S}{\partial x} = 2y + 2y\frac{\partial z}{\partial x} + 2z + 2x\frac{\partial z}{\partial x}$.

Also from (2), $yz + xy\frac{\partial z}{\partial x} = 0$ *i.e.*, $\frac{\partial z}{\partial x} = -\frac{z}{x}$.

$\therefore \quad \frac{\partial S}{\partial x} = 2y - \frac{2yz}{x} + 2z - 2z = 2y - \frac{2yz}{x}$

and $\frac{\partial^2 S}{\partial x^2} = \frac{2yz}{x^2} - \frac{2y}{x}\cdot\frac{\partial z}{\partial x} = \frac{2yz}{x^2} + \frac{2yz}{x^2} = \frac{4yz}{x^2} = 4$ at $x = y = z$.

Similarly by symmetry $\frac{\partial^2 S}{\partial y^2} = 4$ at $x = y = z$.

Also $\frac{\partial^2 S}{\partial x\,\partial y} = 2 - \frac{2z}{x} - \frac{2y}{x}\frac{\partial z}{\partial y}$.

But differentiating (2) partially w.r.t. y taking x as constant, we get

$$xz + xy\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad \frac{\partial z}{\partial y} = -\frac{z}{y}.$$

$\therefore \quad \frac{\partial^2 S}{\partial x\,\partial y} = 2 - \frac{2z}{x} - \frac{2y}{x}\left(-\frac{z}{y}\right) = 2 - \frac{2z}{x} + \frac{2z}{x} = 2$.

Thus at the stationary point $x = y = z = V^{1/3}$, we have

$$r = \frac{\partial^2 S}{\partial x^2} = 4, s = \frac{\partial^2 S}{\partial x\,\partial y} = 2 \text{ and } t = \frac{\partial^2 S}{\partial y^2} = 4.$$

$\therefore \quad rt - s^2 = 4 \times 4 - 2^2 = 12$ which is > 0.

Also $r = 4$ which is > 0.

Hence the stationary value of S given by $x = y = z = V^{1/3}$ is a minimum.

Thus of all rectangular paralleopipeds of the same volume, the cube has the least surface.

Example 8: *Discuss the maxima and minima of the function* $u = \sin x \sin y \sin z$, *where* x, y, z *are the angles of a triangle.*

Solution: We have $u = \sin x \sin y \sin z$, ...(1)

where $x + y + z = \pi$. ...(2)

For a maximum or a minimum of u, we must have

$$du = \cos x \sin y \sin z\, dx + \sin x \cos y \sin z\, dy + \sin x \sin y \cos z\, dz = 0. \quad \text{...(3)}$$

Also from (2), we have

$$dx + dy + dz = 0 . \qquad ...(4)$$

Multiplying (3) by 1 and (4) by λ and adding and then equating to zero the coefficients of dx, dy, dz, we get

$$\cos x \sin y \sin z + \lambda = 0 ,$$

$$\sin x \cos y \sin z + \lambda = 0 ,$$

and $$\sin x \sin y \cos z + \lambda = 0 .$$

From these, we get

$$-\lambda = \cos x \sin y \sin z = \sin x \cos y \sin z = \sin x \sin y \cos z$$

or $$\cot x = \cot y = \cot z \qquad [\text{Dividing by } \sin x \sin y \sin z]$$

i.e., $$x = y = z = \pi / 3 . \qquad [\text{From (2)}]$$

Thus u is stationary when $x = y = z = \pi / 3$.

Let us now find the nature of this stationary value of u.

Since variables x, y and z are connected by the relation (2), only two of them may be regarded as independent.

Let us regard x and y as independent and z to be dependent on them by the relation (2).

Then from (1),

$$\frac{\partial u}{\partial x} = \sin y \sin z \cos x + \sin x \sin y \cos z \frac{\partial z}{\partial x} .$$

Also form (2),

$$1 + \frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \frac{\partial z}{\partial x} = -1 .$$

$$\therefore \quad \frac{\partial u}{\partial x} = \sin y \sin z \cos x - \sin x \sin y \cos z$$

and $$\frac{\partial^2 u}{\partial x^2} = -\sin y \sin z \sin x + \sin y \cos x \cos z \frac{\partial z}{\partial x}$$

$$- \cos x \sin y \cos z + \sin x \sin y \sin z \frac{\partial z}{\partial x}$$

$$= -2 \sin x \sin y \sin z - 2 \sin y \cos x \cos z .$$

Also $$\frac{\partial^2 u}{\partial x \partial y} = \cos y \sin z \cos x + \sin y \cos x \cos z \frac{\partial z}{\partial y}$$

$$- \sin x \cos y \cos z + \sin x \sin y \sin z \frac{\partial z}{\partial y}$$

$$= \cos y \sin z \cos x - \sin y \cos x \cos z - \sin x \cos y \cos z .$$

$$- \sin x \sin y \sin z$$

Hence putting $x = y = z = \pi / 3$, we get

$$r = \frac{\partial^2 u}{\partial x^2} = -2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$= -\frac{3\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = -\sqrt{3},$$

$$s = \frac{\partial^2 u}{\partial x\, \partial y} = \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8} - \frac{3\sqrt{3}}{8} = -\frac{\sqrt{3}}{2}.$$

Also by symmetry, $t = \dfrac{\partial^2 u}{\partial y^2} = -\sqrt{3}$.

$\therefore$ at the stationary point $x = y = z = \pi/3$, we have

$$rt - s^2 = 3 - (3/4) = 9/4 \text{ which is} > 0$$

and $r = -\sqrt{3}$ which is < 0.

$\therefore$ at the stationary point $x = y = z = \pi/3$, u is maximum.

Hence u is maximum when $x = y = z = \pi/3$ and the maximum value of

$$u = \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}.$$

Example 9: *Show that the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \textit{is} \quad \frac{8abc}{3\sqrt{3}}.$$

Solution: Let (x, y, z) denote the coordinates of the vertex of the rectangular parallelopiped which lies in the positive octant and let V denote its volume. Then, we have to find the maximum value of

$$V = 8xyz \qquad \ldots(1)$$

subject to the condition

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \qquad \ldots(2)$$

For a maximum or a minimum of V, we have

$$dV = 8\,yz\,dx + 8zx\,dy + 8xy\,dz = 0$$

i.e.,
$$yz\,dx + zx\,dy + xy\,dz = 0. \qquad \ldots(3)$$

Also differentiating (2), we get

$$\frac{2x}{a^2}\,dx + \frac{2y}{b^2}\,dy + \frac{2z}{c^2}\,dz = 0$$

i.e.,
$$\frac{x}{a^2}\,dx + \frac{y}{b^2}\,dy + \frac{z}{c^2}\,dz = 0. \qquad \ldots(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating the coefficients of dx, dy, dz to zero, we get

$$yz + \frac{\lambda x}{a^2} = 0 \ , \ zx + \frac{\lambda y}{b^2} = 0 \text{ and } xy + \frac{\lambda z}{c^2} = 0 \ .$$

From these, we get

$$\frac{x}{a^2} = -\frac{yz}{\lambda} \ , \ \frac{y}{b^2} = -\frac{zx}{\lambda} \ , \ \frac{z}{c^2} = -\frac{xy}{\lambda}$$

or $$\frac{x^2}{a^2} = -\frac{xyz}{\lambda} \ , \ \frac{y^2}{b^2} = -\frac{xyz}{\lambda} \ , \ \frac{z^2}{c^2} = -\frac{xyz}{\lambda}$$

or $$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{x^2 / a^2 + y^2 / b^2 + z^2 / c^2}{3} = \frac{1}{3} \text{ , using (2)}$$

or $$x = a / \sqrt{3} \ , \ y = b / \sqrt{3} \ , z = c / \sqrt{3} \ .$$

Thus V is stationary when $x = a / \sqrt{3}$, $y = b / \sqrt{3}$, $z = c / \sqrt{3}$.

Now regard x and y as independent variables and z as a function of x and y given by (2).

Then from (1), $\dfrac{\partial V}{\partial x} = 8yz + 8xy \dfrac{\partial z}{\partial x}$.

Differentiating (2) partially w.r.t. x taking y as constant, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \frac{\partial z}{\partial x} = -\frac{c^2 x}{a^2 z} \ .$$

$$\therefore \quad \frac{\partial V}{\partial x} = 8yz + 8xy \cdot \left(-\frac{c^2 x}{a^2 z}\right) = 8yz - \frac{8c^2 x^2 y}{a^2 z}$$

and so $$\frac{\partial^2 V}{\partial x^2} = 8y \frac{\partial z}{\partial x} - \frac{16c^2 x y}{a^2 z} + \frac{8c^2 x^2 y}{a^2 z^2} \cdot \frac{\partial z}{\partial x}$$

$$= 8y \cdot \left(\frac{-c^2 x}{a^2 z}\right) - \frac{16c^2 x y}{a^2 z} - \frac{8c^2 x^2 y}{a^2 z} \cdot \frac{c^2 x}{a^2 z} \ ,$$

which is –ive when $x = a / \sqrt{3}$, $y = b / \sqrt{3}$, $z = c / \sqrt{3}$.

Hence V is maximum when $x = a / \sqrt{3}$, $y = b / \sqrt{3}$, $z = c / \sqrt{3}$ and the maximum value of $V = \dfrac{8abc}{3\sqrt{3}}$.

Note: In complicated problems to show that whether the stationary value of a function is maximum or minimum, it will be sufficient to see whether the second partial differential coefficient of the function w.r.t. any of the independent variables is negative or positive.

Example 10: *A rectangular box open at the top is to have a given capacity. Find the dimensions of the box requiring least material for its construction.*

Solution: Let the given capacity of the box be V, its three edges be x, y, z and its surface be S . Then

$$S = xy + 2yz + 2zx \qquad \ldots(1)$$

and $xyz = V$ . ...(2)

For a maximum or a minimum of S , we have

$$dS = (y + 2z)\,dx + (x + 2z)\,dy + 2(x + y)\,dz = 0 . \quad ...(3)$$

Also from (2), since V is constant, we have $yz\,dx + zx\,dy + xy\,dz = 0$ . ...(4)

Multiplying (3) by 1 and (4) by λ , and adding and then equating to zero the coefficients of dx , dy and dz , we get

$$(y + 2z) + \lambda yz = 0 , \quad ...(5)$$

$$(x + 2z) + \lambda zx = 0 , \quad ...(6)$$

and $$2x + 2y + \lambda xy = 0 . \quad ...(7)$$

Multiplying (5) by x , (6) by y and subtracting, we get

$$2zx - 2zy = 0 \text{ or } 2z(x - y) = 0 , \text{ or } x = y .$$

[The root $z = 0$ is inadmissible because the depth of the box cannot be zero.]

Similarly, from the equations (6) and (7), we get $y = 2z$.

Hence the dimensions of the box for a stationary value of S are

$$x = y = 2z = (2V)^{1/3} , \text{ from (2).}$$

Let us now find the nature of this stationary value of S .

Regard x and y as independent variables and z as a function of x and y given by (2).

Then from (1), $\dfrac{\partial S}{\partial x} = y + 2y\dfrac{\partial z}{\partial x} + 2z + 2x\dfrac{\partial z}{\partial x}$.

Differentiating (2) partially w.r.t. x taking y as constant, we get

$$yz + xy\frac{\partial z}{\partial x} = 0 \text{ } i.e., \frac{\partial z}{\partial x} = -\frac{z}{x} .$$

$$\therefore \quad \frac{\partial S}{\partial x} = y - \frac{2yz}{x} + 2z - 2z = y - \frac{2yz}{x}$$

and so $$\frac{\partial^2 S}{\partial x^2} = \frac{2yz}{x^2} - \frac{2y}{x}\cdot\frac{\partial z}{\partial x} = \frac{2yz}{x^2} + \frac{2yz}{x^2} = \frac{4yz}{x^2} = 2 \text{ at } x = y = 2z .$$

Thus at the stationary point $x = y = 2z = (2V)^{1/3}$, we have

$$r = \frac{\partial^2 S}{\partial x^2} = 2 , \text{ which is positive.}$$

Similarly we can find $s = \dfrac{\partial^2 S}{\partial x\,\partial y}$ and $t = \dfrac{\partial^2 S}{\partial y^2}$

at the stationary point $x = y = 2z = (2V)^{1/3}$ and can show that $rt - s^2$ is positive.

Since at the stationary point $x = y = 2z = (2V)^{1/3}$, $rt - s^2 > 0$ and $r > 0$, therefore the stationary value of S at this point is a minimum.

Hence the dimensions of the box requiring least meterial for its construction are given by $x = y = 2z = (2V)^{1/3}$.

Comprehensive Exercise 2

1. Find the maximum and minimum values of
$$u = a^2x^2 + b^2y^2 + c^2z^2,$$
where $x^2 + y^2 + z^2 = 1$ and $lx + my + nz = 0$.

2. Show that the maximum and minimum values of
$$u = x^2 + y^2 + z^2$$
subject to the conditions
$$px + qy + rz = 0 \text{ and } x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$
are given by the equation
$$\frac{a^2p^2}{u - a^2} + \frac{b^2q^2}{u - b^2} + \frac{c^2r^2}{u - c^2} = 0.$$

3. Find the maximum and minimum values of u^2 when
$$u^2 = a^2x^2 + b^2y^2 + c^2z^2$$
while $x^2 + y^2 + z^2 = 1$ and $lx + my + nz = 0$.

4. Find the maximum value of $x^m y^n z^p$ subject to the condition
$$x + y + z = a.$$

5. Find the minimum value of $x^2 + y^2 + z^2$, given that
$$ax + by + cz = p.$$

6. Find the minimum value of $x + y + z$, subject to the condition
$$(a/x) + (b/y) + (c/z) = 1.$$

7. Find the maximum or minimum value of $x^p y^q z^r$ subject to the condition
$$ax + by + cz = p + q + r.$$

8. Find the maximum or minimum value of $x^p y^q z^r$ subject to the condition
$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1.$$

9. Find the minimum value of $x^4 + y^4 + z^4$, where $xyz = c^3$.

10. Divide a number a into three parts such that their product will be maximum.

11. In a plane triangle ABC, find the maximum value of
$$u = \cos A \cos B \cos C.$$

12. Find a plane triangle ABC such that
$$u = \sin^m A \sin^n B \sin^p C$$

has maximum value.

13. Show that if the perimeter of a triangle is constant, its area is a maximum when it is equilateral.

14. Find the triangle of maximum area inscribed in a circle.

15. Prove that the rectangular solid of maximum volume which can be inscribed in a sphere is a cube.

16. Given $u = 5xyz / (x + 2y + 4z)$. Find the values of x, y, z for which u is maximum subject to the condition $xyz = 8$.

17. Show that the maximum and minimum of the radii vectors of the sections of the surface

$$(x^2 + y^2 + z^2)^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

by the plane $\lambda x + \mu y + \nu z = 0$ are given by the equation

$$\frac{a^2\lambda^2}{1 - a^2 r^2} + \frac{b^2\mu^2}{1 - b^2 r^2} + \frac{c^2\nu^2}{1 - c^2 r^2} = 0 .$$

18. If two variables x and y are connected by the relation $ax^2 + by^2 = ab$, show that the maximum and minimum values of the function $u = x^2 + y^2 + xy$ will be the roots of the equation

$$4(u - a)(u - b) = ab .$$

19. Find the maximum or minimum value of $x^2 + y^2 + z^2$, subject to the conditions

$$lx + my + nz = 1, l'x + m'y + n'z = 1 .$$

20. Show that the maximum and minimum values of

$$u = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

subject to the conditions $lx + my + nz = 0$ and $x^2 + y^2 + z^2 = 1$ are given by the equation

$$\begin{vmatrix} a - u & h & g & l \\ h & b - u & f & m \\ g & f & c - u & n \\ l & m & n & 0 \end{vmatrix} = 0 .$$

21. Find the maximum value of u, when

$$u = x^2 y^3 z^4 \text{ and } 2x + 3y + 4z = a .$$

22. Find the points where $u = ax^p + by^q + cz^r$ has extreme values subject to the condition

$$x^l + y^m + z^n = k .$$

23. Determine the maximum value of OP, O being the origin of coordinates where P describes the curve $x^2 + y^2 + 2z^2 = 5$, $x + 2y + z = 5$.

24. Prove that if $x + y + z = 1$, $ayz + bzx + cxy$ has an extreme velue equal to

$$\frac{abc}{2bc + 2ca + 2ab - a^2 - b^2 - c^2}.$$

Prove also that if a, b, c are all positive and c lies between $a + b - 2\sqrt{(ab)}$ and $a + b + 2\sqrt{(ab)}$ this value is true maximum and that if a, b, c are all negative and c lies between $a + b \pm 2\sqrt{(ab)}$, it is true minimum.

25. Find the maxima and minima of $x^2 + y^2$ subject to the condition

$$ax^2 + 2hxy + by^2 = 1.$$

Answers 2

1. The maximum or minimum values of u are the roots of the equation

$$\frac{l^2}{u - a^2} + \frac{m^2}{u - b^2} + \frac{n^2}{u - c^2} = 0$$

3. The required values are the roots of the equation

$$\frac{l^2}{u^2 - a^2} + \frac{m^2}{u^2 - b^2} + \frac{n^2}{u^2 - c^2} = 0$$

4. $m^m n^n p^p a^{m+n+p} / (m + n + p)^{m+n+p}$

5. $p^2 / (a^2 + b^2 + c^2)$

6. A minimum when $\frac{x}{\sqrt{a}} = \frac{y}{\sqrt{b}} = \frac{z}{\sqrt{c}} = \sqrt{a} + \sqrt{b} + \sqrt{c}$

and the minimum value $= (\sqrt{a} + \sqrt{b} + \sqrt{c})^2$

7. A maximum value is $(p / a)^p (q / b)^q (r / c)^r$

8. A minimum when $\frac{px}{a} = \frac{qy}{b} = \frac{rz}{c} = p + q + r$

9. A minimum when $x = y = z = c$ and the minimum value $= 3c^4$

10. $a / 3, a / 3, a / 3$.

11. u is maximum when $A = B = C = \frac{\pi}{3}$ and the maximum value of $u = \frac{1}{8}$

12. A maximum when

$$\frac{\tan A}{m} = \frac{\tan B}{n} = \frac{\tan C}{p}$$

14. An equilateral triangle

16. $x = 4, y = 2, z = 1$

19. The minimum value $\Sigma (l - l')^2 \,/\, \Sigma(mm' - m'n)^2$.

21. $(a/9)^9$.

22. The values of x, y, z are given by

$$\frac{x^{p-l}}{l/pa} = \frac{y^{q-m}}{m/qb} = \frac{z^{r-n}}{n/rc}.$$

23. $\sqrt{5}$.

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The function $u = \sin x \sin y \sin z$, where x, y, z are the angles of a triangle, is stationary at the point

(a) $x = \frac{\pi}{2}, y = \frac{\pi}{4}, z = \frac{\pi}{4}$

(b) $x = y = z = \frac{\pi}{3}$

(c) $x = \frac{\pi}{4}, y = \frac{\pi}{2}, z = \frac{\pi}{4}$

(d) $x = 0, y = \frac{\pi}{2}, z = \frac{\pi}{2}$

2. The maximum and minimum values of $u = a^2x^2 + b^2y^2 + c^2z^2$ where $x^2 + y^2 + z^2 = 1$ and $lx + my + nz = 0$ are the roots of the equation

(a) $\frac{l^2}{u - a^2} + \frac{m^2}{u - b^2} + \frac{n^2}{u - c^2} = 0$ (b) $\frac{l}{u - a^2} + \frac{m}{u - b^2} + \frac{n}{u - c^2} = 0$

(c) $\frac{l^2}{u - a} + \frac{m^2}{u - b} + \frac{n^2}{u - c} = 0$ (d) $\frac{l}{u - a} + \frac{m}{u - b} + \frac{n}{u - c} = 0$

Fill In The Blank(s)

Fill in the blanks "....." so that the following statements are complete and correct.

1. Let $f(x, y, z)$ be a function of three independent variables x, y and z. The necessary conditions for the existence of a maximum or a minimum of $f(x, y, z)$ at $x = a, y = b$ and $z = c$ are

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0 \quad \text{and} \quad \frac{\partial f}{\partial z} = \ldots \text{at } x = a, y = b, z = c.$$

2. For a maximum or a minimum of $u = x^2 + y^2 + z^2 + x - 2z - xy$, we must have

$$2x - y + 1 = 0, \ldots, \text{ and } 2z - 2 = 0.$$

True or False

Write 'T' for true and 'F' for false statement.

1. The function $u = x^2 + y^2 + z^2 + x - 2z - xy$ has a maximum at the point

$$x = -\frac{2}{3}, y = -\frac{1}{3}, z = 1.$$

2. The volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ is } \frac{8abc}{3\sqrt{3}}.$$

3. The maximum or minimum values of $u = x^2 + y^2 + z^2$ subject to the conditions

$$ax^2 + by^2 + cz^2 = 1 \text{ and } lx + my + nz = 0$$

are the roots of the equation $\frac{l^2}{au - 1} + \frac{m^2}{bu - 1} + \frac{n^2}{cu - 1} = 0$.

4. The minimum value of $x + y + z$, subject to the condition $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$, is $(a + b + c)^2$.

Answers

Multiple Choice Questions

1. (b) 2. (a)

Fill in the Blank(s)

1. 0 2. $-x + 2y = 0$.

True or False

1. *F* 2. *T* 3. *T* 4. *F*

□

Chapter

The Riemann Integral

7.1 Introduction

In elementary treatments, the process of integration is generally introduced as the inverse of differentiation. If $F'(x) = f(x)$ for all x belonging to the domain of the function f, F is called an integral of the given function f.

Historically, however, the subject of integral arose in connection with the problem of finding areas of plane regions in which the area of a plane region is calculated as the limit of a sum. This notion of integral as summation is based on geometrical concepts.

A German mathematician G.F.B. Riemann gave the first rigorous arithmetic treatment of definite integral free from geometrical concepts. Riemann's definition covered only bounded functions. It was Cauchy who extended this definition to unbounded functions. Later on early in the twentieth century Lebesgue introduced the integral on a firm foundation with many refinements and generalisations.

In the present chapter we shall study the Riemann integral of real valued, bounded functions defined on some closed interval.

7.2 Partitions and Riemann Sums

Definition 1: *Let $I = [a, b]$ be a closed and bounded interval. Then by a* **partition** *(or a* **dissection** *or a* **net***) of I we mean a finite set of real numbers $P = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ having the property that $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$.*

The closed sub-intervals

$$I_1 = [x_0, x_1], I_2 = [x_1, x_2], \ldots, I_n = [x_{n-1}, x_n]$$

determined by P constitute the **segments** *of the partition.*

By changing the set of points, the partition can be changed. Thus there can be an infinite number of partitions of the interval $[a, b]$. The family of all partitions of $[a, b]$ shall be denoted by $\mathrm{P}[a, b]$.

We write $\Delta x_r = x_r - x_{r-1}$ for $r = 1, 2, \ldots, n$ so that Δx_r is the length of the segment $[x_{r-1}, x_r]$. The **norm** (or **mesh**) of a partition P is the greatest of the lengths of the segments of a partition P and it is denoted by $\|P\|$. Thus

$$\|P\| = \max. (\Delta x_r : r = 1, 2, \ldots, n).$$

Sometimes the norm of a partition P is also denoted by $\mu(P)$.

Definition 2: *A partition P^* is called a* **refinement** *of another partition P or we say that P^* is* **finer than** *P iff $P^* \supset P$, i.e., if every point of P is used in the construction of P^*.*

If $P^* = P_1 \cup P_2$, then P^* is called the **common refinement** *of the given two partitions P_1 and P_2.*

Definition 3: *Let f be a bounded function defined on a bounded interval $[a, b]$. Also let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be any partition of $[a, b]$. Also let m_r and M_r be the infimum and supremum respectively of the function f on I_r, for $r = 1, 2, \ldots, n$ i.e.,*

$$m_r = \inf \{ f(x) : x_{r-1} \le x \le x_r \}$$

and

$$M_r = \sup \{ f(x) : x_{r-1} \le x \le x_r \}.$$

Let us now form two sums

$$L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r \text{ and } U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r .$$

Then $L(P, f)$ is called the **lower Riemann sum** *(or* **lower Darboux sum***) of f on $[a, b]$ with respect to the partition P and $U(P, f)$ is called the* **upper Riemann sum** *(or* **upper Darboux sum***) of f on $[a, b]$ with respect to the partition P.*

In brief, we shall refer to these sums as the *lower and upper R-sums* of f with respect to P. Obviously $L(P, f) \le U(P, f)$.

Theorem 1: *Let f be a bounded function defined on $[a, b]$ and let m and M be the infimum and supremum of $f(x)$ in $[a, b]$. Then for any partition P of $[a, b]$, we have*

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a).$$

Proof: Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be any partition of $[a, b]$. Then $I_r = [x_{r-1}, x_r]$, $r = 1, 2, \ldots, n$ are the subintervals of $[a, b]$. Let m_r and M_r be the infimum and supremum of $f(x)$ in $[x_{r-1}, x_r]$. Then for every value of r, we have

$$m \leq m_r \leq M_r \leq M$$

$$\Rightarrow \quad m \Delta x_r \leq m_r \Delta x_r \leq M_r \Delta x_r \leq M \Delta x_r \qquad [\because \Delta x_r > 0]$$

$$\Rightarrow \quad \sum_{r=1}^{n} m \Delta x_r \leq \sum_{r=1}^{n} m_r \Delta x_r \leq \sum_{r=1}^{n} M_r \Delta x_r \leq \sum_{r=1}^{n} M \Delta x_r . \qquad \ldots(1)$$

Now $$\sum_{r=1}^{n} m \Delta x_r = m \sum_{r=1}^{n} \Delta x_r = m \sum_{r=1}^{n} (x_r - x_{r-1})$$

$$= m(x_1 - x_0 + x_2 - x_1 + \ldots + x_n - x_{n-1})$$

$$= m(x_n - x_0) = m(b-a).$$

Similarly $$\sum_{r=1}^{n} M \Delta x_r = M(b-a).$$

Also $$\sum_{r=1}^{n} m_r \Delta x_r = L(P, f) \quad \text{and} \quad \sum_{r=1}^{n} M_r \Delta x_r = U(P, f).$$

Hence, from (1), we conclude that

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a) \ \forall P \in \boldsymbol{P}[a, b].$$

It follows that the sets of upper sums and lower sums are bounded.

Theorem 2: *If $f : [a, b] \to \mathrm{R}$ is a bounded function, then*

$$U(P, -f) = -L(P, f) \quad \text{and} \quad L(P, -f) = -U(P, f). \qquad \textbf{(Meerut 2012)}$$

Proof: Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be any partition of $[a, b]$. Let M_r and m_r be the supremum and infimum of f in I_r.

Now f is bounded on $[a, b] \Rightarrow -f$ is bounded on $[a, b]$.

Again M_r, m_r are supremum and infimum of f in I_r

$\Rightarrow$ $-m_r$, $-M_r$ are supremum and infimum of $-f$ in I_r.

We have $U(P, -f) = \sum_{r=1}^{n} (-m_r) \Delta x_r$, by definition of upper R-sum

$$= -\sum_{r=1}^{n} m_r \Delta x_r = -L(P, f).$$

Also $L(P, -f) = \sum_{r=1}^{n} (-M_r) \Delta x_r$, by definition of lower R-sum

$$= -\sum_{r=1}^{n} M_r \Delta x_r = -U(P, f).$$

Theorem 3: *Let f be a bounded function defined on $[a, b]$ and let P be a partition of $[a, b]$. If P^* is a refinement of P, then*

$$L(P^*, f) \geq L(P, f) \text{ and } U(P^*, f) \leq U(P, f).$$

Proof: Let $P = \{a = x_0, x_1, x_2, \ldots, x_{r-1}, x_r, \ldots, x_n = b\}$

and $P^* = \{a = x_0, x_1, x_2, \ldots, x_{r-1}, y_1, x_r, \ldots, x_n = b\}$,

so that P^* has one more partition point y_1 than P.

Let m_r and M_r be the infimum and supremum of f in $[x_{r-1}, x_r]$. Let M_r', M_r'' be the suprema of f in $[x_{r-1}, y_1]$, $[y_1, x_r]$ and m_r', m_r'' be the infima of f in $[x_{r-1}, y_1]$, $[y_1, x_r]$ respectively.

Then $M_r \geq M_r'$, M_r'' and $m_r \leq m_r'$, m_r''.

Since the rth subinterval only of P is split into two more subintervals of P^* and the remaining subintervals are identical in P and P^*, therefore, we have

$$U(P, f) - U(P^*, f^*)$$
$$= M_r(x_r - x_{r-1}) - \{M_r'(y_1 - x_{r-1}) + M_r''(x_r - y_1)\}.$$

But $M_r'(y_1 - x_{r-1}) + M_r''(x_r - y_1) \leq M_r(y_1 - x_{r-1}) + M_r(x_r - y_1)$
$$= M_r(x_r - x_{r-1}).$$

It gives $U(P, f) - U(P^*, f) \geq M_r(x_r - x_{r-1}) - M_r(x_r - x_{r-1}) = 0$

i.e., $U(P, f) \geq U(P^*, f)$ or $U(P^*, f) \leq U(P, f)$.

If P^* contains p more partition points than P then repeating p times the above argument we can show that

$$U(P^*, f) \leq U(P, f). \quad \ldots(1)$$

In a similar manner we can show that

$$L(P^*, f) \geq L(P, f). \quad \ldots(2)$$

Note: We know that for any partition P^*,

$$L(P^*, f) \leq U(P^*, f). \quad \ldots(3)$$

Thus from (1), (2) and (3), we get

$$L(P, f) \leq L(P^*, f) \leq U(P^*, f) \leq U(P, f).$$

Theorem 4: *If P_1 and P_2 be any two partitions of $[a, b]$ then*

$$U(P_1, f) \geq L(P_2, f).$$

Proof: Let $P = P_1 \cup P_2$. Then P is the common refinement of both the partitions P_1 and P_2. Therefore by the theorem 3 above, we get

$$L(P_2, f) \leq L(P, f) \text{ and } U(P, f) \leq U(P_1, f).$$

But we have $L(P, f) \leq U(P, f)$.

Thus $L(P_2, f) \leq L(P, f) \leq U(P, f) \leq U(P_1, f)$.

It follows that $L(P_2, f) \leq U(P_1, f)$ or $U(P_1, f) \geq L(P_2, f)$, *that is, every upper sum for f is greater than or equal to every lower sum for f.*

Theorem 5: *Let f, g be bounded functions defined on $[a, b]$ and let P be any partition of $[a, b]$. Then*

$$L(P, f + g) \geq L(P, f) + L(P, g)$$

and $$U(P, f + g) \leq U(P, f) + U(P, g).$$

Proof: Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of $[a, b]$. Since f and g are bounded functions on $[a, b]$, $f + g$ is also bounded on $[a, b]$.

Let m_r', M_r' be the infimum and supremum of f on I_r,

m_r'', M_r'' be the infimum and supremum of g on I_r

and m_r, M_r be the infimum and supremum of $f + g$ on I_r.

By definition of infimum we find that

$$f(x) \geq m_r', \; g(x) \geq m_r'' \; \forall \, x \in I_r$$

$$\Rightarrow \quad f(x) + g(x) \geq m_r' + m_r'' \; \forall \, x \in I_r$$

$$\Rightarrow \quad (f + g)(x) \geq m_r' + m_r'' \; \forall \, x \in I_r$$

$\Rightarrow$ $m_r' + m_r''$ is a lower bound of $f + g$ on I_r.

But m_r is the greatest lower bound of $f + g$ on I_r.

$$\therefore \quad m_r \geq m_r' + m_r'' \Rightarrow m_r \, \Delta x_r \geq m_r' \, \Delta x_r + m_r'' \, \Delta x_r$$

$$\Rightarrow \quad \sum_{r=1}^{n} m_r \, \Delta x_r \geq \sum_{r=1}^{n} m_r' \, \Delta x_r + \sum_{r=1}^{n} m_r'' \, \Delta x_r$$

$$\Rightarrow \quad L(P, f + g) \geq L(P, f) + L(P, g).$$

Similarly we can prove the other result.

7.3 Lower and Upper Riemann Integrals

(Purvanchal 2009, 12)

Let f be a real valued bounded function defined on $[a, b]$. We know that the set of all numbers $L(P, f)$ with respect to all possible partitions P of $[a, b]$ is bounded above by $M(b - a)$ and hence there exists a supremum of $L(P, f)$. The **lower Riemann integral (lower *R*-integral)** of f over $[a, b]$ is the supremum of $L(P, f)$ over all partitions $P \in \boldsymbol{P}[a, b]$. It is denoted by

$$\underline{\int_a^b} f(x) \, dx.$$

Similarly the set of numbers $U(P, f)$ is bounded below by $m(b - a)$ and so it possesses an infimum. The **upper Riemann integral (upper *R*-integral)** of f over $[a, b]$ is the infimum of $U(P, f)$ over all partitions $P \in \boldsymbol{P}[a, b]$. It is denoted by

$$\overline{\int_a^b} f(x) \, dx.$$

Thus $$\underline{\int_a^b} f(x) \, dx = \sup \{L(P, f) : P \text{ is a partition of } [a, b]\}$$

and $\overline{\int_a^b} f(x)\,dx = \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}$.

We denote the lower and upper integrals of f simply by

$$\underline{\int_a^b} f \quad \text{and} \quad \overline{\int_a^b} f .$$

Since $L(P, -f) = -U(P, f)$ and $U(P, -f) = -L(P, f)$, it gives that

$$\underline{\int_a^b}(-f) = -\overline{\int_a^b} f \quad \text{and} \quad \overline{\int_a^b}(-f) = -\underline{\int_a^b} f.$$

Theorem 1: *The lower R-integral cannot exceed the upper R-integral, i.e.,*

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

(Purvanchal 2007, 09; Rohilkhand 11)

Proof: If P_1 and P_2 are any two partitions of $[a, b]$, then by theorem 4 of article 7.2, we have

$$L(P_1, f) \leq U(P_2, f). \qquad \ldots(1)$$

First, keeping P_2 fixed and taking the supremum over all partitions P_1, (1) gives

$$\underline{\int_a^b} f \leq U(P_2, f). \qquad \ldots(2)$$

Now taking infimum over all partitions P_2, (2) gives

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

Theorem 2: (Darboux Theorem): *Let f be a bounded function defined on $[a, b]$. Then to every $\varepsilon > 0$, there corresponds $\delta > 0$ such that*

$$U(P, f) < \overline{\int_a^b} f + \varepsilon \quad \text{and} \quad L(P, f) > \underline{\int_a^b} f - \varepsilon$$

for all partitions P with $\|P\| \leq \delta$.

Proof: Let $\varepsilon > 0$ be given. Since $\overline{\int_a^b} f$ is the infimum of $U(P, f)$ and $\underline{\int_a^b} f$ is the supremum of $L(P, f)$ for all partitions P, therefore, for given $\varepsilon > 0$ there exist partitions P_1 and P_2 such that

$$U(P_1, f) < \overline{\int_a^b} f + \varepsilon \qquad \ldots(1)$$

and

$$L(P_2, f) > \underline{\int_a^b} f - \varepsilon \ . \qquad \ldots(2)$$

Let P_3 be the common refinement of P_1 and P_2. Then by theorem 3 of article 7.2, we get

$$U(P_3, f) \leq U(P_1, f) \text{ and } L(P_3, f) \geq L(P_2, f). \qquad \ldots(3)$$

Therefore, from (1), (2) and (3), we get

$$U(P, f) < \overline{\int_a^b} f + \varepsilon \quad \text{and} \quad L(P, f) > \underline{\int_a^b} f - \varepsilon$$

for all partitions P of $[a, b]$ with $\|P\| \leq \delta$, where $\delta = \|P_3\| > 0$.

Corollary: *If f is bounded on $[a, b]$ and P is a partition of $[a, b]$, then*

(i) $\lim_{\|P\| \to 0} L(P, f) = \underline{\int}_a^b f$

(ii) $\lim_{\|P\| \to 0} U(P, f) = \overline{\int}_a^b f.$

Proof: (i) Since $\underline{\int}_a^b f$ is the supremum of $L(P, f)$ for all partitions P, therefore we have

$$L(P, f) \leq \underline{\int}_a^b f. \qquad \ldots(1)$$

Using the above theorem, we see that for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$L(P, f) > \underline{\int}_a^b f - \varepsilon \qquad \ldots(2)$$

for all partitions P with $\|P\| \leq \delta$.

From (1) and (2), we get

$$\underline{\int}_a^b f - \varepsilon < L(P, f) \leq \underline{\int}_a^b f < \underline{\int}_a^b f + \varepsilon$$

or $$\underline{\int}_a^b f - \varepsilon < L(P, f) < \underline{\int}_a^b f + \varepsilon.$$

By definition of limit, this implies that

$$\lim_{\|P\| \to 0} L(P, f) = \underline{\int}_a^b f.$$

(ii) Since $\overline{\int}_a^b f$ is the infimum of $U(P, f)$ for all partitions P, therefore we have

$$U(P, f) \geq \overline{\int}_a^b f. \qquad \ldots(3)$$

Using the above theorem, we see that for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$U(P, f) < \overline{\int}_a^b f + \varepsilon \qquad \ldots(4)$$

for all partitions P with $\|P\| \leq \delta$.

From (3) and (4), we get

$$\overline{\int}_a^b f \leq U(P, f) < \overline{\int}_a^b f + \varepsilon$$

or $$\overline{\int}_a^b f - \varepsilon < U(P, f) < \overline{\int}_a^b f + \varepsilon.$$

By definition of limit, this implies that

$$\lim_{\|P\| \to 0} U(P, f) = \overline{\int}_a^b f.$$

7.4 *R*-Integrability

(Purvanchal 2012)

Definition: *Let f be a bounded function defined on the bounded interval $[a, b]$, then f is called* **Riemann integrable** *(or simply R-integrable) on $[a, b]$ iff*

$$\underline{\int}_a^b f = \overline{\int}_a^b f$$

and their common value is called the R-integral of f on $[a, b]$ and is denoted by $\int_a^b f$.

The class of all bounded functions f which are Riemann integrable on $[a, b]$ is denoted by R $[a, b]$. The numbers a and b are called the **lower and upper limits of integration** respectively.

If $\underline{\int}_a^b f \neq \overline{\int}_a^b f$ then f is not Riemann integrable on $[a, b]$.

Note 1: The concept of integrability of a function over an interval as introduced here is subject to two very important limitations, *viz.*

(i) The function is bounded.

(ii) The interval of integration is finite *i.e.*, neither of the end points is infinite.

Note 2: It is not necessary that every bounded function is integrable *i.e.* there may exist a bounded function f for which

$$\underline{\int}_a^b f \neq \overline{\int}_a^b f .$$

Note 3: The statement that $\int_a^b f$ exists indicates that the function f is bounded and integrable over $[a, b]$.

7.5 Another Definition of Riemann Integral

A function f defined on $[a, b]$ is said to be Riemann integrable over $[a, b]$ iff for every $\varepsilon > 0$ there exists a $\delta > 0$ and a number I such that for every partition

$$P = \{a = x_0, x_1, \ldots, x_n = b\}$$

with $\| P \| \leq \delta$ and for every choice of $\xi_r \in [x_{r-1}, x_r]$,

$$\left| \sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1}) - I \right| < \varepsilon .$$

In such a case I is said to be *Riemann integral* of f over $[a, b]$ *i.e.*,

$$I = \int_a^b f(x)\, dx.$$

Theorem: *Definitions of article 7.4 and article 7.5 are equivalent.*

Proof: (i) Let f be integrable according to definition of article 7.4. Then f is bounded and we have

$$\underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f . \qquad ...(1)$$

Let $\varepsilon > 0$ be given. Then by Darboux theorem, there exists $\delta > 0$ such that for every partition P with $\| P \| \le \delta$,

$$L(P, f) > \underline{\int_a^b} f - \varepsilon = \int_a^b f - \varepsilon. \qquad ...(2)$$

and

$$U(P, f) < \overline{\int_a^b} f + \varepsilon = \int_a^b f + \varepsilon. \qquad ...(3)$$

If ξ_r is any point of the interval $[x_{r-1}, x_r]$, then

$$L(P, f) \le \sum_{r=1}^{n} f(\xi_r) \Delta x_r \le U(P, f). \qquad ...(4)$$

From (2), (3) and (4), we conclude that for every partition P with $\| P \| \le \delta$,

$$\int_a^b f - \varepsilon < \sum_{r=1}^{n} f(\xi_r) \Delta x_r < \int_a^b f + \varepsilon$$

i.e.,

$$\left| \sum_{r=1}^{n} f(\xi_r) \Delta x_r - \int_a^b f \right| < \varepsilon$$

or

$$\left| \sum_{r=1}^{n} f(\xi_r) \Delta x_r - I \right| < \varepsilon, \text{ where } I = \int_a^b f.$$

$\therefore f$ is Riemann integrable according to the definition of article 7.5.

Thus the definition of article 7.4 $\Rightarrow$ the definition of article 7.5.

(ii) Let f be integrable according to the definition of article 7.5.

Then for $\varepsilon = 1 > 0$ there exists $\delta > 0$ and a number I such that for every partition P with $\| P \| \le \delta$ and for every choice of $\xi_r \in [x_{r-1}, x_r]$,

$$\left| \sum_{r=1}^{n} f(\xi_r) \Delta x_r - I \right| < 1 \quad i.e., \quad \left| \sum_{r=1}^{n} f(\xi_r) \Delta x_r \right| < |I| + 1 \qquad ...(1)$$

We have to show that f is bounded on $[a, b]$ and the lower and upper integrals of f over $[a, b]$ are equal.

Suppose f is not bounded on $[a, b]$, then f must not be bounded in at least one subinterval of P, say, $[x_{m-1}, x_m]$. Hence there exists a point $\xi_m \in [x_{m-1}, x_m]$ such that $f(\xi_m)$ is infinite.

Now, to form the sum $\sum_{r=1}^{n} f(\xi_r) \Delta x_r$ we have to choose points ξ_r in each subinterval $[x_{r-1}, x_r]$. Choose that ξ_m in the interval $[x_{m-1}, x_m]$ for which $f(\xi_m)$ is infinite.

In that case $\left| \sum_{r=1}^{n} f(\xi_r) \Delta x_r \right| > |I| + 1$,

which is a contradiction to (1). Hence f is bounded on $[a, b]$.

Again, by definition of article 7.5, for any $\varepsilon > 0$ there exists $\delta > 0$ and a number $I \in R$ such that for every partition P of $[a, b]$ with $\|P\| \leq \delta$ and for every choice of $\xi_r \in [x_{r-1}, x_r]$,

$$\left| \sum_{r=1}^{n} f(\xi_r) \Delta x_r - I \right| < \frac{1}{2}\varepsilon$$

i.e.,
$$I - \frac{1}{2}\varepsilon < \sum_{r=1}^{n} f(\xi_r) \Delta x_r < I + \frac{1}{2}\varepsilon. \quad \ldots(2)$$

Let m_r , M_r be the infimum and supremum of f on $[x_{r-1}, x_r]$, so that there exist points $\alpha_r, \beta_r \in [x_{r-1}, x_r]$ such that

$$f(\alpha_r) > M_r - \frac{\varepsilon}{2(b-a)} \quad \text{and} \quad f(\beta_r) < m_r + \frac{\varepsilon}{2(b-a)}.$$

$$\therefore \quad \sum_{r=1}^{n} f(\alpha_r) \Delta x_r > \sum_{r=1}^{n} M_r \Delta x_r - \sum_{r=1}^{n} \frac{\varepsilon}{2(b-a)} \Delta x_r$$

and
$$\sum_{r=1}^{n} f(\beta_r) \Delta x_r < \sum_{r=1}^{n} m_r \Delta x_r + \sum_{r=1}^{n} \frac{\varepsilon}{2(b-a)} \Delta x_r$$

i.e.
$$\sum_{r=1}^{n} f(\alpha_r) \Delta x_r > U(P, f) - \frac{\varepsilon}{2} \quad \ldots(3)$$

and
$$\sum_{r=1}^{n} f(\beta_r) \Delta x_r < L(P, f) + \frac{\varepsilon}{2} \quad \ldots(4)$$

Now from (2) and (3), we get

$$I + \frac{1}{2}\varepsilon > U(P, f) - \frac{1}{2}\varepsilon \quad \ldots(5)$$

and from (2) and (4), we get

$$I - \frac{1}{2}\varepsilon < L(P, f) + \frac{1}{2}\varepsilon \quad \ldots(6)$$

i.e.,
$$I + \varepsilon > U(P, f) \text{ and } I - \varepsilon < L(P, f) \quad \ldots(7)$$

But
$$L(P, f) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(P, f). \quad \ldots(8)$$

From (7) and (8), we get

$$I - \varepsilon < \underline{\int_a^b} f \leq \overline{\int_a^b} f < I + \varepsilon \quad \ldots(9)$$

or
$$0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f < 2\varepsilon \quad \text{or} \quad 0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq 0,$$

as $\varepsilon > 0$ is arbitrary

or
$$\overline{\int_a^b} f - \underline{\int_a^b} f = 0 \quad i.e., \quad \overline{\int_a^b} f = \underline{\int_a^b} f. \quad \ldots(10)$$

$\therefore$ f is Riemann integrable according to the definition of article 7.4. Now from (9) and (10), we have

$$I - \varepsilon < \int_a^b f < I + \varepsilon.$$

Since ε is arbitrary, $I = \int_a^b f$.

Thus the definition of article 7.5 $\Rightarrow$ the definition of article 7.4.

Hence the two definitions of article 7.4 and article 7.5 are equivalent.

7.6 Riemann's Necessary and Sufficient Condition for R-Integrability

Oscillatory sum: With usual notations, we have

$$L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r \,, U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r \,.$$

Let $\omega_r = M_r - m_r$ so that ω_r is the oscillation of f on $[x_{r-1}, x_r]$.

$$\therefore \quad U(P, f) - L(P, f) = \sum_{r=1}^{n} (M_r - m_r) \Delta x_r = \sum_{r=1}^{n} \omega_r \Delta x_r \,.$$

The sum $\sum_{r=1}^{n} \omega_r \Delta x_r$ is called the **oscillatory sum** for the function f corresponding to the partition P and is denoted by $\omega(P, f)$.

Theorem: *A necessary and sufficient condition for R-integrability of a bounded function $f : [a, b] \to \mathbf{R}$ over $[a, b]$ is that for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that for P and all its refinements*

$$0 \le U(P, f) - L(P, f) < \varepsilon. \qquad \textbf{(Rohilkhand 2009)}$$

Proof: The condition is necessary. Let $f \in \boldsymbol{R}[a, b]$ so that

$$\underline{\int_a^b} f = \overline{\int_a^b} f. \qquad \ldots(1)$$

By Darboux theorem, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all partitions P of $[a, b]$ with $\| P \| \le \delta$,

$$U(P, f) < \overline{\int_a^b} f + \frac{\varepsilon}{2} \qquad \ldots(2)$$

and

$$L(P, f) > \underline{\int_a^b} f - \frac{\varepsilon}{2}. \qquad \ldots(3)$$

Adding the inequalities (2) and (3), we get

$$U(P, f) + \underline{\int_a^b} f - \frac{\varepsilon}{2} < L(P, f) + \overline{\int_a^b} f + \frac{\varepsilon}{2}.$$

In view of (1), this gives

$$U(P, f) < L(P, f) + \varepsilon \quad i.e., \quad U(P, f) - L(P, f) < \varepsilon \qquad \ldots(4)$$

Since $U(P, f) \ge L(P, f)$, the inequality (4) can be written as

$$0 \le U(P, f) - L(P, f) < \varepsilon.$$

Hence the condition is necessary.

The condition is sufficient: Let for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that for P and all its refinements,

$$0 \le U(P, f) - L(P, f) < \varepsilon. \qquad ...(5)$$

By the definition of upper and lower integrals, we have

$$\overline{\int_a^b} f \le U(P, f) \text{ and } \underline{\int_a^b} f \ge L(P, f)$$

or $$-\underline{\int_a^b} f \le -L(P, f).$$

$$\therefore \quad \overline{\int_a^b} f - \underline{\int_a^b} f \le U(P, f) - L(P, f) < \varepsilon \qquad ...(6)$$

or $$\overline{\int_a^b} f - \overline{\int_a^b} f < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\overline{\int_a^b} f - \underline{\int_a^b} f \le 0. \qquad ...(7)$$

Also we know that the lower Riemann integral can never exceed the upper Riemann integral

i.e., $$\overline{\int_a^b} f - \underline{\int_a^b} f \ge 0. \qquad ...(8)$$

From (7) and (8), we get

$$\overline{\int_a^b} f - \underline{\int_a^b} f = 0 \quad \text{or} \quad \overline{\int_a^b} f = \underline{\int_a^b} f$$

i.e., the function f is Riemann integrable over $[a, b]$.

Note: **Another statement of the above theorem:** *A necessary and sufficient condition for a bounded function f to be integrable over $[a, b]$ is that for each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that the oscillatory sum $\omega(P, f) < \varepsilon$*

or $\lim \omega(P, f) = 0$ *as* $\| P \|$ *tends to zero.*

Illustrative Examples

Example 1: *Show that if f is defined on $[a, b]$ by*

$$f(x) = k \quad \forall \quad x \in [a, b]$$

where k is a constant, then $f \in \boldsymbol{R}[a, b]$ and $\int_a^b k = k(b - a)$.

Solution: Obviously the given function is bounded over $[a, b]$.

Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of $[a, b]$. Then for any subinterval $[x_{r-1}, x_r]$, we have $m_r = k$, $M_r = k$.

Now, $$U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r = \sum_{r=1}^{n} k \Delta x_r = k \sum_{r=1}^{n} \Delta x_r$$

$$= k\,[\Delta x_1 + \Delta x_2 + \ldots + \Delta x_n]$$
$$= k\,[(x_1 - x_0) + (x_2 - x_1) + \ldots + (x_n - x_{n-1})]$$
$$= k\,(x_n - x_0) = k\,(b - a)$$

and $L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r = \sum_{r=1}^{n} k \Delta x_r = k\,(b - a).$

Hence $\overline{\int_a^b} f = \inf U(P, f) = \inf \{k\,(b - a)\} = k\,(b - a)$

and $\underline{\int_a^b} f = \sup L(P, f) = \sup \{k\,(b - a)\} = k\,(b - a).$

Thus $\overline{\int_a^b} f = \underline{\int_a^b} f = k\,(b - a).$

Hence $f \in R[a, b]$ and $\int_a^b f = k\,(b - a).$

Example 2: *Let $f(x) = x$ on $[0, 1]$. Calculate $\underline{\int_0^1} x\,dx$ and $\overline{\int_0^1} x\,dx$ by dissecting $[0, 1]$ into n equal parts and hence show that $f \in R[0, 1]$.* **(Purvanchal 2011, Meerut 12)**

Solution: Let $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{r-1}{n}, \frac{r}{n}, \ldots, \frac{n}{n} = 1\right\}$. Here

$$m_r = \frac{r-1}{n},\ M_r = \frac{r}{n} \text{ and } \Delta x_r = \frac{1}{n} \text{ for } r = 1, 2, \ldots, n.$$

Now, we have

$$L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r = \sum_{r=1}^{n} \frac{r-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{r=1}^{n} (r - 1)$$
$$= \frac{1}{n^2}\,[1 + 2 + 3 + \ldots + (n - 1)] = \frac{(n-1)\,.\,n}{2\,n^2} = \frac{n-1}{2\,n}$$

and $U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r = \sum_{r=1}^{n} \frac{r}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{r=1}^{n} r$

$$= \frac{1}{n^2}\,[1 + 2 + 3 + \ldots + n] = \frac{n\,(n+1)}{2\,n^2} = \frac{n+1}{2\,n}.$$

Hence by Corollary 1, theorem 2 of article 7.3, we get

$$\underline{\int_0^1} x\,dx = \lim_{\|P\| \to 0} L(P, f) = \lim_{n \to \infty} \frac{n-1}{2\,n} = \frac{1}{2}$$

and $$\overline{\int_0^1} x\,dx = \lim_{\|P\| \to 0} U(P, f) = \lim_{n \to \infty} \frac{n+1}{2\,n} = \frac{1}{2}.$$

Since $\underline{\int_0^1} f = \overline{\int_0^1} f,\ f \in R[0, 1]$ and $\int_0^1 x\,dx = \frac{1}{2}.$

Example 3: *Let $f(x) = x^2$ on $[0, a]$, $a > 0$. Show that $f \in R[0, a]$ and find $\int_0^a f$.*

(Rohilkhand 2010)

Solution: Let $P = \left\{0, \frac{a}{n}, \frac{2a}{n}, \ldots, \frac{(n-1)a}{n}, \frac{na}{n} = a\right\}$ be the partition of $[0, a]$ obtained by dissecting $[0, a]$ into n equal parts. Then

$$\Delta x_r = a / n, r = 1, 2, \ldots, n.$$

Also $I_r = r$th subinterval $= \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right]$.

Since $f(x) = x^2$ is an increasing function in $[0, a]$,

$$m_r = \frac{(r-1)^2 a^2}{n^2} \quad \text{and} \quad M_r = \frac{r^2 a^2}{n^2}, r = 1, 2 \ldots, n.$$

Now $$L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r = \sum_{r=1}^{n} \frac{(r-1)^2 a^2}{n^2} \cdot \frac{a}{n}$$

$$= \frac{a^3}{n^3} \sum_{r=1}^{n} (r-1)^2 = \frac{a^3}{n^3} \cdot \frac{(n-1)\, n\, (2n-1)}{6}.$$

$$\therefore \quad \int_0^a x^2\, dx = \lim_{n \to \infty} L(P, f) = \lim_{n \to \infty} \frac{a^3}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) = \frac{a^3}{3}.$$

Again $$U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r = \sum_{r=1}^{n} \frac{r^2 a^2}{n^2} \cdot \frac{a}{n}$$

$$= \frac{a^3}{n^3} \sum_{r=1}^{n} r^2 = \frac{a^3}{n^3} \cdot \frac{n\,(n+1)\,(2n+1)}{6}.$$

$$\therefore \quad \overline{\int_0^a} x^2\, dx = \lim_{n \to \infty} U(P, f) = \lim_{n \to \infty} \frac{a^3}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) = \frac{a^3}{3}.$$

Since $\overline{\int_0^a} f = \underline{\int_0^a} f, f \in \boldsymbol{R}[0, a]$ and $\int_0^a x^2\, dx = \frac{a^3}{3}$.

Example 4: *If a function f is defined on* $[0, a], a > 0$ *by* $f(x) = x^3$, *then show that f is Riemann integrable on* $[0, a]$ *and* $\int_0^a f(x)\, dx = \frac{a^4}{4}$.

Solution: Let $P = \left\{0, \frac{a}{n}, \frac{2a}{n}, \ldots, \frac{(n-1)a}{n}, \frac{na}{n} = a\right\}$ be the partition of $[0, a]$ obtained by dissecting $[0, a]$ into n equal parts. Then

$$I_r = r \text{ th sub-interval} = \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right]$$

and $$\Delta x_r = \text{length of } I_r = \frac{a}{n}; r = 1, 2, \ldots, n.$$

Let m_r and M_r be respectively the infimum and supremum of f in I_r.
Since $f(x) = x^3$ is an increasing function in $[0, a]$, therefore

$$m_r = \frac{(r-1)^3 a^3}{n^3} \quad \text{and} \quad M_r = \frac{r^3 a^3}{n^3}, r = 1, 2, \ldots, n.$$

Now $L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r = \sum_{r=1}^{n} \left[\frac{(r-1)^3 a^3}{n^3} \cdot \frac{a}{n}\right] = \frac{a^4}{n^4} \sum_{r=1}^{n} (r-1)^3$

$$= \frac{a^4}{n^4}[1^3 + 2^3 + \ldots + (n-1)^3] = \frac{a^4}{n^4} \cdot \left[\frac{(n-1)\,n}{2}\right]^2 = \frac{a^4}{4}\left(1 - \frac{1}{n}\right)^2.$$

$\therefore \quad \int_0^a f(x)\,dx = \lim_{n \to \infty} L(P, f) = \lim_{n \to \infty} \frac{a^4}{4}\left(1 - \frac{1}{n}\right)^2 = \frac{a^4}{4}.$

Again $U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r = \sum_{r=1}^{n} \left[\frac{r^3 a^3}{n^3} \cdot \frac{a}{n}\right] = \frac{a^4}{n^4} \sum_{r=1}^{n} r^3$

$$= \frac{a^4}{n^4}(1^3 + 2^3 + \ldots + n^3) = \frac{a^4}{n^4} \cdot \left[\frac{n(n+1)}{2}\right]^2 = \frac{a^4}{4}\left(1 + \frac{1}{n}\right)^2.$$

$\therefore \quad \overline{\int_0^a} f(x)\,dx = \lim_{n \to \infty} U(P, f) = \lim_{n \to \infty} \frac{a^4}{4}\left(1 + \frac{1}{n}\right)^2 = \frac{a^4}{4}.$

Since $\underline{\int_0^a} f = \overline{\int_0^a} f$, f is Riemann integrable on $[0, a]$ and

$$\int_0^a f(x)\,dx = \int_0^a x^3\,dx = \frac{a^4}{4}.$$

Example 5: *Let f be the function defined on $[0, 1]$ by*

$$f(x) = \begin{cases} 0 & \text{when } x \text{ is irrational,} \\ 1 & \text{when } x \text{ is rational.} \end{cases}$$

Calculate $\underline{\int_0^1} f$ and $\overline{\int_0^1} f$ and hence show that $f \notin R[0, 1]$.

(Purvanchal 2008, 09, Rohilkhand 11)

Solution: First, we observe that f is bounded, for evidently

$$0 \le f(x) \le 1 \ \forall\, x \in [0, 1].$$

Let P be any partition of $[0, 1]$. Then for any subinterval $[x_{r-1}, x_r]$ of P, we have $m_r = 0$ and $M_r = 1$, because every subinterval will contain rational as well as irrational numbers. Note that rational as well as irrational points are everywhere dense.

It follows that

$$L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r = \sum_{r=1}^{n} 0 \cdot \Delta x_r = 0$$

and $U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r = \sum_{r=1}^{n} 1 \cdot \Delta x_r = \sum_{r=1}^{n} \Delta x_r = 1.$

$\therefore \quad \underline{\int_0^1} f = \lim_{n \to \infty} L(P, f) = 0$ and $\overline{\int_0^1} f = \lim_{n \to \infty} U(P, f) = 1.$

Since $\underline{\int_0^1} f \ne \overline{\int_0^a} f$, we have $f \notin R[0, 1]$.

Example 6: *Give an example of a bounded function which is not R-integrable over* $[0, 1]$.

Solution: See example 5 above. As another example, consider the function f defined on $[0, 1]$ as follows :

$$f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational.} \end{cases}$$

Evidently f is bounded on $[0, 1]$.

If P is any partition of $[0, 1]$, then for any subinterval $[x_{r-1}, x_r]$ of P, we have $m_r = -1$ and $M_r = 1, r = 1, 2, \ldots, n$.

$$\therefore \quad L(P, f) = \sum_{r=1}^{n} m_r \, \Delta x_r = \sum_{r=1}^{n} -1 \, . \, \Delta x_r = -1 \sum_{r=1}^{n} \Delta x_r$$

$$= -1 \, . \, (1 - 0) = -1 . 1 = -1,$$

$$U(P, f) = \sum_{r=1}^{n} M_r \, \Delta x_r = 1 . \sum_{r=1}^{n} \Delta x_r = 1 \, . \, (1 - 0) = 1.$$

Hence $\underline{\int}_0^1 f = \lim_{n \to \infty} L(P, f) = -1, \overline{\int}_0^1 f = \lim_{n \to \infty} U(P, f) = 1.$

Thus $\underline{\int}_0^1 f \neq \overline{\int}_0^1 f$ and consequently $f \notin R[0, 1]$.

Example 7: *Show that* $f(x) = \sin x$ *is integrable on* $\left[0, \frac{1}{2}\pi\right]$ *and* $\int_0^{\pi/2} \sin x \, dx = 1$.

Solution: Let $P = \left\{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \ldots, \frac{(r-1)\pi}{2n}, \frac{r\pi}{2n}, \ldots, \frac{n\pi}{2n} = \frac{\pi}{2}\right\}$

be the partition of $[0, \pi/2]$ obtained by dissecting $[0, \frac{1}{2}\pi]$ into n equal parts. The length of each subinterval $= \pi / 2n$ and the r th subinterval $I_r = \left[\frac{(r-1)\pi}{2n}, \frac{r\pi}{2n}\right]$.

Since $f(x) = \sin x$ is increasing in $[0, \frac{1}{2}\pi]$, we have

$$m_r = \sin \frac{(r-1)\pi}{2n} \text{ and } M_r = \sin \frac{r\pi}{2n}, \; r = 1, 2, \ldots, n.$$

$$\text{Now } U(P, f) = \sum_{r=1}^{n} M_r \, \Delta x_r = \sum_{r=1}^{n} \left(\sin \frac{r\pi}{2n}\right) \cdot \frac{\pi}{2n}$$

$$= \frac{\pi}{2n}\left[\sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin \frac{n\pi}{2n}\right].$$

We know that

$$\sin a + \sin(a + d) + \ldots + \sin\{a + (n-1)d\} = \frac{\sin\left(a + \frac{n-1}{2} d\right) \sin \frac{nd}{2}}{\sin(d/2)}.$$

$$\therefore U(P, f) = \frac{\pi}{2n}\left[\frac{\sin\left(\frac{\pi}{2n} + \frac{n-1}{2}\cdot\frac{\pi}{2n}\right)\sin\frac{n\pi}{4n}}{\sin\frac{\pi}{4n}}\right] = \frac{\frac{\pi}{2n}\sin\frac{(n+1)\pi}{4n}\cdot\sin\frac{\pi}{4}}{\sin\frac{\pi}{4n}}$$

$$= \frac{\frac{\pi}{2n}\cdot\sin\left(\frac{\pi}{4} + \frac{\pi}{4n}\right)\cdot\frac{1}{\sqrt{2}}}{\sin\frac{\pi}{4n}} = \frac{\frac{\pi}{2\sqrt{2}n}\left\{\sin\frac{\pi}{4}\cos\frac{\pi}{4n} + \cos\frac{\pi}{4}\sin\frac{\pi}{4n}\right\}}{\sin\left(\frac{\pi}{4n}\right)}$$

$$= \frac{\pi}{2\sqrt{2}n}\cdot\frac{1}{\sqrt{2}}\left(\cot\frac{\pi}{4n} + 1\right) = \frac{\pi}{4n}\left(\cot\frac{\pi}{4n} + 1\right).$$

Similarly, we can find that

$$L(P, f) = \frac{\pi}{4n}\left(\cot\frac{\pi}{4n} - 1\right).$$

Now $$\int_{-0}^{\pi/2} f = \lim_{n\to\infty} L(P, f) = \lim_{n\to\infty}\frac{\pi}{4n}\left(\cot\frac{\pi}{4n} - 1\right)$$

$$= \lim_{n\to\infty}\frac{(\pi/4n)}{\tan(\pi/4n)} - \lim_{n\to\infty}\frac{\pi}{4n} = 1 - 0 = 1$$

$$\left[\because \lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1\right]$$

and $$\overline{\int}_0^{\pi/2} f = \lim_{n\to\infty} U(P, f) = \lim_{n\to\infty}\frac{\pi}{4n}\left(\cot\frac{\pi}{4n} + 1\right) = 1.$$

Since $$\int_{-0}^{\pi/2} f = \overline{\int}_0^{\pi/2} f, f \in \boldsymbol{R}\left[0, \frac{\pi}{2}\right] \text{ and } \int_0^{\pi/2} f = 1.$$

Example 8: *Let $f(x)$ be a function defined on $[0, \frac{1}{4}\pi]$ by*

$$f(x) = \begin{cases} \cos x, & \text{if } x \text{ is rational} \\ \sin x, & \text{if } x \text{ is irrational}. \end{cases}$$

Show that f is not Riemann integrable over $\left[0, \frac{1}{4}\pi\right]$.

Solution: Consider the partition

$$P = \left\{0, \frac{\pi}{4n}, \frac{2\pi}{4n}, \ldots, \frac{(r-1)\pi}{4n}, \frac{r\pi}{4n}, \ldots, \frac{n\pi}{4n} = \frac{\pi}{4}\right\}$$

obtained by dissecting $\left[0, \frac{1}{4}\pi\right]$ into n equal parts.

Then for any subinterval $\left[\frac{(r-1)\pi}{4n}, \frac{r\pi}{4n}\right]$, $r = 1, 2, \ldots, n$, we have

$$m_r = \sin\frac{(r-1)\pi}{4n} \text{ and } M_r = \cos\frac{(r-1)\pi}{4n}, \Delta x_r = \frac{\pi}{4n}.$$

Hence $L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r = \sum_{r=1}^{n} \sin \frac{(r-1)\pi}{4n} \cdot \frac{\pi}{4n}$

$$= \frac{\pi}{4n}\left[\sin \frac{\pi}{4n} + \ldots + \sin \frac{(n-1)\pi}{4n}\right]$$

$$= \frac{\pi}{4n} \cdot \frac{\sin\left(\frac{\pi}{4n} + \frac{n-2}{2} \cdot \frac{\pi}{4n}\right) \cdot \sin \frac{n\pi}{8n}}{\sin \frac{\pi}{8n}}$$ [summing up the series]

$$= \frac{(\pi/8n)}{\sin(\pi/8n)} \cdot 2 \sin^2 \frac{\pi}{8}$$

and $U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r = \sum_{r=1}^{n} \cos \frac{(r-1)\pi}{4n} \cdot \frac{\pi}{4n}$

$$= \frac{\pi}{4n}\left[\cos 0 + \cos \frac{\pi}{4n} + \ldots + \cos \frac{(n-1)\pi}{4n}\right]$$

$$= \frac{\pi}{4n} \cdot \frac{\cos\left(\frac{n-1}{2} \cdot \frac{\pi}{4n}\right) \sin \frac{n\pi}{8n}}{\sin \frac{\pi}{8n}}$$ [summing up the series]

$$= \frac{(\pi/8n)}{\sin(\pi/8n)} \cdot 2 \cos \frac{(n-1)\pi}{8n} \cdot \sin \frac{\pi}{8}.$$

Hence $\int_0^{\pi/4} f = \lim_{\|P\| \to 0} L(P, f) = \lim_{n \to \infty} L(P, f)$

$$= \lim_{n \to \infty} \frac{(\pi/8n)}{\sin(\pi/8n)} \cdot 2 \sin^2 \frac{\pi}{8} = 2 \sin^2 \frac{\pi}{8} = \left(1 - \cos \frac{\pi}{4}\right)$$

$$= 1 - \frac{1}{\sqrt{2}}$$

and $\bar{\int}_0^{\pi/4} f = \lim_{n \to \infty} U(P, f)$

$$= \lim_{n \to \infty} \frac{(\pi/8n)}{\sin(\pi/8n)} \cdot 2 \cos \frac{(n-1)\pi}{8n} \cdot \sin \frac{\pi}{8}$$

$$= \lim_{n \to \infty} \frac{(\pi/8n)}{\sin(\pi/8n)} \cdot 2 \cos \frac{\pi}{8}\left(1 - \frac{1}{n}\right) \sin \frac{\pi}{8}$$

$$= 2 \cos \frac{\pi}{8} \sin \frac{\pi}{8} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Since $\int_0^{\pi/4} f \neq \bar{\int}_0^{\pi/4} f$, f is not Riemann integrable over $\left[0, \frac{\pi}{4}\right]$.

Example 9: *Let f be a function on $[0, 1]$ defined by*

$$f(x) = \begin{cases} 1 & \text{if } x \neq \frac{1}{2} \\ 0 & \text{if } x = \frac{1}{2}. \end{cases}$$

Show that $f \in \boldsymbol{R}[0, 1]$ and find $\int_0^1 f$.

Solution: Obviously $f(x)$ is bounded in $[0, 1]$ since

$$0 \leq f(x) \leq 1 \; \forall \; x \in [0, 1].$$

Let P be a partition of $[0, 1]$ such that the point $\frac{1}{2}$ belongs to the open interval $]x_{s-1}, x_s[$. Then we have (with usual notations)

$$m_r = M_r = 1 \text{ for } r = 1, 2, \ldots, n$$

and $r \neq s, m_s = 0, M_s = 1.$

Now
$$U(P, f) - L(P, f)$$

$$= \sum_{\substack{r=1 \\ r \neq s}}^{n} (M_r - m_r) \Delta x_r + (M_s - m_s)(x_s - x_{s-1})$$

$$= \sum_{r=1}^{n} (1 - 1)(x_r - x_{r-1}) + (1 - 0)(x_s - x_{s-1})$$

$$= x_s - x_{s-1}. \qquad \ldots(1)$$

Let $\varepsilon > 0$ be given. Then we choose a partition P such that the point $\frac{1}{2}$ is an interior point of one of the subintervals whose length is less than ε. Then, it follows from (1) that

$$U(P, f) - L(P, f) < \varepsilon.$$

Hence, by theorem of article 7.6, $f \in \boldsymbol{R}[0, 1]$.

Now to find $\int_0^1 f$, it is enough to find $\int_{_0}^{1} f$ or $\bar{\int}_0^1 f$.

We calculate $\bar{\int}_0^1 f$. For any partition P, we have

$$U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r = \sum_{r=1}^{n} 1 . \Delta x_r = \sum_{r=1}^{n} \Delta x_r$$

$$= \text{length of the interval } [0, 1] = 1.$$

Hence $$\bar{\int}_0^1 f = \lim_{n \to \infty} U(P, f) = 1.$$

Thus $$f \in \boldsymbol{R}[0, 1] \text{ and } \int_0^1 f = \bar{\int}_0^1 f = 1.$$

Example 10: *If $f(x) = x + x^2$ for rational values of x in the interval $[0, 2]$ and $f(x) = x^2 + x^3$ for irrational values of x in the same interval, evaluate the upper and the lower Riemann integrals of f over $[0, 2]$.* **(Purvanchal 2007)**

Solution: We have $(x + x^2) - (x^2 + x^3) = x - x^3 = x(1 - x^2)$

so that $(x + x^2) - (x^2 + x^3) > 0$ if $0 < x < 1$

and < 0 if $1 < x < 2$.

If P is any partition of $[0, 2]$, then any subinterval of P, however small it may be, will contain rational as well as irrational points.

With usual notations, we have for all values of r

$$M_r = x + x^2, \quad \text{if } 0 < x < 1$$
$$= x^2 + x^3, \quad \text{if } 1 < x < 2$$

and
$$m_r = x^2 + x^3, \quad \text{if } 0 < x < 1$$
$$= x + x^2, \quad \text{if } 1 < x < 2.$$

Hence $$\overline{\int_0^2} f(x)\,dx = \int_0^1 (x + x^2)\,dx + \int_1^2 (x^2 + x^3)\,dx$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3} + \frac{x^4}{4}\right]_1^2$$

$$= \left(\frac{1}{2} + \frac{1}{3}\right) - 0 + \left(\frac{8}{3} + \frac{16}{4}\right) - \left(\frac{1}{3} + \frac{1}{4}\right) = \frac{83}{12} = 6\frac{11}{12}$$

and $$\underline{\int_0^2} f(x)\,dx = \int_0^1 (x^2 + x^3)\,dx + \int_1^2 (x + x^2)\,dx$$

$$= \left[\frac{x^3}{3} + \frac{x^4}{4}\right]_0^1 + \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{3} + \frac{1}{4} + \frac{4}{2} + \frac{8}{3} - \frac{1}{2} - \frac{1}{3} = \frac{53}{12} = 4\frac{5}{12}.$$

Example 11: *Find the upper and lower R-integrals for the function f defined on $[0, 1]$ as follows :*

$$f(x) = \begin{cases} \sqrt{(1 - x^2)}, & \text{if } x \text{ is rational} \\ (1 - x), & \text{if } x \text{ is irrational}. \end{cases}$$

Solution: Here we have

$$(1 - x^2) - (1 - x)^2 = 2x - 2x^2 = 2x(1 - x) > 0 \quad \forall\, x \in]0, 1[$$

i.e., $$\sqrt{(1 - x^2)} > (1 - x) \quad \forall\, x \in]0, 1[.$$

With usual notations, we have for all values of r

$$M_r = \sqrt{(1 - x^2)} \text{ and } m_r = (1 - x).$$

Hence $\underline{\int_0^1} f = \int_0^1 (1-x)\,dx = \left[x - \frac{x^2}{2}\right]_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$

and $\overline{\int_0^1} f = \int_0^1 \sqrt{(1-x^2)}\,dx = \left[\frac{1}{2}x\sqrt{(1-x^2)} + \frac{1}{2}\sin^{-1}x\right]_0^1$

$= \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}.$

Thus $\underline{\int_0^1} f \neq \overline{\int_0^1} f$. If follows that $f \notin \boldsymbol{R}[0, 1]$.

7.7 Some Classes of Integrable Functions

(Purvanchal 2007; 08, 10; Rohilkhand 11)

Theorem 1: *If f is continuous on $[a, b]$, then $f \in \boldsymbol{R}[a, b]$.*

Proof: Since f is continuous on $[a, b]$, f is bounded on $[a, b]$. Also since f is continuous on a closed and bounded interval $[a, b]$, f is uniformly continuous on $[a, b]$. Hence for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all points x', x'' of $[a, b]$

$$|f(x') - f(x'')| < \frac{\varepsilon}{b-a} \text{ whenever } |x' - x''| < \delta. \qquad ...(1)$$

Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition of $[a, b]$ with

$\|P\| < \delta$.

Since f is continuous on $[x_{r-1}, x_r]$ it attains its infimum m_r and supremum M_r at some points c_r and d_r of $[x_{r-1}, x_r]$ respectively so that

$$m_r = f(c_r) \text{ and } M_r = f(d_r). \qquad ...(2)$$

Since $|c_r - d_r| < \delta$, therefore from (1), we get

$$|f(c_r) - f(d_r)| < \varepsilon/(b-a).$$

But $|f(c_r) - f(d_r)| = f(d_r) - f(c_r)$ $[\because f(d_r) \geq f(c_r)]$

$= M_r - m_r$.

Thus $M_r - m_r < \varepsilon/(b-a)$, $r = 1, 2, \ldots, n$.

Now for the partition P of $[a, b]$, we have

$$0 \leq U(P, f) - L(P, f) = \sum_{r=1}^{n} (M_r - m_r)\,\Delta x_r$$

$$< \sum_{r=1}^{n} \{\varepsilon/(b-a)\}\,\Delta x_r$$

$$= \frac{\varepsilon}{b-a}\sum_{r=1}^{n} \Delta x_r = \frac{\varepsilon}{b-a}\cdot(b-a) = \varepsilon.$$

Hence by theorem of article 7.6, f is R-integrable over $[a, b]$,

i.e., $f \in \boldsymbol{R}[a, b]$.

Note: There exist functions which are integrable but not continuous. So, continuity is a sufficient condition but not necessary for integrability.

Theorem 2: *If f is monotonic on $[a, b]$, then $f \in R[a, b]$.*

(Purvanchal 2008, 12; Rohilkhand 10)

Proof: Suppose f is monotonic increasing on $[a, b]$.

Then $f(a) \le f(x) \le f(b) \quad \forall \; x \in [a, b]$.

$\therefore$ f is bounded on $[a, b]$ and $\inf f = f(a)$, $\sup f = f(b)$.

Let $\varepsilon > 0$ be given and $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be a partition of $[a, b]$ with $\|P\| \le \varepsilon / [f(b) - f(a) + 1]$.

If m_r and M_r be the inf and sup of f on I_r, then $m_r = f(x_{r-1})$ and $M_r = f(x_r)$ because f is monotonic increasing on $[a, b]$.

Hence
$$U(P, f) - L(P, f) = \sum_{r=1}^{n} (M_r - m_r)\,\Delta x_r$$

$$= \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})]\,\Delta x_r$$

$$\le \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})] \cdot \frac{\varepsilon}{f(b) - f(a) + 1}$$

$$= \frac{\varepsilon}{f(b) - f(a) + 1} \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})]$$

$$= \frac{\varepsilon}{f(b) - f(a) + 1} [f(x_1) - f(x_0) + f(x_2) - f(x_1) + \ldots + f(x_n) - f(x_{n-1})]$$

$$= \frac{\varepsilon}{f(b) - f(a) + 1} [f(b) - f(a)] \qquad [\because f(x_0) = f(a), f(x_n) = f(b)]$$

$$< \varepsilon.$$

It follows from the theorem of article 7.6 that $f \in R[a, b]$.

If f is monotonic decreasing on $[a, b]$, then $-f$ is monotonic increasing on $[a, b]$ and so $-f \in R[a, b]$.

$\therefore$ $$\underline{\int_a^b} -f(x)\,dx = \overline{\int_a^b} -f(x)\,dx$$

i.e., $$-\underline{\int_a^b} f(x)\,dx = -\overline{\int_a^b} f(x)\,dx$$

or $$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$$

i.e., f is R-integrable on $[a, b]$.

Hence if f is monotonic on $[a, b]$, $f \in R[a, b]$.

Note: If we had taken $f(b) - f(a)$ instead of $f(b) - f(a) + 1$, the proof would not have been valid when $f(b) - f(a) = 0$ *i.e.*, when f is a constant function. It is to include this case that we have used this artifice.

Theorem 3: *If the set of points of discontinuity of a bounded function f defined on $[a, b]$ is finite, then $f \in \mathbf{R}[a, b]$.*

Proof: Since f is discontinuous on $[a, b]$, the supremum M and the infimum m of f in $[a, b]$ are not equal *i.e.*, $M - m \neq 0$.

Let $\{c_1, c_2, \ldots, c_p\}$ be the finite set of points of discontinuity of f in $[a, b]$, where $c_1 < c_2 < \ldots < c_p$. Let $\varepsilon > 0$ be given. We enclose the points $c_1, c_2, \ldots, c_p$ respectively in p mutually disjoint intervals

$$[a_1, b_1], [a_2, b_2], \ldots, [a_p, b_p] \qquad \ldots(1)$$

such that the sum of their lengths is $< \varepsilon / 2 (M - m)$.

Now f is continuous on each of the subintervals

$$[a, a_1], [b_1, a_2], [b_2, a_3], \ldots, [b_p, b]. \qquad \ldots(2)$$

Consequently, there exist partitions P_r, $r = 1, 2, \ldots, p + 1$ respectively, of the subintervals in (2) such that, using theorem 1 of article 7.7,

$$\omega(P_r, f) < \frac{\varepsilon}{2(p+1)} \text{ for } r = 1, 2, \ldots, p + 1.$$

Now consider the partition P of $[a, b]$ defined by

$$P = \cup \{P_r : r = 1, 2, \ldots, p + 1\}.$$

The subintervals of P can be divided into two groups :

(i) all the subintervals of P_r, $r = 1, 2, \ldots, p + 1$

(ii) all the subintervals given in (1).

Corresponding to these two groups of subintervals we have two contributions to the oscillatory sum $\omega(P, f)$.

The total contribution to the oscillatory sum $\omega(P, f)$ of the subintervals in (i) is

$$< \frac{\varepsilon}{2(p+1)}(p+1) = \frac{\varepsilon}{2}.$$

Also since the oscillation of f in each of the subintervals in (ii) is $\leq M - m$, the total contribution to the oscillatory sum $\omega(P, f)$ of the subintervals in (ii) is

$$< \frac{\varepsilon}{2(M-m)} \cdot (M - m) = \frac{\varepsilon}{2}.$$

$\therefore$ for the partition P of $[a, b]$, we have

$$\omega(P, f) < \varepsilon / 2 + \varepsilon / 2 = \varepsilon.$$

Thus for each $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that

$$0 \leq \omega (P, f) < \varepsilon.$$

Hence $f \in \boldsymbol{R} [a, b]$, by theorem of article 7.6.

Theorem 4: *If the set of points of discontinuity of a bounded function f defined on $[a, b]$ has only a finite number of limit points then $f \in \boldsymbol{R} [a, b]$.*

Proof: Since f is discontinuous on $[a, b]$, the supremum M and the infimum m of f in $[a, b]$ are not equal *i.e.*, $M - m \neq 0$.

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ be the finite set of limit points of the set of the points of discontinuity of f in $[a, b]$, where $\alpha_1 < \alpha_2 < \ldots < \alpha_p$. Let $\varepsilon > 0$ be given. We enclose the points $\alpha_1, \alpha_2, \ldots, \alpha_p$ respectively in p mutually disjoint intervals

$$[a_1, b_1], [a_2, b_2], \ldots, [a_p, b_p] \qquad \ldots(1)$$

such that the sum of their lengths is $< \varepsilon / 2 (M - m)$.

In each of the remaining $p + 1$ subintervals

$$[a, a_1], [b_1, a_2], [b_2, a_3], \ldots, [b_p, b] \qquad \ldots(2)$$

f has only a finite number of points of discontinuity because none of these $p + 1$ sub-intervals contains a limit point of the set of points of discontinuity of f.

Hence, by the previous theorem, there exist partitions P_r, $r = 1, 2, \ldots, p + 1$ respectively of the subintervals in (2) such that

$$\omega (P_r, f) < \frac{\varepsilon}{2 (p + 1)} \text{ for } r = 1, 2, \ldots, p + 1.$$

Now consider the partition P of $[a, b]$ determined by

$$P = \cup \{P_r : r = 1, 2, \ldots, p + 1\}.$$

The subintervals of P can be divided into two groups :

(i) all the subintervals of P_r, $r = 1, 2, \ldots, p + 1$.

(ii) all the subintervals given in (1).

The total contribution to the oscillatory sum $\omega (P, f)$ of the subintervals in (i) is

$$< \{\varepsilon / 2 (p + 1)\}. (p + 1) = \varepsilon / 2.$$

Also since the oscillation of f in each of the subintervals in (ii) is $\leq M - m$, the total contribution to the oscillatory sum $\omega (P, f)$ of the subintervals in (ii) is

$$< \{\varepsilon / 2 (M - m)\}. (M - m) = \varepsilon / 2.$$

Thus for any $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$\omega (P, f) < \varepsilon / 2 + \varepsilon / 2 = \varepsilon.$$

Hence, by theorem of article 7.6, $f \in \boldsymbol{R} [a, b]$.

Illustrative Examples

Example 12: *A function f is defined on $[0, 1]$ by*

$$f(x) = 1/2^n \text{ for } 1/2^{n+1} < x \le 1/2^n, n = 0, 1, 2, \ldots.$$

and $f(0) = 0$. Show that $f \in \boldsymbol{R}[0, 1]$ and calculate the value of

$$\int_0^x f(t)\, dt$$

where x lies between $1/2^m$ and $1/2^{m-1}$.

Solution: Here, for $n = 1, 2, 3, \ldots$, we have $f\left(\frac{1}{2^n} + 0\right) = \frac{1}{2^{n-1}}$ and $f\left(\frac{1}{2^n} - 0\right) = \frac{1}{2^n}$, which shows that the function $f(x)$ is discontinuous at $x = 1/2^n$, $n = 1, 2, 3, \ldots$ Also for $n = 0$, $f\left(\frac{1}{2^n}\right) = f(1) = 1$

and $f\left(\frac{1}{2^n} - 0\right) = 1$ so that $f(x)$ is continuous at $x = \frac{1}{2^0} = 1$.

Thus the points of discontinuity of f are

$$\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n}, \ldots$$

Since the set of these points of discontinuity of f has only one limiting point at $x = 0$, it follows from theorem 4 of article 7.7 that $f \in \boldsymbol{R}[0, 1]$.

Now $\int_0^x f(t)\, dt = \int_{1/2^m}^x f + \int_{1/2^{m+1}}^{1/2^m} f + \int_{1/2^{m+2}}^{1/2^{m+1}} f + \ldots$

$$= \int_{1/2^m}^x \frac{1}{2^{m-1}} + \int_{1/2^{m+1}}^{1/2^m} \frac{1}{2^m} + \int_{1/2^{m+2}}^{1/2^{m+1}} \frac{1}{2^{m+1}} + \ldots$$

$$= \frac{1}{2^{m-1}}\left[x - \frac{1}{2^m}\right] + \frac{1}{2^m}\left[\frac{1}{2^m} - \frac{1}{2^{m+1}}\right] + \frac{1}{2^{m+1}}\left[\frac{1}{2^{m+1}} - \frac{1}{2^{m+2}}\right] + \ldots$$

$$= \frac{1}{2^{m-1}}\left[x - \frac{1}{2^m}\right] + \frac{1}{2^{2m+1}} + \frac{1}{2^{2m+3}} + \frac{1}{2^{2m+5}} + \ldots$$

$$= \frac{x}{2^{m-1}} - \frac{1}{2^{2m-1}} + \frac{1/2^{2m+1}}{1 - \frac{1}{4}} \quad \text{(Summing up the infinite G.P.)}$$

$$= \frac{x}{2^{m-1}} - \frac{1}{2^{2m-1}} + \frac{1}{3.2^{2m-1}} = \frac{x}{2^{m-1}} - \frac{1}{3.2^{2m-2}}.$$

Example 13: *Let the function f be defined on $[0, 1]$ as follows :*

$$f(x) = 2rx \quad \text{when} \quad \frac{1}{r+1} < x \leq \frac{1}{r}, \quad r = 1, 2, 3, \ldots$$

Prove that f is R-integrable in $[0, 1]$ and evaluate $\int_0^1 f(x)\,dx$. **(Rohilkhand 2010)**

Solution: The given function f is not defined for $x = 0$. We may, however, define f at this point in any manner we like provided f remains bounded.

Here $f(x) = 2x$ when $\frac{1}{2} < x \leq 1$

$f(x) = 4x$ when $\frac{1}{3} < x \leq \frac{1}{2}$

... ...

$f(x) = 2(r-1)x$ when $\frac{1}{r} < x \leq \frac{1}{r-1}$

$f(x) = 2rx$ when $\frac{1}{r+1} < x \leq \frac{1}{r}$

... ...

For $r = 2, 3, 4, \ldots$, we have

$$f\left(\frac{1}{r} + 0\right) = \lim_{h \to 0} 2(r-1)\left(\frac{1}{r} + h\right) = 2 - \frac{2}{r}$$

and $$f\left(\frac{1}{r} - 0\right) = \lim_{h \to 0} 2r\left(\frac{1}{r} - h\right) = 2.$$

Since $f\left(\frac{1}{r} + 0\right) \neq f\left(\frac{1}{r} - 0\right)$, f is not continuous at $x = 1/r$.

Also $f(1) = 2$ and $f(1 - 0) = \lim_{h \to 0} 2(1 - h) = 2$, so that f is continuous at $x = 1$.

Thus the given function f is not continuous at $x = 1/r, r = 2, 3, 4, \ldots$ and the set of points of discontinuity $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ of f has only one limit point at $x = 0$. Consequently, by theorem 4 of article 7.7, the given function is R-integrable.

Now $\int_{1/(n+1)}^{1} f(x)\,dx = \int_{1/2}^{1} f + \int_{1/3}^{1/2} f + \int_{1/4}^{1/3} f + \ldots + \int_{1/(n+1)}^{1/n} f$

$$= \sum_{r=1}^{n} \int_{1/(r+1)}^{1/r} f. \qquad \ldots(1)$$

We have $\int_{1/(r+1)}^{1/r} f(x)\,dx = \int_{1/(r+1)}^{1/r} 2rx\,dx = [rx^2]_{1/(r+1)}^{1/r}$

$$= r\left[\frac{1}{r^2} - \frac{1}{(r+1)^2}\right]. \qquad \ldots(2)$$

Putting $r = 1, 2, \ldots, n$ in (2) and then adding the partial integrals, in view of (1), we have

$$\int_{1/(n+1)}^{1} f(x)\,dx = 1\left[\frac{1}{1^2} - \frac{1}{2^2}\right] + 2\left[\frac{1}{2^2} - \frac{1}{3^2}\right] + 3\left[\frac{1}{3^2} - \frac{1}{4^2}\right] + \ldots + n\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right]$$

$$= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2} - \frac{n}{(n+1)^2}$$

$$= \sum_{r=1}^{n} \frac{1}{r^2} - \frac{1/n}{\left(1 + \frac{1}{n}\right)^2}.$$

Letting n tend to infinity, we get

$$\int_0^1 f(x)\,dx = \sum_{1}^{\infty} \frac{1}{r^2} - 0 = \sum_{1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}.$$

Example 14: *A function f is defined on $[0, 1]$ by*

$$f(x) = 1/n \text{ for } 1/(n+1) < x \le 1/n, n = 1, 2, 3, \ldots \text{ and } f(0) = 0.$$

Prove that $f \in \boldsymbol{R}[0, 1]$ and evaluate $\int_0^1 f(x)\,dx$.

Solution: It can be easily seen that the points of discontinuity of f are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. Since the set of points of discontinuity of f has only one limit point at $x = 0$, it follows from theorem 4 of article 7.7 that $f \in \boldsymbol{R}[0, 1]$.

Now as in the previous example,

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} \int_{1/(r+1)}^{1/r} \frac{1}{r}\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{r}\left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$$= \lim_{n\to\infty} \left[\left(1 - \frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \frac{1}{n}\left(\frac{1}{n} - \frac{1}{n+1}\right)\right]$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2}\right) - \left(\frac{1}{2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n.(n+1)}\right)\right]$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2}\right) - \left(\frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n} - \frac{1}{(n+1)}\right)\right]$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2}\right) - \left(1 - \frac{1}{n+1}\right)\right] = \frac{\pi^2}{6} - 1.$$

$$\left[\because \text{the series } 1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2} + \ldots \text{converges to } \frac{\pi^2}{6}\right]$$

Example 15: *Let a function f be defined on $[0, 1]$ as follows :*

If x is irrational let $f(x) = 0$, if x is a rational number p/q in its lowest terms, let $f(x) = 1/q$, also let $f(0) = f(1) = 0$. Show that f is integrable over $[0, 1]$ and $\int_0^1 f(x)\,dx = 0$.

Solution: Evidently, the function f is bounded on $[0, 1]$. Let $\varepsilon > 0$ be given. Then there exist only a finite number of fractions $\frac{1}{q}$ such that $\frac{1}{q} > \frac{\varepsilon}{2}$, for $\frac{1}{q} > \frac{\varepsilon}{2}$ iff $q < \frac{2}{\varepsilon}$ and ε being given, $\frac{2}{\varepsilon}$ is finite. We enclose these exceptional points, in order, in mutually disjoint closed intervals

$$[a_1, b_1], [a_2, b_2], \ldots, [a_m, b_m] \qquad \ldots(1)$$

such that the sum of their lengths is less than $\varepsilon/2$. On the remaining sub-intervals

$$[b_0 = 0, a_1], [b_1, a_2], [b_2, a_3], \ldots, [b_m, a_{m+1} = 1] \qquad \ldots(2)$$

at each point the value of f is $< \varepsilon/2$.

It is obvious that the oscillation of f in each of the subintervals (1) cannot exceed 1 and the oscillation of f in each of the sub-intervals (2) is less than $\varepsilon/2$. Thus for the partition

$$P = \{0 = b_0, a_1, b_1, a_2, b_2, \ldots, a_m, b_m, b_{m+1} = 1\}$$

we have, $\omega(P, f) = U(P, f) - L(P, f)$

$$< \sum_{r=1}^{m} 1 \cdot (b_r - a_r) + \sum_{r=0}^{m} \frac{\varepsilon}{2}\,(a_{r+1} - b_r)$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

$$\left[\because \sum_{r=1}^{m} (b_r - a_r) < \frac{\varepsilon}{2},\ \sum_{r=0}^{m} (a_{r+1} - b_r) < 1\right]$$

Thus for a given $\varepsilon > 0$ there exists a partition P such that $\omega(P, f) < \varepsilon$.

Hence $f \in \boldsymbol{R}[0, 1]$.

Also since for every partition P, $L(P, f) = 0$, we have

$$\int_0^1 f = \int_{-0}^{1} f = 0.$$

7.8 Algebra of Integrable Functions

Definition: *If $b < a$, we define $\int_a^b f$ to be $-\int_b^a f$ provided that $f \in \boldsymbol{R}[a, b]$.*

Also by definition, we write $\int_a^a f = 0$.

Theorem 1: *Let $f \in \boldsymbol{R}[a, b]$ and let m, M be bounds of f on $[a, b]$. Then*

$$m(b - a) \le \int_a^b f(x)\,dx \le M(b - a) \text{ if } b \ge a,$$

$$m(b - a) \ge \int_a^b f(x)\,dx \ge M(b - a) \text{ if } b \le a.$$

(Purvanchal 2008)

Proof: If $a = b$, the result is trivial. Let $b > a$. Then by theorem 1 of article 7.2, we have for all partitions P of $[a, b]$,

$$m(b-a) \le L(P, f) \le U(P, f) \le M(b-a)$$

or $$m(b-a) \le L(P, f) \le M(b-a)$$

or $$m(b-a) \le \sup L(P, f) \le M(b-a)$$

or $$m(b-a) \le \underline{\int_a^b} f \le M(b-a)$$

or $$m(b-a) \le \int_a^b f \le M(b-a). \qquad \left[\because \underline{\int_a^b} f = \int_a^b f\right]$$

Now let $b < a$ so that $a > b$. Hence, as proved above we get

$$m(a-b) \le \int_b^a f \le M(a-b)$$

$\Rightarrow$ $$-m(b-a) \le -\int_a^b f \le -M(b-a), \quad \text{by the definition mentioned above}$$

$\Rightarrow$ $$m(b-a) \ge \int_a^b f \ge M(b-a).$$

Corollary 1: *Let $f \in \boldsymbol{R}[a, b]$. Then there exists a number μ lying between the bounds m and M of f such that* $\int_a^b f(x)\,dx = \mu(b-a)$. **(Purvanchal 2011)**

This result is known as the **first mean value theorem of integral calculus.**

Corollary 2: *Let f be continuous on $[a, b]$. Then there exists a point $c \in [a, b]$ such that*

$$\int_a^b f(x)\,dx = (b-a) f(c).$$

Proof: Since f is continuous, $f \in \boldsymbol{R}[a, b]$. Therefore, by Corollary 1, there exists a number μ lying between m and M, such that

$$\int_a^b f(x)\,dx = \mu(b-a).$$

Now f being continuous on $[a, b]$, so it takes every value between its bounds m and M *i.e.*, in particular it takes the value μ lying between m and M. Consequently there is a point $c \in [a, b]$ such that $f(c) = \mu$ and hence

$$\int_a^b f(x)\,dx = (b-a) f(c).$$

Corollary 3: *Let $f \in \boldsymbol{R}[a, b]$ and let K be a number such that*

$$|f(x)| \le K \ \forall\ x \in [a, b].$$

Then $$\left|\int_a^b f(x)\,dx\right| \le K|b-a|.$$ **(Purvanchal 2010)**

Proof: The result is trivial when $a = b$. Let m, M be the bounds of f on $[a, b]$. Let $b > a$. We have, for all $x \in [a, b]$

$$|f(x)| \le K \Rightarrow -K \le f(x) \le K \quad \Rightarrow -K \le m \le f(x) \le M \le K$$

$$\Rightarrow \quad -K(b-a) \le m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \le K(b-a)$$

$$\Rightarrow \quad \left|\int_a^b f(x)\,dx\right| \le K(b-a). \qquad \ldots(1)$$

Now let $b < a$ so that $a > b$.

Then we get from (1) that

$$\left|\int_b^a f(x)\,dx\right| \le K(a-b) \Rightarrow \left|-\int_a^b f(x)\,dx\right| \le K(a-b)$$

$$\Rightarrow \quad \left|\int_a^b f(x)\,dx\right| \le K(a-b). \qquad \ldots(2)$$

Hence, from (1) and (2), we get

$$\left|\int_a^b f(x)\,dx\right| \le K|b-a|.$$

Corollary 4: *Let $f \in \boldsymbol{R}[a, b]$ and let $f(x) \ge 0 \ \forall\, x \in [a, b]$. Then*

$$\int_a^b f(x)\,dx \ge 0 \text{ if } b \ge a \text{ and } \int_a^b f(x)\,dx \le 0 \text{ if } b \le a.$$

Proof: Since $f(x) \ge 0 \ \forall\, x \in [a, b]$, hence, $m \ge 0$.

If $b \ge a$, then from the first result of the above theorem, we get

$$\int_a^b f(x)\,dx \ge m(b-a) \ge 0$$

and if $b \le a$, then from the second result of the above theorem, we get

$$\int_a^b f(x)\,dx \le m(b-a) \le 0. \qquad [\because\ m \ge 0 \text{ and } b-a \le 0]$$

Corollary 5: *Let $f, g \in \boldsymbol{R}[a, b]$. Then*

$$f \ge g \Rightarrow \begin{cases} \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx \text{ if } b \ge a \\ \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx \text{ if } b \le a. \end{cases}$$

Proof: We have

$$f \ge g \Rightarrow [f(x) - g(x)] \ge 0 \ \forall\ x \in [a, b]$$

$$\Rightarrow \quad \int_a^b [f(x) - g(x)]\,dx \ge 0 \text{ or } \le 0$$

according as $b \ge a$ or $b \le a$ by Corollary 4,

$$\Rightarrow \quad \left[\int_a^b f(x)\,dx - \int_a^b g(x)\,dx\right] \ge 0 \text{ or } \le 0$$

$$\Rightarrow \quad \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx \text{ or } \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

according as $b \ge a$ or $b \le a$.

Theorem 2: *If $f \in \boldsymbol{R}[a, b]$ and $K \in \boldsymbol{R}$, then $Kf \in \boldsymbol{R}[a, b]$ and*

$$\int_a^b Kf(x)\,dx = K\int_a^b f(x)\,dx.$$

Proof: If $K = 0$, the theorem is obvious. Suppose that $K \neq 0$. Since $f \in \boldsymbol{R}[a, b]$, f is bounded on $[a, b]$ and

$$\underline{\int}_a^b f(x)\,dx = \overline{\int}_a^b f(x)\,dx = \int_a^b f(x)\,dx. \qquad \ldots(1)$$

We know that $|Kf| = |K| \cdot |f|$, so that Kf is bounded on $[a, b]$.

Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be any partition of $[a, b]$ and m_r and M_r be the infimum and supremum of f in $[x_{r-1}, x_r]$.

Case I: Let $K > 0$. Then $K m_r$ and $K M_r$ are the inf and sup of Kf in $[x_{r-1}, x_r]$.

So we have

$$\begin{aligned}
\underline{\int}_a^b Kf(x)\,dx &= \sup\left[\sum_{r=1}^{n} K m_r \Delta x_r\right] \\
&= K \sup\left[\sum_{r=1}^{n} m_r \Delta x_r\right] = K \underline{\int}_a^b f(x)\,dx \qquad \ldots(2) \\
&= K \overline{\int}_a^b f(x)\,dx, \quad \text{using (1)} \\
&= K.\inf\left[\sum_{r=1}^{n} M_r \Delta x_r\right] = \inf\left[\sum_{r=1}^{n} K M_r \Delta x_r\right] \\
&= \underline{\int}_a^b Kf(x)\,dx.
\end{aligned}$$

Thus $$\underline{\int}_a^b Kf(x)\,dx = \overline{\int}_a^b Kf(x)\,dx.$$

Hence $Kf \in \boldsymbol{R}[a, b]$.

Also from (2), we have

$$\underline{\int}_a^b Kf(x)\,dx = K \underline{\int}_a^b f(x)\,dx$$

$$\Rightarrow \quad \int_a^b Kf(x)\,dx = K \int_a^b f(x)\,dx. \qquad [\because f \in \boldsymbol{R}[a, b] \text{ and } Kf \in \boldsymbol{R}[a, b]]$$

Case II: Let $K < 0$. Then $K M_r$ and $K m_r$ respectively denote the inf and sup of Kf in $[x_{r-1}, x_r]$. We have

$$\begin{aligned}
\underline{\int}_a^b Kf(x)\,dx &= \sup\left[\sum_{r=1}^{n} K M_r \Delta x_r\right] \\
&= K \inf\left[\sum_{r=1}^{n} M_r \Delta x_r\right], \text{ as } K < 0 \\
&= K \overline{\int}_a^b f(x)\,dx = K \underline{\int}_a^b f(x)\,dx \qquad \ldots(3) \\
&= K \sup\left[\sum_{r=1}^{n} m_r \Delta x_r\right] = \inf\left[\sum_{r=1}^{n} K m_r \Delta x_r\right], \text{ as } K < 0
\end{aligned}$$

$$= \bar{\int}_a^b K f(x)\, dx.$$

Thus in this case, $\underline{\int}_a^b K f(x)\, dx = \bar{\int}_a^b K f(x)\, dx$.

Hence $Kf \in \boldsymbol{R}[a, b]$. Also from (3), we have

$$\underline{\int}_a^b K f(x)\, dx = K \underline{\int}_a^b f(x)\, dx$$

$$\Rightarrow \quad \int_a^b K f(x)\, dx = K \int_a^b f(x)\, dx.$$

$[\because f \in \boldsymbol{R}[a, b]$ and $Kf \in \boldsymbol{R}[a, b]]$

Theorem 3: *Let $a < c < b$. Then $f \in \boldsymbol{R}[a, b]$ iff $f \in \boldsymbol{R}[a, c]$ and $f \in \boldsymbol{R}[c, b]$. In either case*

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Proof: Obviously f is bounded on $[a, c]$ and $[c, b]$ iff f is bounded on $[a, b]$.

Let $f \in \boldsymbol{R}[a, b]$. Then for a given $\varepsilon > 0$, there is a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon. \qquad \ldots(1)$$

Let $P^* = P \cup \{c\}$. Then P^* is also a partition of $[a, b]$ and it is a refinement of P so that

$$U(P^*, f) - L(P^*, f) \leq U(P, f) - L(P, f) < \varepsilon. \qquad \ldots(2)$$

Let P_1 and P_2 be the partitions of $[a, c]$ and $[c, b]$ respectively such that $P^* = P_1 \cup P_2$. Then

$$U(P^*, f) - L(P^*, f) = [U(P_1, f) + U(P_2, f)] - [L(P_1, f) + L(P_2, f)]$$

$$= [U(P_1, f) - L(P_1, f)] + [U(P_2, f) - L(P_2, f)]$$

$< \varepsilon$, by (2).

Since each of $[U(P_1, f) - L(P_1, f)]$ and $[U(P_2, f) - L(P_2, f)]$ is ≥ 0, each of them is less than ε.

Hence $f \in \boldsymbol{R}[a, c]$ and $f \in \boldsymbol{R}[c, b]$.

Also $U(P^*, f) = U(P_1, f) + U(P_2, f)$

$$\Rightarrow \quad \inf U(P^*, f) = \inf U(P_1, f) + \inf U(P_2, f)$$

$$\Rightarrow \quad \bar{\int}_a^b f(x)\, dx = \bar{\int}_a^a f(x)\, dx + \bar{\int}_c^b f(x)\, dx$$

$$\Rightarrow \quad \int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

$[\because f \in \boldsymbol{R}[a, b], f \in \boldsymbol{R}[a, c], f \in \boldsymbol{R}[c, b]]$

Conversely, let $f \in \boldsymbol{R}[a, c]$ and $f \in \boldsymbol{R}[c, b]$.

Then given $\varepsilon > 0$, there exist partitions P_1 and P_2 of $[a, c]$ and $[c, b]$ respectively such that

$$U(P_1, f) - L(P_1, f) < \varepsilon / 2$$

and $$U(P_2, f) - L(P_2, f) < \varepsilon / 2.$$

Let $P = P_1 \cup P_2$, then P is partition of $[a, b]$.

Now $U(P, f) - L(P, f) = [U(P_1, f) + U(P_2, f)] - [L(P_1, f) + L(P_2, f)]$

$$= [U(P_1, f) - L(P_1, f)] + [U(P_2, f) - L(P_2, f)]$$

$$< \varepsilon / 2 + \varepsilon / 2 = \varepsilon.$$

Hence $f \in \boldsymbol{R}[a, b]$. The remaining part of the theorem can be proved as before.

Theorem 4: *If $f, g \in \boldsymbol{R}[a, b]$ then $f \pm g \in \boldsymbol{R}[a, b]$*

and $$\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx.$$

Proof: Since $f, g \in \boldsymbol{R}[a, b]$, for a given $\varepsilon > 0$, there exist partitions P_1 and P_2 of $[a, b]$ such that

$$\left.\begin{aligned} U(P_1, f) - L(P_1, f) < \varepsilon / 2 \\ \text{and} \quad U(P_2, g) - L(P_2, g) < \varepsilon / 2 \end{aligned}\right\} \quad \ldots(1)$$

Let $P = P_1 \cup P_2$, then P is a common refinement of P_1 and P_2 and it is a partition of $[a, b]$. We have

$$U(P, f + g) - L(P, f + g)$$

$$\leq [U(P, f) + U(P, g)] - [L(P, f) + L(P, g)],$$

[by theorem 5, article 7.2]

$$= [U(P, f) - L(P, f)] + [U(P, g) - L(P, g)]$$

$$< \varepsilon / 2 + \varepsilon / 2 = \varepsilon, \text{ by (1)}.$$

Since $$U(P, f + g) - L(P, f + g) < \varepsilon, f + g \in \boldsymbol{R}[a, b].$$

Again, by the second definition of Riemann integrability, $f, g \in \boldsymbol{R}[a, b] \Rightarrow$ for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every partition P of $[a, b]$ with $\| P \| \leq \delta$ and for every choice of $\xi_r \in [x_{r-1}, x_r]$,

$$\left| \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r - \int_a^b f(x)\, dx \right| < \varepsilon / 2 \quad \ldots(2)$$

and $$\left| \sum_{r=1}^{n} g(\xi_r)\, \Delta x_r - \int_a^b g(x)\, dx \right| < \varepsilon / 2 \quad \ldots(3)$$

From (2) and (3), we get

$$\left| \sum_{r=1}^{n} [f(\xi_r) + g(\xi_r)]\, \Delta x_r - \left[\int_a^b f(x)\, dx + \int_a^b g(x)\, dx \right] \right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

Similarly we can prove the result for the difference $f - g$ of the functions f and g.

Lemma: *Let f be a bounded function with bounds m and M defined on $[a, b]$. Then the oscillation $M - m$ of f on $[a, b]$ is the supremum of the set*

$$\{|f(x) - f(y)| : x, y \in [a, b]\}$$

of numbers.

Theorem 5: *If $f \in \boldsymbol{R}[a, b]$, $g \in \boldsymbol{R}[a, b]$, then $fg \in \boldsymbol{R}[a, b]$.*

(Purvanchal 2010; Bundelkhand 11)

Proof: Since $f, g \in \boldsymbol{R}[a, b]$, they are bounded on $[a, b]$.

So there exists a positive real number M such that

$$|f(x)| \leq M \text{ and } |g(x)| \leq M \quad \forall \ x \in [a, b].$$

It follows that $|f(x)\, g(x)| \leq M^2 \quad \forall\, x \in [a, b]$.

Thus fg is bounded on $[a, b]$.

Again $f, g \in \cup [a, b] \Rightarrow$ for a given $\varepsilon > 0$, there exist partitions P_1 and P_2 of $[a, b]$ such that

$$U(P_1, f) - L(P_1, f) < \varepsilon / 2M$$

and $$U(P_2, g) - L(P_2, g) < \varepsilon / 2M .$$

Let $P = P_1 \cup P_2 = \{a = x_0, x_1, \ldots, x_n = b\}$ so that P is refinement of both P_1 and P_2. It follows by theorem 3 of article 7.2 that

$$\left.\begin{aligned} U(P, f) - L(P, f) &< \varepsilon / 2M \\ \text{and} \quad U(P, g) - L(P, g) &< \varepsilon / 2M \end{aligned}\right\} \quad \ldots(1)$$

Let m_r, M_r; m_r', M_r' and m_r'', M_r'' be respectively the bounds of fg, f and g in the r th interval $I_r = [x_{r-1}, x_r]$. Then for all $x, y \in [x_{r-1}, x_r]$, we have

$$\begin{aligned} |(fg)(y) - (fg)(x)| &= |f(y)\, g(y) - f(x)\, g(x)|, \text{ by the def. of } fg \\ &= |g(y)[f(y) - f(x)] + f(x)[g(y) - g(x)]| \\ &\leq |g(y)|\,|f(y) - f(x)| + |f(x)|\,|g(y) - g(x)| \\ &\leq M|f(y) - f(x)| + M|g(y) - g(x)| \quad \ldots(2) \end{aligned}$$

Taking suprema of both sides of (2), we have by the lemma given above

$$M_r - m_r \leq M(M_r' - m_r') + M(M_r'' - m_r'')$$

$$\Rightarrow \quad \sum_{r=1}^{n} (M_r - m_r)\, \Delta x_r \leq M \sum_{r=1}^{n} (M_r' - m_r')\, \Delta x_r + M \sum_{r=1}^{n} (M_r'' - m_r'')\, \Delta x_r$$

or
$$\begin{aligned} U(P, fg) - L(P, fg) &\leq M[U(P, f) - L(P, f)] + M[U(P, g) - L(P, g)] \\ &< M(\varepsilon / 2M) + M(\varepsilon / 2M), \text{ by (1)} \\ &= \varepsilon / 2 + \varepsilon / 2 = \varepsilon . \end{aligned}$$

Hence $fg \in \boldsymbol{R}[a, b]$ by theorem of article 7.6.

Corollary: *If* $f \in \boldsymbol{R}[a, b]$ *then* $f^2 \in \boldsymbol{R}[a, b]$.

Theorem 6: *If* $f \in \boldsymbol{R}[a, b]$ *then* $|f| \in \boldsymbol{R}[a, b]$. (Purvanchal 2008)

Show that $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.

Proof: Since $f \in \boldsymbol{R}[a, b]$, f is bounded on $[a, b]$ so that there exists a positive number k such that $|f(x)| \le k\ \forall x \in [a, b]$ *i.e.* $|f(x)|$ is bounded on $[a, b]$.

Again, since $f \in \boldsymbol{R}[a, b]$, for given $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon \qquad \ldots(1)$$

Let m_r, M_r and m_r', M_r' be the bounds of $|f|$ and f respectively in $[x_{r-1}, x_r]$. Then for all $x, y \in [x_{r-1}, x_r]$, we have

$$\Big|\,|f(y)| - |f(x)|\,\Big| \le |f(y) - f(x)| \qquad \ldots(2)$$

Taking suprema of both sides of (2), we have by the lemma mentioned above

$$(M_r - m_r) \le (M_r' - m_r')$$

$$\Rightarrow \quad \sum_{r=1}^{n} (M_r - m_r)\,\Delta x_r \le \sum_{r=1}^{n} (M_r' - m_r')\,\Delta x_r$$

$$\Rightarrow \quad U(P, |f|) - L(P, |f|) \le U(P, f) - L(P, f) < \varepsilon, \text{ by (1)}.$$

Hence $|f| \in \boldsymbol{R}[a, b]$.

Now, let $f_1(x) = \frac{1}{2}\{|f(x)| + f(x)\}$

and $f_2(x) = \frac{1}{2}\{|f(x)| - f(x)\}$.

Then $|f(x)| = f_1(x) + f_2(x)$; $f(x) = f_1(x) - f_2(x)$.

Since $f_1(x) \ge 0$ and $f_2(x) \ge 0\ \forall x \in [a, b]$, we have

$$\int_a^b f_1(x)\,dx \ge 0 \text{ and } \int_a^b f_2(x)\,dx \ge 0.$$

Hence, we get

$$\left|\int_a^b f(x)\,dx\right| = \left|\int_a^b f_1(x)\,dx - \int_a^b f_2(x)\,dx\right|$$

$$\le \left|\int_a^b f_1(x)\,dx\right| + \left|\int_a^b f_2(x)\,dx\right|$$

$$= \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx$$

$$= \int_a^b [f_1(x) + f_2(x)]\,dx = \int_a^b |f(x)|\,dx.$$

Thus $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.

Note: The converse of the above theorem need not be true. Consider the function f defined on $[0, 1]$ by

$$f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational.} \end{cases}$$

We have seen earlier that $f \notin \boldsymbol{R}[0, 1]$.

But $|f| \in \boldsymbol{R}[0, 1]$. For, we have $|f(x)| = 1 \ \forall x \in [0, 1]$ and we know that every constant function is R-integrable.

7.9 Fundamental Theorem of Integral Calculus

(Meerut 2012)

In this section we shall establish the close relationship between the derivative and the integral in a rigorous manner. In fact, we shall prove that integration and differentiation are, in a certain sense, inverse operations.

Integral function: Let $f \in \boldsymbol{R}[a, b]$. We define a new real valued function F with domain $[a, b]$ by setting

$$F(x) = \int_a^x f(t)\, dt, a < x \leq b, F(a) = 0.$$

The function F is called an **integral function** or an **indefinite integral** of the integrable function f. The function F is well defined on $[a, b]$, as $f \in \boldsymbol{R}[a, x]$ where $a < x \leq b$ and the condition $F(a) = 0$ is consistent with our previous definition that $\int_a^a f = 0$.

Primitive: Definition: *A differentiable function ϕ defined on $[a, b]$ such that its derivative ϕ' equals a given function f on $[a, b]$ is called a* **primitive** *or* **anti-derivative** *of f on $[a, b]$.*

If ϕ is a primitive of f then $\phi + c$ will also be a primitive of f where c is any constant. Hence the primitive of a function is not unique.

Theorem 1: *Let $f \in \boldsymbol{R}[a, b]$. Then the function F defined on $[a, b]$ by*

$$F(x) = \int_a^x f(t)\, dt$$

is continuous on $[a, b]$.

Proof: Since $f \in \boldsymbol{R}[a, b]$, f is bounded on $[a, b]$. It follows that there exists a positive number M such that

$$|f(t)| \leq M \ \forall t \in [a, b].$$

Let $a \leq x < y \leq b$. Then we have

$$|F(y) - F(x)| = \left| \int_a^y f(t)\, dt - \int_a^x f(t)\, dt \right|$$

$$= \left| \int_a^y f(t)\, dt + \int_x^a f(t)\, dt \right| \qquad \left[\because \int_a^b f = -\int_b^a f\right]$$

$$= \left| \int_x^y f(t)\,dt \right| \qquad \text{[by theorem 3 of article 7. 8]}$$

$$\leq M \mid y - x \mid \qquad \text{[by Cor. 3 of theorem 1 of article 7.8]}$$

$$= M(y - x). \qquad \ldots(1)$$

Let $\varepsilon > 0$ be given. Then, if $|y - x| < \varepsilon / M$, we conclude from (1) that $|F(y) - F(x)| < \varepsilon$.

Thus given $\varepsilon > 0$, there exists $\delta\,(= \varepsilon / M) > 0$ such that

$$|F(y) - F(x)| < \varepsilon \text{ whenever } |y - x| < \delta \;\forall\, x, y \in [a, b].$$

Consequently the function F is uniformly continuous on $[a, b]$ and hence continuous on $[a, b]$.

Note: The above theorem can also be stated as follows :

The integral of an integrable function is continuous.

Theorem 2: *Let f be continuous on $[a, b]$ and let*

$$F(x) = \int_a^x f(t)\,dt \;\forall x \in [a, b].$$

Then $F'(x) = f(x) \;\forall\, x \in [a, b]$.

Proof: Let $x \in [a, b]$ be fixed. Choose $h \neq 0$ such that

$$x + h \in [a, b].$$

Then, we have

$$F(x + h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt$$

$$= \int_x^{x+h} f(t)\,dt.$$

Since f is continuous on $[a, b]$ there exists a number c in the interval $[x, x + h]$, such that

$$\int_x^{x+h} f(t)\,dt = h\,f(c) \qquad \text{[By Cor. 2 of theorem 1, article 7.8].}$$

We see that if $h \to 0$, then $c \to x$.

Thus $F(x + h) - F(x) = h\,f(c)$

$$\Rightarrow \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} = \lim_{h \to 0} f(c)$$

$$= \lim_{c \to x} f(c) = f(x), \text{ as } f \text{ is continuous.}$$

Hence, we get $F'(x) = f(x) \;\forall\, x \in [a, b]$.

Note: The following theorem is an improvement on the above theorem for only the R-integrability of f and the continuity of f at the point x_0 is assumed.

Theorem 3: *Let $f \in R[a, b]$ and let f be continuous at $x_0 \in [a, b]$.*

If $F(x) = \int_a^x f(t)\,dt,\ a \le x \le b$, *then*

$$F'(x_0) = f(x_0).$$

Proof: Since f is continuous at x_0, for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x_0 + h) - f(x_0)| < \varepsilon \qquad \ldots(1)$$

provided $|h| < \delta$ and $a \le x_0 + h \le b$.

We have $$F(x_0 + h) - F(x_0) = \int_a^{x_0 + h} f(t)\,dt - \int_a^{x_0} f(t)\,dt$$

$$= \int_a^{x_0 + h} f(t)\,dt$$

and $$\int_{x_0}^{x_0 + h} f(x_0)\,dt = f(x_0).[(x_0 + h) - x_0] = h f(x_0).$$

Now $$\left|\frac{F(x_0 + h) - F(x_0)}{h} - f(x_0)\right|$$

$$= \left|\frac{1}{h}\int_{x_0}^{x_0 + h} f(t)\,dt - \frac{1}{h}\int_{x_0}^{x_0 + h} f(x_0)\,dt\right|$$

$$= \left|\frac{1}{h}\int_{x_0}^{x_0 + h} [f(t) - f(x_0)]\,dt\right| \qquad \ldots(2)$$

By virtue of (1) and (2), we get on using Cor. 3 of theorem 1 of article 7.8

$$\left|\frac{F(x_0 + h) - F(x_0)}{h} - f(x_0)\right| < \frac{1}{|h|} \cdot |h|\,\varepsilon = \varepsilon \text{ for all } h \text{ with } |h| < \delta.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\lim_{h \to 0} \frac{F(x_0 + h) - F(x_0)}{h} = f(x_0) \text{ i.e., } F'(x_0) = f(x_0).$$

Theorem 4: (Fundamental theorem of Integral Calculus): *Let f be a continuous function on $[a, b]$ and let ϕ be a differentiable function on $[a, b]$ such that $\phi'(x) = f(x)$ for all $x \in [a, b]$. Then*

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a).$$

Proof: Let $F(x) = \int_a^x f(t)\,dt$. Then by theorem 2 of article 7.9, we have

$$F'(x) = f(x) \quad \forall\ x \in [a, b]. \qquad \ldots(1)$$

By hypothesis, $\phi'(x) = f(x)\ \forall\ x \in [a, b]$. $\qquad \ldots(2)$

From (1) and (2), we have $\forall\ x \in [a, b]$,

$$F'(x) = \phi'(x) \quad \text{or} \quad F'(x) - \phi'(x) = 0$$

$\Rightarrow$ $$(F - \phi)'(x) = 0 \Rightarrow (F - \phi)(x) = c \text{ for some } c \in R$$

$\Rightarrow$ $$F(x) - \phi(x) = c.$$

Thus $$F(x) = \phi(x) + c \quad \forall\ x \in [a, b].$$

Now $F(b) - F(a) = [\phi(b) + c] - [\phi(a) + c] = \phi(b) - \phi(a)$. ...(3)

Also $F(a) = \int_a^a f(t)\,dt = 0$ and $F(b) = \int_a^b f(t)\,dt$.

Putting these values in (3), we get

$$\int_a^b f(t)\,dt = \phi(b) - \phi(a)$$

or $$\int_a^b f(x)\,dx = \phi(b) - \phi(a).$$

Note: The following theorem is an improvement on the above theorem since only the R-integrability of f is assumed. In fact theorem 4 is a corollary of theorem 5.

Theorem 5: (Fundamental theorem of Integral Calculus): *Let $f \in R[a, b]$ and let ϕ be a differentiable function on $[a, b]$ such that $\phi'(x) = f(x)$ for all $x \in [a, b]$. Then* $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$.

(Rohilkhand 2011, 12)

Proof: Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of $[a, b]$. Now ϕ is differentiable on $[a, b]$ implies that ϕ is differentiable on each subinterval $[x_{r-1}, x_r]$. Hence by the mean value theorem of differential calculus, we find that there exists ξ_r in $[x_{r-1}, x_r]$, $r = 1, 2, \ldots, n$, such that

$$\phi(x_r) - \phi(x_{r-1}) = (x_r - x_{r-1})\,\phi'(\xi_r) = (\phi)'(\xi_r)\,.\,\Delta x_r$$

or $$\phi(x_r) - \phi(x_{r-1}) = f(\xi_r)\,.\,\Delta x_r \qquad [\because \phi'(\xi_r) = f(\xi_r)]$$

or $$\sum_{r=1}^{n} [\phi(x_r) - \phi(x_{r-1})] = \sum_{r=1}^{n} f(\xi_r)\,\Delta x_r\,. \qquad ...(1)$$

Now $$\sum_{r=1}^{n} [\phi(x_r) - \phi(x_{r-1})] = \phi(x_1) - \phi(x_0) + \phi(x_2) - \phi(x_1) + \ldots$$

$$\ldots + \phi(x_n) - \phi(x_{n-1})$$

$$= \phi(x_n) - \phi(x_0) = \phi(b) - \phi(a).$$

It follows from (1) that

$$\phi(b) - \phi(a) = \sum_{r=1}^{n} f(\xi_r)\,\Delta x_r \qquad ...(2)$$

Taking limit as $\| P \| \to 0$,

we get $$\phi(b) - \phi(a) = \int_a^b f(x)\,dx,$$

since $\sum_{r=1}^{n} f(\xi_r)\,\Delta x_r$ tends to $\int_a^b f(x)\,dx$ as $\| P \| \to 0$.

The result of the above theorem is usually written in the form

$$\int_a^b \phi'(x)\,dx = \phi(b) - \phi(a).$$

Note: Some authors call theorem 2 or theorem 3 as the **first fundamental theorem** and the theorem 4 or theorem 5 as the **second fundamental theorem** of integral calculus.

7.10 Mean Value Theorems of Integral Calculus

Theorem 1: (First Mean Value Theorem): *Let $f \in R[a, b]$. Then there exists a number μ lying between the bounds m and M of f such that*

$$\int_a^b f(x)\,dx = \mu\,(b-a).$$

Moreover if f is continuous, then

$$\int_a^b f(x)\,dx = (b-a)\,f(c),\ a \le c \le b.$$

(Purvanchal 2012)

Proof: See Cor. 1 and Cor. 2 of theorem 1 of article 7.8.

Theorem 2: (Second Mean Value Theorem): *Let*

$$f \in \boldsymbol{R}[a, b] \text{ and } g \in \boldsymbol{R}[a, b],$$

and $g(x) \ge 0$ *or* $\le 0\ \forall\ x \in [a, b]$.

Then there exists a number μ with $m \le \mu \le M$ such that

$$\int_a^b f(x)\,g(x)\,dx = \mu \int_a^b g(x)\,dx$$

where m, M are the bounds of f on $[a, b]$.

Proof: First let $g(x) \ge 0\ \forall\ x \in [a, b]$. (Purvanchal 2012)

Then $mg(x) \le f(x)\,g(x) \le Mg(x)\ \forall\ x \in [a, b]$.

f follows from Cor. 5 of theorem 1 of article 7.8 that

$$m\int_a^b g(x)\,dx \le \int_a^b f(x)\,g(x)\,dx \le M\int_a^b g(x)\,dx$$

or $$m\int_a^b g(x)\,dx \ge \int_a^b f(x)\,g(x)\,dx \ge M\int_a^b g(x)\,dx$$

according as $a \le b$ or $a \ge b$.

Hence there exists a number μ with $m \le \mu \le M$ such that

$$\int_a^b f(x)\,g(x)\,dx = \mu \int_a^b g(x)\,dx.$$

Now let $g(x) \le 0\ \ \forall\ x \in [a, b]$.

Then $-g(x) \ge 0\ \forall\ x \in [a, b]$.

Hence by the above result for some $\mu \in [m, M]$, we have

$$\int_a^b f(x)\,[-g(x)]\,dx = \mu \int_a^b [-g(x)]\,dx$$

or $$\int_a^b f(x)\,g(x)\,dx = \mu \int_a^b g(x)\,dx.$$

Corollary: *If, in addition to the conditions of the theorem, f is continuous as well, then there exists $\xi \in [a, b]$ such that*

$$\int_a^b f(x)\,g(x)\,dx = f(\xi)\int_a^b g(x)\,dx.$$

Proof: Since f is continuous on $[a, b]$, it takes every value between m and M. Hence by the above theorem there exists a point $\xi \in [a, b]$ such that

$$\int_a^b f(x)\, g(x)\, dx = f(\xi) \int_a^b g(x)\, dx.$$

Theorem 3: (Bonnet's Mean Value Theorem): *Let $g \in R[a, b]$ and let f be monotonic and non-negative on $[a, b]$. Then for some ξ or $\eta \in [a, b]$*

$$\int_a^b f(x)\, g(x)\, dx = f(a) \int_a^{\xi} g(x)\, dx$$

or

$$\int_a^b f(x)\, g(x)\, dx = f(b) \int_{\eta}^b g(x)\, dx$$

according as f is monotonically non-increasing or non-decreasing on $[a, b]$.

Proof: If $a = b$, the result is trivial. Let $b > a$ and let f be non-negative and monotonically non-increasing on $[a, b]$.

Let $P = \{a = x_0, x_1, \ldots\ldots, x_n = b\}$

be any partition of $[a, b]$. Let m_r, M_r be the bounds of g on $[x_{r-1}, x_r]$ and ξ_r any point on $[x_{r-1}, x_r]$. Then

$$m_r \Delta x_r \le g(\xi_r)\, \Delta x_r \le M_r \Delta x_r$$

and

$$m_r \Delta x_r \le \int_{x_{r-1}}^{x_r} g \le M_r \Delta x_r$$

[Theorem 1 of article 7.8].

On summing for each $r = 1, 2, \ldots, p \le n$, we get

$$\sum_1^p m_r\, \Delta x_r \le \sum_1^p g(\xi_r)\, \Delta x_r \le \sum_1^p M_r\, \Delta x_r \qquad \ldots(1)$$

and

$$\sum_1^p m_r\, \Delta x_r \le \int_a^{x_p} g \le \sum_1^p M_r \Delta x_r\,. \qquad \ldots(2)$$

Then (1) and (2) give

$$\left| \int_a^{x_p} g - \sum_1^p g(\xi_r)\, \Delta x_r \right| \le \sum_1^p (M_r - m_r)\, \Delta x_r$$

$$\le \sum_1^n (M_r - m_r)\, \Delta x_r = U(P, g) - L(P, g)$$

$$= \omega(P, g). \qquad \ldots(3)$$

[Note that if $b < a$, the inequalities (1) and (2) are reversed but (3) remains the same].

Now by theorem 1 of article 7.9, $\int_a^x g$ is continuous on $[a, b]$ and hence is bounded on $[a, b]$. Let m, M be its bounds on $[a, b]$. Then (3) gives

$$m - \omega(P, f) \le \sum_1^p g(\xi_r)\, \Delta x_r \le M + \omega(P, f).$$

Using Abel's lemma*, we get

$$f(a)[m - \omega(P, f)] \leq \sum_{1}^{p} f(\xi_r)\, g(\xi_r)\, \Delta x_r \leq f(a)[M - \omega(P, f)]. \quad \ldots(4)$$

Since f is monotonic, we have $f \in \boldsymbol{R}[a, b]$.

Also $g \in \boldsymbol{R}[a, b]$.

Hence $fg \in \boldsymbol{R}[a, b]$.

Now $f \in \boldsymbol{R}[a, b] \Rightarrow \omega(P, f) \to 0$ as $\| P \| \to 0$.

Hence (4) gives $f(a)\, m \leq \int_a^b fg \leq f(a)\, M$.

Thus $$\int_a^b fg = f(a)\, \mu$$

where μ lies between the bounds m, M of the continuous function $\int_a^x g$ on $[a, b]$.

Hence $\int_a^x g$ must take the value μ at some point $\xi \in [a, b]$

so that $$\mu = \int_a^{\xi} g(x)\, dx.$$

Therefore $$\int_a^b f(x)\, g(x)\, dx = f(a) \int_a^{\xi} g(x)\, dx.$$

If f is monotonically non-decreasing on $[a, b]$, then it is monotonically non-increasing on $[b, a]$ and so as above, we get

$$\int_a^b fg = f(b) \int_a^{\eta} g \text{ for some } \eta \in [a, b]$$

or $$\int_a^b fg = f(b) \int_{\eta}^{b} g$$

i.e., $$\int_a^b f(x)\, g(x)\, dx = f(b) \int_{\eta}^{b} g(x)\, dx.$$

Theorem 4: Weierstrass's (Second) Mean Value Theorem.

Let $g \in \boldsymbol{R}[a, b]$ and let f be bounded and monotonic on $[a, b]$.

Then $$\int_a^b fg = f(a) \int_a^{\xi} g + f(b) \int_{\xi}^{b} g.$$

Proof: We assume that f is monotonically non-increasing on $[a, b]$. Then $f(x) - f(b)$ is monotonically non-increasing and non-negative on $[a, b]$. Hence by the Bonnet's theorem (theorem 3 of article 7.10), there exists some $\xi \in [a, b]$ such that

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* **Abel's Lemma.** If $v_1 \geq v_2 \geq \ldots \geq v_n \geq 0$ and numbers $u_1, u_2, \ldots, u_n$ and k_1, k_2 are such that

$$k_1 \leq \sum_{r=1}^{n} u_r \leq k_2, \text{ then } k_1 v_1 \leq \sum_{r=1}^{n} u_r\, v_r \leq k_2 v_1.$$

In our case, $u_r = g(\xi_r)\, \Delta x_r$ and $v_r = f(\xi_r)$.

$$\int_a^b [f(x) - f(b)]\, g(x)\, dx = [f(a) - f(b)] \int_a^\xi g(x)\, dx$$

or $$\int_a^b f(x)\, g(x)\, dx - f(b) \int_a^b g(x)\, dx = f(a) \int_a^\xi g(x)\, dx - f(b) \int_a^\xi g(x)\, dx$$

or $$\int_a^b f(x)\, g(x)\, dx = f(a) \int_a^\xi g(x)\, dx + f(b) \left[\int_a^b g(x)\, dx - \int_a^\xi g(x)\, dx\right]$$

$$= f(a) \int_a^\xi g(x)\, dx + f(b) \int_\xi^b g(x)\, dx.$$

Now let f be monotonically non-decreasing on $[a, b]$. Then $-f$ is monotonically non-increasing on $[a, b]$. Hence by the above result, we have

$$\int_a^b [-f(x)]\, g(x)\, dx = [-f(a)] \int_a^\xi g(x)\, dx + [-f(b)] \int_\xi^b g(x)\, dx$$

or $$\int_a^b f(x)\, g(x)\, dx = f(a) \int_a^\xi g(x)\, dx + f(b) \int_\xi^b g(x)\, dx$$ as before.

Remark: Note that the condition that f is monotone on $[a, b]$ cannot be dropped in the above theorem. Consider, for example, the functions $f(x) = \cos x$ and $g(x) = x^2$ and the interval $[-\pi/2, \pi/2]$. In this case

$$f\left(-\frac{\pi}{2}\right) \int_{-\pi/2}^\xi g(x)\, dx + f\left(\frac{\pi}{2}\right) \int_\xi^{\pi/2} g(x)\, dx$$

$$= \cos\left(-\frac{\pi}{2}\right) \int_{-\pi/2}^\xi x^2\, dx + \cos\frac{\pi}{2} \int_\xi^{\pi 2} x^2\, dx = 0.$$

But $$\int_{-\pi/2}^{\pi/2} x^2 \cos x\, dx > 0.$$

Thus the theorem does not hold.

Illustrative Examples

Example 16: *Compute* $\int_1^2 x^3\, dx$.

Solution: Let $f(x) = x^3$, $1 \le x \le 2$. Then f is continuous on $[1, 2]$. Moreover, if $\phi(x) = x^4/4$ $(1 \le x \le 2)$, then $\phi'(x) = x^3 = f(x)$, $(1 \le x \le 2)$.

Hence by the fundamental theorem of integral calculus, we have

$$\int_1^2 x^3\, dx = \phi(2) - \phi(1) = \frac{2^4}{4} - \frac{1^4}{4} = \frac{15}{4}.$$

Example 17: *(i) Taking* $f(x) = x$, $g(x) = e^{x}$, *verify the second mean value theorem in* $[-1, 1]$. [*Theorem 2 of article 7.10*].

(ii) Also verify Bonnet's mean value theorem in $[-1, 1]$ *for the functions*

$$f(x) = e^{x} \text{ and } g(x) = x.$$

Solution: (i) Since f and g are continuous on $[-1, 1]$, we have

$$f, g \in \boldsymbol{R}[-1, 1].$$

Also $g(x) > 0$ for all $x \in [-1, 1]$. Hence the conditions of theorem 2 of article 7.10 are satisfied. Now

$$\int_{-1}^{1} f(x)\, g(x)\, dx = \int_{-1}^{1} xe^{x}\, dx = [xe^{x} - e^{x}]_{-1}^{1} = \frac{2}{e}. \qquad \ldots(1)$$

and $$\int_{-1}^{1} g(x)\, dx = \int_{-1}^{1} e^{x}\, dx = [e^{x}]_{-1}^{1} = e - e^{-1} = \frac{e^2 - 1}{e}.$$

Since f is continuous on $[-1, 1]$, it takes every value between $f(-1) = -1$ and $f(1) = 1$. Let $\mu = 2/(e^2 - 1)$. Since $e > 2$, we have $e^2 > 4 \Rightarrow e^2 - 1 > 3$ so that $0 < \mu < 1$.

It follows that there is a point ξ in $[-1, 1]$ such that

$$f(\xi) = 2/(e^2 - 1).$$

Accordingly, we have

$$f(\xi) \int_{-1}^{1} g(x)\, dx = \frac{2}{e^2 - 1} \cdot \frac{e^2 - 1}{e} = \frac{2}{e}. \qquad \ldots(2)$$

From (1) and (2), we have

$$\int_{-1}^{1} f(x)\, g(x)\, dx = f(\xi) \int_{-1}^{1} g(x)\, dx.$$

Thus the second mean value theorem is verified.

(ii) Since $g(x) = x$ is continuous on $[-1, 1]$, we have

$$g \in \boldsymbol{R}[-1, 1].$$

Also $f(x) = e^{x}$ is monotonically non-decreasing and positive on $[-1, 1]$. Hence all the conditions of the Bonnet's mean value theorem are satisfied. As in (i), we have

$$\int_{-1}^{1} f(x)\, g(x)\, dx = \int_{-1}^{1} e^{x}\, x\, dx = \frac{2}{e}.$$

Now $$\int_{\eta}^{1} g(x)\, dx = \int_{\eta}^{1} x\, dx = \frac{1}{2}[1 - \eta^2].$$

$$\therefore \quad f(1) \int_{\eta}^{1} g(x)\, dx = \frac{e}{2}(1 - \eta^2).$$

We choose η such that

$$\frac{2}{e} = \frac{e}{2}(1 - \eta^2) \text{ i.e., } \eta^2 = \frac{e^2 - 4}{e^2}.$$

Also it is easy to see that $0 < \eta < 1$, where $\eta = \frac{\sqrt{(e^2 - 4)}}{e}$.

For this value of η, we then have

$$\int_{-1}^{1} f(x)\, g(x)\, dx = f(1) \int_{\eta}^{1} g(x)\, dx.$$

Hence Bonnet's mean value theorem is verified.

Example 18: *Show that the Bonnet's mean value theorem does not hold on* $[-1, 1]$ *for* $f(x) = g(x) = x^2$.

Solution: The function $f(x) = x^2$ is not monotonic on $[-1, 1]$ since for the interval $[-1, 0]$ it is non-increasing and for $[0, 1]$ it is non-decreasing. Thus the conditions of the Bonnet's mean value theorem are not satisfied and hence the theorem does not hold in $[-1, 1]$.

Example 19: *Prove Bonnet's form of the second mean value theorem that if* f' *is continuous and of constant sign, and* $f(b)$ *has the same sign as* $f(a) - f(b)$, *then*

$$\int_a^b f(x)\, g(x)\, dx = f(a) \int_a^{\xi} g(x)\, dx$$

where ξ *lies between* a *and* b. *Show that if* $q > p > 0$, *then*

$$\left| \int_p^q \frac{\sin x}{x}\, dx \right| < \frac{2}{p}.$$

Solution: Let $\int_a^x g(t)\, dt = F(x)$

so that $F'(x) = g(x)$.

Then $\int_a^b f(x)\, g(x)\, dx = \int_a^b f(x)\, F'(x)\, dx$

$= [f(x)\, F(x)]_a^b - \int_a^b f'(x)\, F(x)\, dx$ [Integrating by parts]

$= f(b)\, F(b) - \int_a^b f'(x)\, F(x)\, dx$ [$\because F(a) = 0$]

$= f(b)\, F(b) - F(x_0) \int_a^b f'(x)\, dx$ [$a \le x_0 \le b$]

[by Cor. of theorem article 2 7.10]

$= f(b)\, F(b) - F(x_0)\, [f(b) - f(a)]$ ...(1)

$= f(a).\mu$ where μ lies between $F(x_0)$ and $F(b)$.

Since F is continuous, there exists a point ξ between x_0 and b such that

$$\mu = F(\xi) = \int_a^{\xi} f(x)\, dx.$$

It follows that $\int_a^b f(x)\, g(x)\, dx = f(a) \int_a^{\xi} g(x)\, dx$.

Writing $f(x) = 1/x$, $g(x) = \sin x$, we find

$$\int_p^q \frac{\sin x}{x}\, dx = \frac{1}{p}\int_p^{\xi} \sin x\, dx, \ (p < \xi < q)$$

$$= \frac{1}{p}[-\cos x]_p^{\xi} = \frac{1}{p}[\cos p - \cos \xi].$$

Hence $$\left|\int_p^q \frac{\sin x}{x}\, dx\right| = \left|\frac{1}{p}(\cos p - \cos \xi)\right| < \frac{2}{p}.$$

Remark: From (1), we get

$$\int_a^b f(x)\, g(x)\, dx = f(a)\, F(x_0) + f(b)\, [F(b) - F(x_0)]$$

$$= f(a) \int_a^{x_0} g(x)\, dx + f(b)\left[\int_a^b g(x)\, dx - \int_a^{x_0} g(x)\, dx\right]$$

$$= f(a) \int_a^{x_0} g(x)\, dx + f(b) \int_{x_0}^b g(x)\, dx$$

which is the theorem 4 of article 7.10.

Comprehensive Exercise 1

1. (i) Let $f(x) = x\ (0 \le x \le 1)$. Let P be the partition $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$ of $[0, 1]$. Compute $U(P, f)$ and $L(P, f)$.
   (ii) Let $f(x) = x\ (0 \le x \le 3)$. Let P be the partition $\{0, 1, 2, 3\}$ of $[0, 3]$. Compute $U(P, f)$ and $L(P, f)$.
2. Show by definition that $\int_0^1 x^4\, dx = \frac{1}{5}$.
3. Let $f(x) = x^{-1/2}$ on $[1, 4]$. Consider the partition obtained by dividing $[1, 4]$ into n equal parts and hence show that $\int_1^4 x^{-1/2}\, dx = 2$.
4. If $f(x) = \cos x\ \forall\, x \in [0, \pi/2]$, show that f is integrable on $[0, \pi/2]$ and $\int_0^{\pi/2} \cos x\, dx = 1$.
5. Show that $f(x) = 3x + 1$ is integrable on $[1, 2]$ and $\int_1^2 (3x + 1)\, dx = \frac{11}{2}$.
6. Give an example to prove that a bounded function need not be R-integrable.
7. Give an example of a discontinuous function which is R-integrable on $[0, 1]$.

8. Let f be defined on $[0, 1]$ by $f(x) = \frac{n}{n+1}$, when

$$\frac{1}{n+1} < x \leq \frac{1}{n}, n = 1, 2, 3, \ldots$$

and $f(x) = 1, x = 0.$

Then, show that f is Riemann integrable on $[0, 1]$ and $\int_0^1 f(x)\,dx = \frac{\pi^2}{6} - 1$.

9. A function $f(x)$ is defined on $[0, 1]$ as follows :

$$f(x) = \frac{n}{n+2}, \text{ when } \frac{1}{n+1} \leq x \leq \frac{1}{n} \; (n = 1, 2, 3, \ldots)$$

and $f(x) = 1$, when $x = 0$. Show that $f(x)$ is R-integrable on $[0, 1]$ and $\int_0^1 f(x)\,dx = \frac{1}{2}$.

10. Let a function $f(x)$ be defined on $[0, 1]$ as follows :

$$f(x) = \frac{1}{a^{r-1}}, \text{when } \frac{1}{a^r} < x < \frac{1}{a^{r-1}},$$

for $r = 1, 2, 3, \ldots$ where a is an integer greater than 1. Show that $\int_0^1 f(x)\,dx$ exists and is equal to $\frac{a}{a+1}$.

11. Calculate the values of upper and lower integrals for the function f defined on $[0, 2]$ as follows :

$f(x) = x^2$ when x is rational

and $f(x) = x^3$ when x is irrational.

12. Let $f(x)$ be a function bounded on $[a, b]$ and let P_1 and P_2 be two partitions of $[a, b]$ such that $P_1 \subset P_2$. Then prove that

$$U(P_1, f) - L(P_1, f) \geq U(P_2, f) - L(P_2, f).$$

Answers 1

1. (i) $U(P, f) = \frac{2}{3}, L(P, f) = \frac{1}{3}$

(ii) $U(P, f) = 6, \; L(P, f) = 3$

11. Upper integral $= \frac{49}{12}$, lower integral $= \frac{31}{12}$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. Let f be a bounded function defined on $[a, b]$ and let P be a partition of $[a, b]$. If P^* is a refinement of P, then
   (a) $L(P^*, f) \le L(P, f)$ (b) $U(P^*, f) \le U(P, f)$
   (c) $U(P^*, f) \ge U(P, f)$ (d) $L(P^*, f) = U(P, f)$.

2. Let f, g be bounded functions defined on $[a, b]$ and let P be any partition of $[a, b]$. Then
   (a) $U(P, f + g) \le U(P, f) + U(P, g)$
   (b) $U(P, f + g) \ge U(P, f) + U(P, g)$
   (c) $U(P, f + g) = U(P, f) + U(P, g)$
   (d) $L(P, f + g) \le L(P, f) + L(P, g)$.

3. Let f be a bounded function defined on the bounded interval $[a, b]$. Then f is Riemann integrable on $[a, b]$ if and only if
   (a) $\underline{\int_a^b} f \le \overline{\int_a^b} f$ (b) $\underline{\int_a^b} f \ge \overline{\int_a^b} f$
   (c) $\underline{\int_a^b} f = \overline{\int_a^b} f$ (d) $\overline{\int_a^b} f + \underline{\int_a^b} f = 0$.

4. If f is Riemann integrable on $[a, b]$, then
   (a) $\left| \int_a^b f(x)\,dx \right| = \int_a^b |f(x)|\,dx$
   (b) $\left| \int_a^b f(x)\,dx \right| \ge \int_a^b |f(x)|\,dx$
   (c) $\left| \int_a^b f(x)\,dx \right| = -\int_a^b |f(x)|\,dx$
   (d) $\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx$.

Fill In The Blank(s)

Fill in the blanks "....." so that the following statements are complete and correct.

1. Let $I = [a, b]$ be a closed and bounded interval. Then by a partition of I we mean a set of real numbers $P = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ having the property that ..... .

2. Let P and P^* be two partitions of a closed and bounded interval $[a, b]$. Then P^* is called a refinement of P if ..... .

3. Let P_1 and P_2 be two partitions of a closed and bounded interval $[a, b]$. If $P^* = P_1 \cup P_2$, then P^* is called the ..... of P_1 and P_2.
4. Let f be a bounded function defined on a bounded interval $[a, b]$. If corresponding to any partition P of $[a, b]$, $L(P, f)$ is the lower Riemann sum of f on $[a, b]$ and $U(P, f)$ is the upper Riemann sum of f on $[a, b]$, then $L(P, f)$ ..... $U(P, f)$.
5. Let f be a bounded function defined on $[a, b]$ and let P be a partition of $[a, b]$. If P^* is a refinement of P, then $L(P^*, f)$ ..... $L(P, f)$.
6. If P_1 and P_2 be any two partitions of $[a, b]$, then $U(P_1, f)$ ..... $L(P_2, f)$.
7. Let f be a real valued bounded function defined on $[a, b]$. The lower Riemann integral of f over $[a, b]$ is the ..... of $L(P, f)$ over all partitions $P \in \boldsymbol{P}[a, b]$.
8. Let f be a real valued bounded function defined on $[a, b]$. The upper Riemann integral of f over $[a, b]$ is the ..... of $U(P, f)$ over all partitions $P \in \boldsymbol{P}[a, b]$.
9. Let f be a real valued bounded function defined on $[a, b]$. Then the lower Riemann integral of f over $[a, b]$ cannot ..... the upper Riemann integral of f over $[a, b]$.
10. Let f be a bounded function defined on the bounded interval $[a, b]$. Then f is called Riemann integrable on $[a, b]$ if $\int_a^b f =$ .....
11. A necessary and sufficient condition for Riemann integrability of a bounded function $f : [a, b] \to \mathbb{R}$ over $[a, b]$ is that for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that for P and all its refinements
$$0 \le U(P, f) - L(P, f) < \text{.....}$$
12. Let f be Riemann integrable on $[a, b]$ and let ϕ be a differentiable function on $[a, b]$ such that $\phi'(x) = f(x)$ for all $x \in [a, b]$. Then $\int_a^b f(x)\,dx =$ .....

True or False

Write 'T' for true and 'F' for false statement.

1. Every continuous function defined on $[a, b]$ is Riemann integrable on $[a, b]$.
2. Every bounded function f defined on $[a, b]$ is Riemann integrable on $[a, b]$.
3. If a function f is discontinuous on $[a, b]$, then f cannot be Riemann integrable on $[a, b]$.
4. If a function f is monotonic on $[a, b]$, then f is Riemann integrable on $[a, b]$.
5. If f is Riemann integrable on $[a, b]$, then $|f|$ may or may not be Riemann integrable on $[a, b]$.
6. If a function f is Riemann integrable on $[a, b]$, then the function F defined on $[a, b]$ by
$$F(x) = \int_a^x f(t)\,dt$$
is uniformly continuous on $[a, b]$.

7. Let f be continuous on $[a, b]$ and let $F(x) = \int_a^x f(t)\,dt \ \forall\, x \in [a, b]$.

   Then $F'(x) = f(x) \ \forall\, x \in [a, b]$.

8. Let f be a continuous function on $[a, b]$ and let ϕ be a differentiable function on $[a, b]$ such that $\phi'(x) = f(x)$ for all $x \in [a, b]$. Then $\int_a^b f(x)\,dx = \phi(a) - \phi(b)$.

9. Let f be a bounded function defined on $[a, b]$, where $b \geq a$, and let m be the infimum of $f(x)$ in $[a, b]$. Then for any partition P of $[a, b]$, we have

   $m(b - a) \geq L(P, f)$.

10. If $f : [a, b] \to \mathbf{R}$ is a bounded function, then $L(P, -f) = -U(P, f)$.

11. Let f be a bounded function defined on $[a, b]$ and let P be a partition of $[a, b]$. If P^* is a refinement of P, then $U(P^*, f) \geq U(P, f)$.

12. Let f be a bounded function defined on $[a, b]$. Then $\underline{\int}_a^b f = \overline{\int}_a^b f$.

13. Let f be a continuous function defined on $[a, b]$. Then $\underline{\int}_a^b f = \overline{\int}_a^b f$.

14. Let f be a bounded function defined on $[a, b]$. If the set of points of discontinuity of f on $[a, b]$ is finite, then $\underline{\int}_a^b f = \overline{\int}_a^b f$.

Answers

Multiple Choice Questions

1. (b) 2. (a) 3. (c) 4. (d)

Fill in the Blank(s)

1. $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$
2. $P^* \supset P$
2. common refinement
4. $\leq$
5. $\geq$
6. $\geq$
7. supremum
8. infimum
9. exceed
10. $\overline{\int}_a^b f$
11. ε
12. $\phi(b) - \phi(a)$

True or False

1. T 2. F 3. F 4. T 5. F
6. T 7. T 8. F 9. F 10. T
11. F 12. F 13. T 14. T

□

Chapter

Convergence of Improper Integrals

8.1 Some Definitions

1. **Infinite Interval:** The interval whose length (range) is infinite is said to be an *infinite interval*. Thus the intervals (a, ∞), $(-\infty, b)$ and $(-\infty, \infty)$ are infinite intervals.

2. **Bounded Functions:** A function $f(x)$ is said to be *bounded* over the interval I if there exist two real numbers a and b $(b > a)$ such that

$$a \leq f(x) \leq b \text{ for all } x \in I.$$

A function $f(x)$ is said to be unbounded at a point, if it becomes infinite at that point. Thus the function

$$f(x) = x / \{(x-1)(x-2)\}$$

is unbounded at each of the points $x = 1$ and $x = 2$.

3. **Monotonic functions:** A real valued function f defined on an interval I is said to be **monotonically** increasing if

$$x > y \Rightarrow f(x) > f(y) \quad \forall \; x, y \in I$$

and **monotonically decreasing** if

$$x > y \Rightarrow f(x) < f(y) \ \forall\, x, y \in I.$$

A function f defined on an interval I is said to be a monotonic function if it is either monotonically decreasing or monotonically increasing on I.

For example the function f defined by $f(x) = \sin x$ is monotonically increasing in the interval $0 \le x \le \frac{1}{2}\pi$ and monotonically decreasing in the interval $\frac{1}{2}\pi \le x \le \pi$.

4. Proper Integral: The definite integral $\int_a^b f(x)\,dx$ is said to be a *proper integral* if the range of integration is finite and the integrand $f(x)$ is bounded. The integral $\int_0^{\pi/2} \sin x\,dx$ is a proper integral. Also $\int_0^1 \frac{\sin x}{x}\,dx$ is an example of a proper integral because $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

5. Improper Integrals: The definite integral $\int_a^b f(x)\,dx$ is said to be an *improper integral* if (*i*) the interval (a, b) is not finite (*i.e.*, is infinite) and the function $f(x)$ is bounded over this interval; or (*ii*) the interval (a, b) is finite and $f(x)$ is not bounded over this interval; or (*iii*) neither the interval (a, b) is finite nor $f(x)$ is bounded over it.

6. Improper integrals of the first kind or infinite integrals: A definite integral $\int_a^b f(x)\,dx$ in which the range of integration is infinite (*i.e.*, either $b = \infty$ or $a = -\infty$ or both) and the integrand $f(x)$ is bounded, is called an improper integral of the first kind or an infinite integral. Thus $\int_0^\infty \frac{dx}{1+x^2}$ is an improper integral of the first kind since the upper limit of integration is infinite and the integrand $1/(1+x^2)$ is bounded. Similarly $\int_{-\infty}^0 e^x\,dx$ is an example of an improper integral of the first kind because here the lower limit of integration is infinite. Also $\int_{-\infty}^\infty \frac{dx}{1+x^2}$ is an improper integral of the first kind.

In case the **interval (a, b) is infinite and the integrand $f(x)$ is bounded**, we define

(*i*) $\int_a^\infty f(x)\,dx = \lim_{x \to \infty} \int_a^x f(x)\,dx,$

provided that the limit exists finitely *i.e.*, the limit is equal to a definite real number.

(*ii*) $\int_{-\infty}^b f(x)\,dx = \lim_{x \to \infty} \int_{-x}^b f(x)\,dx,$

provided that the limit exists finitely.

(iii) $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{x_1 \to \infty} \int_{-x_1}^{c} f(x)\,dx + \lim_{x_2 \to \infty} \int_{c}^{x_2} f(x)\,dx$

provided that both these limits exist finitely.

7. **Improper integrals of the second kind:** A definite integral $\int_a^b f(x)\,dx$ in which the range of integration is finite but the integrand $f(x)$ is unbounded at one or more points of the interval $a \le x \le b$, is called an improper integral of the second kind.

Thus $\int_0^4 \frac{dx}{(x-2)(x-3)}$

and $\int_0^1 \frac{1}{x^2}\,dx$ are improper integrals of the second kind.

In the case of the definite integral

$$\int_a^b f(x)\,dx,$$

if the range of integration (a, b) is finite and the integrand **$f(x)$ is unbounded at one or more points of the given interval**, we define the value of the integral as follows :

(i) If $f(x)$ is unbounded at $x = b$ only *i.e.*, if $f(x) \to \infty$ as $x \to b$ only, then we define

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} f(x)\,dx,$$

provided that the limit exists finitely. Here ε is a small positive number.

(ii) If $f(x) \to \infty$ as $x \to a$ only, then we define

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b} f(x)\,dx,$$

provided that the limit exists finitely.

(iii) If $f(x) \to \infty$ as $x \to c$ only, where $a < c < b$, then we define

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)\,dx + \lim_{\varepsilon' \to 0} \int_{c+\varepsilon'}^{b} f(x)\,dx,$$

provided that both these limits exist finitely.

(iv) If $f(x)$ is unbounded at both the points a and b of the interval (a, b) and is bounded at each other point of this interval, we write

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

where $a < c < b$ and the value of the integral exists only if each of the integrals on the right hand side exists.

8.2 Convergence of Improper Integrals

When the limit of an improper integral as defined above, is a definite finite number, we say that the given integral is **convergent** and the value of the integral is equal to the value of that limit. When the limit is ∞ or $-\infty$, the integral is said to be **divergent** *i.e.*, the value of the integral does not exist.

In case the limit is neither a definite number nor ∞ or $-\infty$, the integral is said to be **oscillatory** and in this case also the value of the integral does not exist *i.e.*, the integral is not convergent. We can define the convergence of the infinite integral $\int_a^\infty f(x)\,dx$ as follows :

Definition: *The integral* $\int_a^\infty f(x)\,dx$ *is said to converge to the value I, if for any arbitrarily chosen positive number* ε*, however small but not zero, there exists a corresponding positive number N such that*

$$\left|\int_a^b f(x)\,dx - I\right| < \varepsilon \text{ for all values of } b \geq N.$$

Similarly we can define the convergence of an integral, when the lower limit is infinite, or when the integrand becomes infinite at the upper or lower limit.

Illustrative Examples

Example 1: *Discuss the convergence of the following integrals by evaluating them*

(*i*) $\int_1^\infty \frac{dx}{\sqrt{x}}$, (*ii*) $\int_1^\infty \frac{dx}{x^{3/2}}$.

Solution: (i) We have

$$\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{x\to\infty} \int_1^x \frac{dx}{\sqrt{x}}, \text{ (By def.)}$$

$$= \lim_{x\to\infty} \int_1^x x^{-1/2}\,dx = \lim_{x\to\infty} \left[\frac{x^{1/2}}{1/2}\right]_1^x$$

$$= \lim_{x\to\infty} [2\sqrt{x} - 2] = \infty.$$

Thus the limit does not exist finitely and therefore the given integral is divergent (*i.e.*, the integral does not exist).

(ii) We have

$$\int_1^\infty \frac{dx}{x^{3/2}} = \lim_{x\to\infty} \int_1^x \frac{dx}{x^{3/2}}, \quad \text{(By def.)}$$

$$= \lim_{x\to\infty} \int_1^x x^{-3/2}\,dx = \lim_{x\to\infty} \left[\frac{x^{-1/2}}{-1/2}\right]_1^x = \lim_{x\to\infty} \left[-\frac{2}{\sqrt{x}}\right]_1^x$$

$$= \lim_{x \to \infty} \left[-\frac{2}{\sqrt{x}} + 2 \right] = 2.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent and its value is 2.

Example 2: *Test the convergence of* $\int_0^\infty e^{-mx}\, dx, \quad (m > 0)$.

Solution: We have $\int_0^\infty e^{-mx}\, dx = \lim_{x \to \infty} \int_0^\infty e^{-mx}\, dx$, (by def.)

$$= \lim_{x \to \infty} \left[\frac{e^{-mx}}{-m} \right]_0^x = \lim_{x \to \infty} \left\{ -\frac{1}{m} (e^{-mx} - 1) \right\}$$

$$= -\frac{1}{m} [0 - 1] = \frac{1}{m}$$

Thus the limit exists and is unique and finite, therefore the given integral is convergent.

Example 3: *Test the convergence of* $\int_0^\infty \frac{4a\, dx}{x^2 + 4a^2}$.

Solution: We have $\int_0^\infty \frac{4a\, dx}{x^2 + 4a^2} = \lim_{x \to \infty} \int_0^x \frac{4a\, dx}{x^2 + (2a)^2}$, (By def.)

$$= \lim_{x \to \infty} \left[4a \cdot \frac{1}{2a} \tan^{-1} \frac{x}{2a} \right]_0^x = 2 \lim_{x \to \infty} \left[\tan^{-1} \frac{x}{2a} \right]_0^x$$

$$= 2 \cdot \lim_{x \to \infty} \left[\tan^{-1} \frac{x}{2a} - 0 \right] = 2 \cdot [\tan^{-1} \infty] = 2 \cdot \frac{\pi}{2} = \pi.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent.

Example 4: *Test the convergence of* (*i*) $\int_{-\infty}^0 e^x\, dx$; (*ii*) $\int_{-\infty}^0 e^{-x}\, dx$.

Solution: (i) We have $\int_{-\infty}^0 e^x\, dx = \lim_{x \to \infty} \int_{-x}^0 e^x\, dx$, (By def.)

$$= \lim_{x \to \infty} |e^x|_{-x}^0 = \lim_{x \to \infty} |1 - e^{-x}| = |1 - 0| = 1.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent.

(ii) We have $\int_{-\infty}^0 e^{-x}\, dx = \lim_{x \to \infty} \int_{-x}^0 e^{-x}\, dx$, (By def.)

$$= \lim_{x \to \infty} \left[\frac{e^{-x}}{-1} \right]_{-x}^0 = -\lim_{x \to \infty} |e^0 - e^x| = \infty.$$

Thus the limit does not exist finitely and therefore the given integral is divergent (*i.e.*, the integral does not exist).

Example 5: *Test the convergence of* $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$. **(Kanpur 2008)**

Solution: We have

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$= \lim_{x \to \infty} \int_{-x}^{0} \frac{dx}{1+x^2} + \lim_{x \to \infty} \int_{0}^{x} \frac{dx}{1+x^2}$$

$$= \lim_{x \to \infty} [\tan^{-1} x]_{-x}^{0} + \lim_{x \to \infty} [\tan^{-1} x]_{0}^{x}$$

$$= \lim_{x \to \infty} [0 - \tan^{-1}(-x)] + \lim_{x \to \infty} [\tan^{-1} x - 0]$$

$$= -(-\pi/2) + \pi/2 = \pi.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent.

Example 6: *Evaluate* $\int_{0}^{1} \frac{dx}{\sqrt{x}}$.

Solution: In the given integral, the integrand $1/\sqrt{x}$ becomes infinite at the lower limit $x = 0$. Therefore we have

$$\int_{0}^{1} \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0} \int_{0+\varepsilon}^{1} \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0} [2\sqrt{x}]_{\varepsilon}^{1}$$

$$= \lim_{\varepsilon \to 0} [2 - 2\sqrt{\varepsilon}] = 2.$$

Hence the given integral is convergent and its value is 2.

Example 7: *Evaluate* $\int_{0}^{1} \frac{dx}{\sqrt{(1-x)}}$.

Solution: Here the integrand *i.e.*, $1/\sqrt{(1-x)}$ becomes unbounded *i.e.*, infinite at the upper limit (*i.e.*, $x = 1$).

$$\therefore \quad \int_{0}^{1} \frac{dx}{\sqrt{(1-x)}} = \lim_{\varepsilon \to 0} \int_{0}^{1-\varepsilon} \frac{dx}{\sqrt{(1-x)}}$$

$$= \lim_{\varepsilon \to 0} [-2\sqrt{(1-x)}]_{0}^{1-\varepsilon} = \lim_{\varepsilon \to 0} [-2\sqrt{\varepsilon} + 2] = 2,$$

which is a definite real number. Hence the given integral is convergent and its value is 2.

Example 8: *Evaluate* $\int_{-1}^{1} \frac{dx}{x^2}$.

Solution: Here the integrand becomes infinite at $x = 0$ and $-1 < 0 < 1$.

$$\therefore \quad \int_{-1}^{1} \frac{dx}{x^2} = \lim_{\varepsilon \to 0} \int_{-1}^{-\varepsilon} \frac{dx}{x^2} + \lim_{\varepsilon' \to 0} \int_{\varepsilon'}^{1} \frac{dx}{x^2}$$

$$= \lim_{\varepsilon \to 0} \left[-\frac{1}{x} \right]_{-1}^{-\varepsilon} + \lim_{\varepsilon' \to 0} \left[-\frac{1}{x} \right]_{\varepsilon'}^{1}$$

$$= \lim_{\varepsilon \to 0} \left[\frac{1}{\varepsilon} - 1 \right] + \lim_{\varepsilon' \to 0} \left[-1 + \frac{1}{\varepsilon'} \right].$$

Since both the limits do not exist finitely, therefore the integral does not exist and is divergent.

Comprehensive Exercise 1

Evaluate the following integrals and discuss their convergence :

1. $\int_1^\infty \frac{dx}{x}$.
2. $\int_3^\infty \frac{dx}{(x-2)^2}$.
3. $\int_0^\infty e^{2x}\, dx$.
4. $\int_0^\infty \frac{dx}{(1+x)^{2/3}}$
5. $\int_{-\infty}^0 \sinh x\, dx$;
6. $\int_{-\infty}^0 \cosh x\, dx$.
7. $\int_0^\infty \cos x\, dx$
8. $\int_{-\infty}^\infty e^{-x}\, dx$.
9. $\int_{-\infty}^\infty \frac{dx}{x^2 + 2x + 2}$.
10. $\int_0^1 \frac{dx}{x^3}$.
11. $\int_0^1 \frac{dx}{1-x}$.
12. $\int_{-1}^1 \frac{dx}{x^{2/3}}$.

Answers 1

1. ∞, divergent
2. 1, convergent
3. ∞, divergent
4. ∞, divergent
5. $-\infty$, divergent
6. ∞, divergent
7. Oscillates and so not convergent
8. ∞, divergent
9. π, convergent
10. ∞, divergent
11. ∞, divergent
12. 6, convergent

8.3 Tests for Convergence of Improper Integrals of the First Kind

i.e., to test the convergence of improper integrals in which the range of integration is infinite and the integrand is bounded.

If an integral of the form $\int_a^\infty f(x)\, dx$ or $\int_{-\infty}^b f(x)\, dx$ cannot be actually integrated, its convergence is determined with the help of the following tests :

8.4 Comparison Test

(Meerut 2012)

Let $f(x)$ and $g(x)$ be two functions which are bounded and integrable in the interval (a, ∞). Also let $g(x)$ be positive and $|f(x)| \leq g(x)$ when $x \geq a$. Then, if $\int_a^\infty g(x)\,dx$ is **convergent**, $\int_a^\infty f(x)\,dx$ *is also* **convergent**.

Similarly if $|f(x)| \geq g(x)$ for all values of x greater than some number $x_0 > a$ and $\int_a^\infty g(x)\,dx$ is **divergent**, *then $\int_a^\infty f(x)\,dx$ is also* **divergent**.

Alternative form of the above comparison test:

If $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ is a definite number, other than zero, the integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either both converge or both diverge.

Note: While applying comparison test, we generally take $g(x) = \frac{1}{x^n}$ *i.e,* $\int_a^\infty \frac{dx}{x^n}$ is generally taken as the comparison integral.

Theorem: *The comparison integral $\int_a^\infty \frac{dx}{x^n}$, where $a > 0$, is convergent when $n > 1$ and divergent when $n \leq 1$.*

Proof: By the definition of an improper integral, we have

$$\int_a^\infty \frac{dx}{x^n} = \lim_{x \to \infty} \int_a^x \frac{dx}{x^n} = \lim_{x \to \infty} \int_a^x x^{-n}\,dx$$

$$= \lim_{x \to \infty} \left[\frac{x^{1-n}}{1-n}\right]_a^x, \text{ if } n \neq 1$$

$$= \lim_{x \to \infty} \left[\frac{x^{1-n}}{1-n} - \frac{a^{1-n}}{1-n}\right]. \qquad \ldots(1)$$

If $n > 1$, then $1 - n$ is negative and so $n - 1$ is positive.

Therefore in this case

$$\lim_{x \to \infty} x^{1-n} = \lim_{x \to \infty} \frac{1}{x^{n-1}} = \frac{1}{\infty} = 0.$$

Hence from (1), we have

$$\int_a^\infty \frac{dx}{x^n} = \frac{a^{1-n}}{n-1}, \text{ if } n > 1.$$

Hence the given integral is convergent when $n > 1$.

If $n < 1$, then $1 - n$ is positive and so $\lim_{x \to \infty} x^{1-n} = \infty$.

$\therefore$ from (1), we have $\int_a^\infty \frac{dx}{x^n} = \infty$.

Hence the given integral is divergent when $n < 1$.

When $n = 1$, we have

$$\int_a^\infty \frac{dx}{x^n} = \int_a^\infty \frac{dx}{x} = \lim_{x \to \infty} \int_a^x \frac{dx}{x} = \lim_{x \to \infty} [\log x]_a^x$$

$$= \lim_{x \to \infty} [\log x - \log a] = \infty - \log a = \infty.$$

Hence the given integral is divergent when $n = 1$.

$\therefore$ $\int_a^\infty \frac{dx}{x^n}$ converges when $n > 1$ and diverges when $n \leq 1$.

In other words $\int_a^\infty \frac{dx}{x^n}$ converges if and only if $n > 1$.

Illustrative Examples

Example 9: *Test the convergence of the integral*

$$\int_0^\infty \frac{\cos mx}{x^2 + a^2} dx.$$

(Rohilkhand 2011)

Solution: Here $f(x) = \frac{\cos mx}{x^2 + a^2}$. Let $g(x) = \frac{1}{x^2 + a^2}$.

Obviously $g(x)$ is positive in the interval $(0, \infty)$.

We have $|f(x)| = \left|\frac{\cos mx}{x^2 + a^2}\right| = \frac{|\cos mx|}{x^2 + a^2} \leq \frac{1}{x^2 + a^2}$, since $|\cos mx| \leq 1$.

Thus $|f(x)| \leq g(x)$ when $x \geq 0$.

$\therefore$ by comparison test, $\int_0^\infty \frac{\cos mx}{x^2 + a^2} dx$ is convergent if $\int_0^\infty \frac{dx}{x^2 + a^2}$ is convergent.

But $\int_0^\infty \frac{dx}{x^2 + a^2} = \lim_{x \to \infty} \int_0^\infty \frac{dx}{x^2 + a^2} = \lim_{x \to \infty} \left[\frac{1}{a} \tan^{-1} \frac{x}{a}\right]_0^x$

$$= \lim_{x \to \infty} \left[\frac{1}{a} \tan^{-1} \frac{x}{a} - 0\right] = \frac{1}{a} \cdot \frac{\pi}{2} = \text{a definite real number.}$$

$\therefore$ $\int_0^\infty \frac{dx}{x^2 + a^2}$ is convergent.

Hence $\int_0^\infty \frac{\cos mx}{x^2 + a^2} dx$ is also convergent.

Example 10: *Test the convergence of the integral*

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx.$$

Solution: Let $a > 0$. Then we can write

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \int_0^a \frac{\sin^2 x}{x^2}\,dx + \int_a^\infty \frac{\sin^2 x}{x^2}\,dx.$$

Since $\lim_{x \to 0} \frac{\sin^2 x}{x^2} = 1$, therefore the integrand $\frac{\sin^2 x}{x^2}$ is bounded throughout the finite interval $(0, a)$.

So $\int_0^a \frac{\sin^2 x}{x^2}\,dx$ is a proper integral and we need to check the convergence of the integral $\int_a^\infty \frac{\sin^2 x}{x^2}\,dx$ only.

Here $f(x) = \frac{\sin^2 x}{x^2}$. Take $g(x) = \frac{1}{x^2}$.

Obviously $g(x)$ is positive in the interval (a, ∞).

We have $|f(x)| = \left|\frac{\sin^2 x}{x^2}\right| = \frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$, since $\sin^2 x \le 1$.

$\therefore$ by comparison test, $\int_a^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent if $\int_a^\infty \frac{dx}{x^2}$ is convergent.

But the comparison integral $\int_a^\infty \frac{dx}{x^2}$ is convergent because here $n = 2$ which is > 1.

$\therefore$ $\int_a^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent.

Hence $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent.

Example 11: *Show that the integral* $\int_a^\infty \frac{dx}{x\sqrt{(1+x^2)}}$ *converges, where* $a > 0$.

Solution: Let $f(x) = \frac{1}{x\sqrt{(1+x^2)}}$.

Then $f(x)$ is bounded in the interval (a, ∞). Take $g(x) = 1/x^2$. Then $g(x)$ is positive in the interval (a, ∞). We have

$$|f(x)| = \left|\frac{1}{x\sqrt{(1+x^2)}}\right| = \frac{1}{x^2\sqrt{\{1+(1/x^2)\}}}$$

$$< \frac{1}{x^2}, \text{ since } \frac{1}{\sqrt{\{1 + (1/x^2)\}}} < 1.$$

$\therefore$ by comparison test, $\int_a^\infty \frac{dx}{x\sqrt{(1+x^2)}}$ is convergent if $\int_a^\infty \frac{dx}{x^2}$ is convergent.

But the comparison integral $\int_a^\infty \frac{dx}{x^2}$ is convergent because here $n = 2$ which is > 1.

Hence $\int_a^\infty \frac{dx}{x\sqrt{(1+x^2)}}$ is also convergent.

Alternative Method : Here $f(x) = \frac{1}{x\sqrt{(1+x^2)}} = \frac{1}{x^2\sqrt{\{1+(1/x^2)\}}}$.

Take $g(x) = \frac{1}{x^2}$.

We have $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{\sqrt{\{1+(1/x^2)\}}} = 1$, which is finite and non-zero.

Therefore $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either both converge or both diverge. But

$\int_a^\infty g(x)\,dx = \int_a^\infty \frac{dx}{x^2}$ is convergent because here $n = 2$ which is > 1.

Hence $\int_a^\infty f(x)\,dx$ *i.e.*, $\int_a^\infty \frac{1}{x\sqrt{(1+x^2)}}\,dx$ is also convergent.

Example 12: *Test the convergence of* $\int_0^\infty e^{-x}\frac{\sin x}{x}\,dx$. **(Kanpur 2011)**

Solution: We can write

$$\int_0^\infty e^{-x}\frac{\sin x}{x}\,dx = \int_0^1 e^{-x}\frac{\sin x}{x}\,dx + \int_1^\infty e^{-x}\frac{\sin x}{x}\,dx.$$

Since $\lim_{x\to 0} e^{-x}\frac{\sin x}{x} = 1$, therefore the integrand $e^{-x}\frac{\sin x}{x}$ is bounded throughout the finite interval $(0, 1)$. So $\int_0^1 e^{-x}\frac{\sin x}{x}\,dx$ is a proper integral and therefore it is convergent. Thus we need to check the convergence of $\int_1^\infty e^{-x}\frac{\sin x}{x}\,dx$ only.

Let $f(x) = e^{-x}\frac{\sin x}{x}$. Then $f(x)$ is bounded in the interval $(1, \infty)$.

Take $g(x) = e^{-x}$. Then $g(x)$ is positive in the interval $(1, \infty)$.

We have

$$|f(x)| = \left| e^{-x}\frac{\sin x}{x} \right| = e^{-x}.|\sin x|\cdot\frac{1}{x}$$

$$\leq e^{-x}, \text{ since } |\sin x| \leq 1 \text{ and } \frac{1}{x} \leq 1.$$

Thus $|f(x)| \leq g(x)$ throughout the interval $(1, \infty)$.

$\therefore$ by comparison test $\int_1^\infty f(x)\,dx$ is convergent if $\int_1^\infty g(x)\,dx$ is convergent.

Now $\int_1^\infty g(x)\,dx = \int_1^\infty e^{-x}\,dx = \lim_{x \to \infty} \int_1^\infty e^{-x}\,dx = \lim_{x \to \infty} [-e^{-x}]_1^x$

$$= \lim_{x \to \infty} [-e^{-x} + e^{-1}] = 0 + e^{-1} = 1/e,$$

which is a definite finite number. Hence $\int_1^\infty g(x)\,dx$ is convergent.

$\therefore$ $\int_1^\infty f(x)\,dx$ is also convergent.

Hence $\int_0^\infty e^{-x} \frac{\sin x}{x}\,dx$ is convergent because the sum of two convergent integrals is also convergent.

Example 13: *Show that the integral* $\int_0^\infty e^{-x^2}\,dx$ *is convergent.*

(Rohilkhand 2010; Meerut 12)

Solution: We have

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx.$$

Obviously $\int_0^1 e^{-x^2}\,dx$ is a proper integral because here the interval of integration $(0, 1)$ is finite and the integrand e^{-x^2} is bounded throughout this interval. Therefore this integral is convergent. So we need to check the convergence of $\int_1^\infty e^{-x^2}\,dx$ only.

Let $f(x) = e^{-x^2}$. Take $g(x) = xe^{-x^2}$ so that $g(x)$ is positive throughout the interval $(1, \infty)$. We have

$$|f(x)| = e^{-x^2} \leq xe^{-x^2}, \text{ since } x \geq 1.$$

Thus $|f(x)| \leq g(x)$ throughout the interval $(1, \infty)$.

$\therefore$ by comparison test $\int_1^\infty e^{-x^2}\,dx$ is convergent if $\int_1^\infty xe^{-x^2}\,dx$ is convergent.

Now $\int_1^\infty xe^{-x^2}\,dx = \lim_{x \to \infty} \int_1^x xe^{-x^2}\,dx$

$$= \lim_{x \to \infty} \left(-\frac{1}{2} e^{-x^2}\right)_1^x$$

$$= \lim_{x \to \infty} \left(-\frac{1}{2} e^{-x^2} + \frac{1}{2} e^{-1}\right)$$

$$= \frac{1}{2} e^{-1}, \text{ which is a definite number.}$$

$\therefore \int_1^\infty xe^{-x^2} dx$ is convergent and so $\int_1^\infty e^{-x^2} dx$ is also convergent.

Hence the given integral $\int_0^\infty e^{-x^2} dx$ is also convergent as it is the sum of two convergent integrals.

8.5 The μ-Test

Let $f(x)$ be bounded and integrable in the interval (a, ∞) where $a > 0$.

If there is a number $\mu > 1$, such that $\lim_{x \to \infty} x^\mu f(x)$ exists, then $\int_a^\infty f(x)\, dx$ is convergent.

If there is a number $\mu \leq 1$, such that $\lim_{x \to \infty} x^\mu f(x)$ exists and is non-zero, then $\int_a^\infty f(x)\, dx$ is divergent and the same is true if $\lim_{x \to \infty} x^\mu f(x)$ is $+\infty$ or $-\infty$.

While applying the μ-test, the value of μ is usually taken to be equal to the highest power of x in the denominator of the integrand minus the highest power of x in the numerator of the integrand.

Illustrative Examples

Example 14: *Examine the convergence of* $\int_1^\infty \frac{dx}{x^{1/3}(1 + x^{1/2})}$.

Solution: Let $f(x) = \frac{1}{x^{1/3}(1 + x^{1/2})} = \frac{1}{x^{1/3} x^{1/2} \{1 + (1 / x^{1/2})\}}$

$$= \frac{1}{x^{5/6} \{1 + (1 / x^{1/2})\}}.$$

Obviously $f(x)$ is bounded in the interval $(1, \infty)$.

Take $\mu = \frac{5}{6} - 0 = \frac{5}{6}$. We have

$$\lim_{x \to \infty} x^\mu f(x) = \lim_{x \to \infty} x^{5/6} \cdot \frac{1}{x^{5/6} \{1 + (1 / x^{1/2})\}}$$

$$= \lim_{x \to \infty} \frac{1}{1 + (1 / x^{1/2})} = 1,$$

which is finite and non-zero. Since $\mu = \frac{5}{6}$ *i.e.*, < 1, it follows from the μ-test that the given integral is divergent.

Example 15: *Examine the convergence of* $\int_0^\infty \frac{x\,dx}{(1+x)^3}$.

(Rohilkhand 2011; Purvanchal 11)

Solution: Let $a > 0$. Then we have

$$\int_0^\infty \frac{x\,dx}{(1+x)^3} = \int_0^a \frac{x\,dx}{(1+x)^3} + \int_a^\infty \frac{x\,dx}{(1+x)^3}.$$

The first integral on the right hand side is convergent because it is a proper integral. We observe that in this integral the range of integration $(0, a)$ is finite and the integrand $x/(1+x)^3$ is bounded throughout the interval $(0, a)$. So we need to check the convergence of $\int_a^\infty \frac{x\,dx}{(1+x)^3}$ only.

Let $f(x) = \frac{x}{(1+x)^3}$. Then $f(x)$ is bounded in the interval (a, ∞).

Take $\mu = 3 - 1 = 2$. Then

$$\lim_{x\to\infty} x^\mu f(x) = \lim_{x\to\infty} x^2 \cdot \frac{x}{(1+x)^3} = \lim_{x\to\infty} \frac{1}{\{1+(1/x)\}^3} = 1,$$

which exists *i.e.*, is equal to a definite real number.

Since $\mu = 2$ *i.e.*, > 1, therefore by μ-test the integral $\int_a^\infty \frac{x\,dx}{(1+x)^3}$ is convergent.

Hence $\int_0^\infty \frac{x\,dx}{(1+x)^3}$ is also convergent because it is the sum of two convergent integrals.

Example 16: *Examine the convergence of* $\int_a^\infty \frac{dx}{x(\log x)^{n+1}}$, *where* $a > 1$.

Solution: Let $\log x = t$ so that $(1/x)\,dx = dt$.

$$\therefore \quad \int_a^\infty \frac{dx}{x(\log x)^{n+1}} = \int_{\log a}^\infty \frac{dt}{t^{n+1}}.$$

Let $f(t) = 1/t^{n+1}$.

Then $f(t)$ is bounded in the interval $(\log a, \infty)$.

Take $\mu = (n+1) - 0 = n+1$. Then

$$\lim_{t\to\infty} t^\mu f(t) = \lim_{t\to\infty} \frac{t^{n+1}}{t^{n+1}} = \lim_{t\to\infty} 1 = 1,$$

which is finite and non-zero.

Therefore by μ-test, the given integral is convergent if

$$\mu > 1 \text{ i.e., } n+1 > 1 \text{ i.e., } n > 0$$

and divergent if $\mu \le 1$ *i.e.*, $n + 1 \le 1$ *i.e.*, $n \le 0$.

Example 17: *Show that the integral* $\int_1^\infty x^{n-1} e^{-x}\, dx$ *is convergent.*

Solution: Let $f(x) = x^{n-1} e^{-x}$. Then $f(x)$ is bounded in the interval $(1, \infty)$. We have

$$\lim_{x \to \infty} x^\mu f(x) = \lim_{x \to \infty} \frac{x^\mu \cdot x^{n-1}}{e^x} = \lim_{x \to \infty} \frac{x^{\mu+n-1}}{1 + x + \frac{x^2}{2!} + \ldots}$$

$= 0$ for all values of μ and n.

Taking $\mu > 1$, we see by μ-test that the integral $\int_1^\infty x^{n-1} e^{-x}\, dx$ is convergent for all values of n.

Example 18: *Test the convergence of the integral* $\int_0^\infty \frac{x^{2m}}{1 + x^{2n}}\, dx$, *where m and n are positive integers.* **(Purvanchal 2007; Rohilkhand 12)**

Solution: Let $a > 0$. We have

$$\int_0^\infty \frac{x^{2m}}{1 + x^{2n}}\, dx = \int_0^a \frac{x^{2m}}{1 + x^{2n}}\, dx + \int_a^\infty \frac{x^{2m}}{1 + x^{2n}}\, dx.$$

The first integral on the right hand side is a proper integral and so it is convergent.

Therefore the given integral is convergent or divergent according as $\int_a^\infty \frac{x^{2m}}{1 + x^{2n}}\, dx$ is convergent or divergent.

To test the convergence of $\int_a^\infty \frac{x^{2m}}{1 + x^{2n}}\, dx$, let us take $\mu = 2n - 2m$.

We have

$$\lim_{x \to \infty} x^\mu \cdot \frac{x^{2m}}{1 + x^{2n}} = \lim_{x \to \infty} x^{2n-2m} \cdot \frac{x^{2m}}{x^{2n}\{1 + (1/x^{2n})\}}$$

$$= \lim_{x \to \infty} \frac{1}{1 + (1/x^{2n})} = 1,$$

which is finite and non-zero.

$\therefore$ by μ-test, the given integral is convergent if $\mu > 1$ *i.e.*, if $2n - 2m > 1$ which is possible if $n > m$ since m and n are positive integers. Also by μ-test, the given integral is divergent if $\mu \leq 1$ *i.e.*, if $2n - 2m \leq 1$ *i.e.*, if $n \leq m$ since n and m are positive integers.

8.6 Abel's Test for the Convergence of Integral of a Product

If $\int_a^\infty f(x)\, dx$ *converges and* $\phi(x)$ *is bounded and monotonic for* $x > a$, *then* $\int_a^\infty f(x)\, \phi(x)\, dx$ *is convergent.* **(Rohilkhand 2008, 11; Purvanchal 07, 10, 11; Kanpur 12)**

Illustrative Examples

Example 19: *Test the convergence of* $\int_a^\infty (1 - e^{-x}) \frac{\cos x}{x^2} dx$, *when* $a > 0$.

(Rohilkhand 2009)

Solution: Let $f(x) = \frac{\cos x}{x^2}$ and $\phi(x) = 1 - e^{-x}$.

We have $\left|\frac{\cos x}{x^2}\right| \leq \frac{1}{x^2}$ as $|\cos x| \leq 1$.

Since $\int_a^\infty \frac{1}{x^2} dx$ is convergent, therefore by comparison test $\int_a^\infty \frac{\cos x}{x^2} dx$ is also convergent.

Again $\phi(x) = 1 - e^{-x}$ is monotonic increasing and bounded function for $x > a$.

Hence by Abel's test $\int_a^\infty (1 - e^{-x}) \frac{\cos x}{x^2} dx$ is convergent.

Example 20: *Test the convergence of* $\int_a^\infty e^{-x} \frac{\sin x}{x^2} dx$, *where* $a > 0$.

Solution: Let $f(x) = \frac{\sin x}{x^2}$ and $\phi(x) = e^{-x}$.

Since $\left|\frac{\sin x}{x^2}\right| \leq \frac{1}{x^2}$ and $\int_a^\infty \frac{1}{x^2} dx$ is convergent, therefore by comparison test $\int_a^\infty \frac{\sin x}{x^2} dx$ is also convergent.

Again e^{-x} is monotonic decreasing and bounded function for $x > a$.

Hence by Abel's test $\int_a^\infty e^{-x} \frac{\sin x}{x^2} dx$ is convergent.

8.7 Dirichlet's test for the Convergence of Integral of a Product

If $f(x)$ *be bounded and monotonic in the interval* $a \leq x < \infty$ *and if* $\lim_{x \to \infty} f(x) = 0$, *then the integral* $\int_a^\infty f(x)\, \phi(x)\, dx$ *converges provided* $\left|\int_a^x \phi(x)\, dx\right|$ *is bounded as* x *takes all finite values.*

(Purvanchal 2012)

Illustrative Examples

Example 21: *Test the convergence of the integral*

$$\int_a^{\infty} \frac{\sin x}{\sqrt{x}}\, dx, \text{ where } a > 0.$$

Solution: Let $f(x) = \frac{1}{\sqrt{x}}$ and $\phi(x) = \sin x$.

Now $\frac{1}{\sqrt{x}}$ is bounded and monotonic decreasing for all $x \geq a$ and $\lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0$.

Also $\left| \int_a^x \phi(x)\, dx \right| = \left| \int_a^x \sin x\, dx \right| = |\cos a - \cos x| \leq 2$, for all finite values of x.

[Note that the value of $\cos x$ lies between -1 and 1].

$\therefore$ $\left| \int_a^x \phi(x)\, dx \right|$ is bounded for all finite values of x.

Hence by Dirichlet's test the integral $\int_a^{\infty} \frac{\sin x}{\sqrt{x}}\, dx$ is convergent.

Example 22: *Show that* $\int_0^{\infty} \sin x^2\, dx$ *is convergent.* **(Agra 2012)**

Solution: We have $\int_0^{\infty} \sin x^2\, dx = \int_0^1 \sin x^2\, dx + \int_1^{\infty} \sin x^2\, dx$.

But $\int_0^1 \sin x^2\, dx$ is a proper integral and hence convergent.

Now it remains to test the convergence of $\int_1^{\infty} \sin x^2\, dx$. We can write

$$\int_1^{\infty} \sin x^2\, dx = \int_1^{\infty} 2x \,.\, (\sin x^2) \,.\, \frac{1}{2x}\, dx.$$

Let $f(x) = \frac{1}{2x}$ and $\phi(x) = 2x \sin x^2$.

The function $f(x) = \frac{1}{2x}$ is bounded and monotonic decreasing for all $x \geq 1$ and

$\lim_{x \to \infty} \frac{1}{2x} = 0$.

Also $\left| \int_1^x \phi(x)\, dx \right| = \int_1^x 2x \sin x^2\, dx$

$= |\cos 1^2 - \cos x^2| \leq 2$, for all finite values of x.

$\therefore$ $\left| \int_1^x \phi(x)\, dx \right|$ is bounded for all finite values of x.

Hence by Dirichlet's test

$$\int_1^{\infty} \frac{1}{2x} \,.\, (\sin x^2)\, 2x\, dx \quad i.e., \int_1^{\infty} \sin x^2\, dx \text{ is convergent.}$$

Since the sum of two convergent integrals is convergent, therefore the integral $\int_0^\infty \sin x^2\, dx$ is convergent.

Example 23: *Show that the integral* $\int_0^\infty \frac{\sin x}{x}\, dx$ *is convergent.* **(Purvanchal 2010, 12)**

Solution: We have $\int_0^\infty \frac{\sin x}{x}\, dx = \int_0^a \frac{\sin x}{x}\, dx + \int_a^\infty \frac{\sin x}{x}\, dx$, where $a > 0$.

Since $\lim_{x \to 0} \frac{\sin x}{x} = 1$, the integral $\int_0^a \frac{\sin x}{x}\, dx$ is a proper integral and hence convergent.

Now to test the convergence of $\int_a^\infty \frac{\sin x}{x}\, dx$.

Let $f(x) = 1/x$ and $\phi(x) = \sin x$.

The function $f(x) = 1/x$ is bounded and monotonic decreasing for all $x \geq a$ and $\lim_{x \to \infty} \frac{1}{x} = 0$.

Also $\left| \int_a^x \phi(x)\, dx \right| = \left| \int_a^x \sin x\, dx \right| = |\cos a - \cos x| \leq 2$, for all finite values of x.

$\therefore$ $\left| \int_a^x \phi(x)\, dx \right|$ is bounded for all finite values of x.

Hence by Dirichlet's test the integral $\int_a^\infty \frac{\sin x}{x}\, dx$ is convergent.

Since the sum of two convergent integrals is convergent, therefore $\int_0^\infty \frac{\sin x}{x}\, dx$ is convergent.

Example 24: *Prove that* $\int_a^\infty \frac{\cos \alpha x - \cos \beta x}{x}\, dx$ *is convergent where* $a > 0$. **(Rohilkhand 2011)**

Solution: We have

$$\int_a^\infty \frac{\cos \alpha x - \cos \beta x}{x}\, dx = \int_a^\infty \frac{\cos \alpha x}{x}\, dx - \int_a^\infty \frac{\cos \beta x}{x}\, dx.$$

The function $f(x) = 1/x$ is bounded and monotonic decreasing for all $x \geq a$ and $\lim_{x \to \infty} \frac{1}{x} = 0$.

Also $\left| \int_a^x \cos \alpha x\, dx \right| = \left| \frac{1}{\alpha} (\sin \alpha x - \sin \alpha a) \right| \leq \frac{2}{|\alpha|}$.

$\therefore$ $\left| \int_a^x \cos \alpha x\, dx \right|$ is bounded for all finite values of x.

Similarly $\left| \int_a^x \cos \beta x\, dx \right|$ is bounded for all finite values of x.

$\therefore$ by Dirichlet's test both the integrals

$$\int_a^\infty \frac{\cos \alpha x}{x}\,dx \text{ and } \int_a^\infty \frac{\cos \beta x}{x}\,dx \text{ are convergent.}$$

Hence the given integral is convergent.

Example 25: *Show that the integral*

$$\int_0^\infty e^{-ax}\,\frac{\sin x}{x}\,dx,\ a \geq 0 \text{ is convergent.}$$

(Kanpur 2009)

Solution: We have

$$\int_0^\infty e^{-ax}\,\frac{\sin x}{x}\,dx = \int_0^\alpha e^{-ax}\,\frac{\sin x}{x}\,dx + \int_\alpha^\infty e^{-ax}\,\frac{\sin x}{x}\,dx, \text{ where } \alpha > 0.$$

Since $\lim_{x \to 0} e^{-ax}\,\frac{\sin x}{x} = 1$, the integral $\int_0^\alpha e^{-ax}\,\frac{\sin x}{x}\,dx$ is a proper integral and hence convergent.

Now it remains to test the convergence of

$$\int_\alpha^\infty e^{-ax}\,\frac{\sin x}{x}\,dx. \quad \text{Let } f(x) = \frac{e^{-ax}}{x} \text{ and } \phi(x) = \sin x.$$

Obviously the function $f(x) = \frac{1}{x\,e^{ax}}$ is bounded and monotonic decreasing for all $x \geq \alpha$ and $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{x\,e^{ax}} = 0.$

Moreover $\left|\int_\alpha^x \phi(x)\,dx\right| = \left|\int_\alpha^x \sin x\,dx\right| = |\cos \alpha - \cos x| \leq 2$, for all finite values of x.

$\therefore$ $\left|\int_\alpha^x \phi(x)\,dx\right|$ is bounded for all finite values of x.

$\therefore$ by Dirichlet's test $\int_\alpha^\infty e^{-ax}\,\frac{\sin x}{x}\,dx$ is convergent.

Since the sum of two convergent integrals is convergent, therefore $\int_0^\infty e^{-ax}\,\frac{\sin x}{x}\,dx$ is convergent.

8.8 Absolute Convergence

The infinite integral $\int_a^\infty f(x)\,dx$ is said to be absolutely convergent if the integral $\int_a^\infty |f(x)|\,dx$ is convergent.

If the integral $\int_a^\infty f(x)\,dx$ is absolutely convergent, it is necessarily convergent.

But if the integral $\int_a^\infty f(x)\,dx$ is convergent, it is not necessarily absolutely

convergent. Thus absolute convergence gives a sufficient but not a necessary condition for the convergence of an infinite integral.

Illustrative Examples

Example 26: *Show that* $\int_1^\infty \frac{\sin x}{x^4}\,dx$ *is absolutely convergent.*

Solution: The integral $\int_1^\infty \frac{\sin x}{x^4}\,dx$ will be absolutely convergent if $\int_1^\infty \left|\frac{\sin x}{x^4}\right| dx$ is convergent.

Let $f(x) = \left|\frac{\sin x}{x^4}\right|$. Then $f(x)$ is bounded in the interval $(1, \infty)$. We have

$$f(x) = \left|\frac{\sin x}{x^4}\right| = \frac{|\sin x|}{x^4} \le \frac{1}{x^4}, \text{ since } |\sin x| \le 1.$$

$\therefore$ by comparison test, $\int_0^\infty f(x)\,dx$ is convergent if $\int_1^\infty \frac{1}{x^4}\,dx$ is convergent. But the comparison integral $\int_1^\infty \frac{1}{x^4}\,dx$ is convergent because here $n = 4$ which is > 1.

Hence $\int_1^\infty f(x)\,dx$ is convergent and so the given integral is absolutely convergent.

Example 27: *Show that* $\int_0^\infty \frac{\sin mx}{a^2 + x^2}\,dx$ *converges absolutely.*

Solution: The integral $\int_0^\infty \frac{\sin mx}{a^2 + x^2}\,dx$ will be absolutely convergent if $\int_0^\infty \left|\frac{\sin mx}{a^2 + x^2}\right| dx$ is convergent.

Let $f(x) = \left|\frac{\sin mx}{a^2 + x^2}\right|$. Then $f(x)$ is bounded in the interval $(0, \infty)$. We have

$$f(x) = \left|\frac{\sin mx}{a^2 + x^2}\right| = \frac{|\sin mx|}{a^2 + x^2} \le \frac{1}{a^2 + x^2}, \text{ since } |\sin mx| \le 1.$$

$\therefore$ by comparison test, $\int_0^\infty f(x)\,dx$ is convergent if $\int_0^\infty \frac{1}{a^2 + x^2}\,dx$ is convergent.

But $\int_0^\infty \frac{dx}{a^2 + x^2} = \lim_{x \to \infty} \int_0^x \frac{dx}{a^2 + x^2} = \lim_{x \to \infty} \left[\frac{1}{a} \tan^{-1} \frac{x}{a}\right]_0^x$

$$= \lim_{x \to \infty} \left[\frac{1}{a} \tan^{-1} \frac{x}{a} - 0\right] = \frac{1}{a} \cdot \frac{\pi}{2},$$

which is a definite real number.

$\therefore \quad \int_0^\infty \frac{dx}{a^2 + x^2}$ is convergent. Hence $\int_0^\infty f(x)\,dx$ is also convergent and so the given integral is absolutely convergent.

Example 28: *Show that the integral* $\int_0^\infty e^{-x} \cos mx\,dx$ *converges absolutley.*

Solution: The integral $\int_0^\infty e^{-x} \cos mx\,dx$ will be absolutely convergent if $\int_0^\infty |e^{-x} \cos mx|\,dx$ is convergent.

Let $f(x) = |e^{-x} \cos mx|$. Then $f(x)$ is bounded in the interval $(0, \infty)$. We have

$$f(x) = |e^{-x} \cos mx| = e^{-x} |\cos mx|$$
$$\leq e^{-x}, \quad \text{since } |\cos mx| \leq 1.$$

$\therefore$ by comparison test, $\int_0^\infty f(x)\,dx$ is convergent if $\int_0^\infty e^{-x}\,dx$ is convergent.

But $\int_0^\infty e^{-x}\,dx = \lim_{x \to \infty} \int_0^x e^{-x}\,dx = \lim_{x \to \infty} [-e^{-x}]_0^\infty$

$$= \lim_{x \to \infty} [-e^{-x} + 1] = 1, \text{ which is a definite real number.}$$

$\therefore \quad \int_0^\infty e^{-x}\,dx$ is convergent.

Hence $\int_0^\infty f(x)\,dx$ is convergent and so the given integral is absolutely convergent.

8.9 Tests for Convergence of Improper Integrals of The Second Kind

Now we shall make a study of the tests for the convergence of a definite integral of the type $\int_a^b f(x)\,dx$ in which the range of integration is finite and the integrand $f(x)$ is unbounded at one or more points of the given interval $[a, b]$. It is sufficient to consider the case when $f(x)$ becomes unbounded at $x = a$ and bounded for all other values of x in the interval $[a, b]$. In this case we have

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b f(x)\,dx.$$

In the articles to follow we give a few important tests for the convergence of the above integral.

8.10 Comparison Test

Consider the improper integral $\int_a^b f(x)\,dx$, *where the range of integration* (a, b) *is finite and* $f(x)$ *is unbounded only at* $x = a$. *Let* $g(x)$ *be positive in the interval* $(a + \varepsilon, b)$ *and* $|f(x)| \leq g(x)$ *in the interval* $(a + \varepsilon, b)$. *Then* $\int_a^b f(x)\,dx$ *is convergent if* $\int_a^b g(x)\,dx$ *is convergent.*

Similarly if $|f(x)| \geq g(x)$ *for all values of* x *in the interval* $(a + \varepsilon, b)$, *then* $\int_a^b f(x)\,dx$ *is divergent provided* $\int_a^b g(x)\,dx$ *is divergent.*

Alternative form of the above comparison test:

If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is a definite number, other than zero, the integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ either both converge or both diverge.

Note: While applying the above comparison test, we generally take $g(x) = \frac{1}{(x-a)^n}$ *i.e.*, $\int_a^b \frac{dx}{(x-a)^n}$ is generally taken as the comparison integral.

Theorem: *The comparison integral* $\int_a^b \frac{dx}{(x-a)^n}$ *is convergent when* $n < 1$ *and divergent when* $n \geq 1$. **(Purvanchal 2011)**

Proof: We have

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b (x-a)^{-n}\,dx$$

$$= \lim_{\varepsilon \to 0} \left[\frac{(x-a)^{-n+1}}{1-n}\right]_{a+\varepsilon}^b, \text{ if } n \neq 1$$

$$= \lim_{\varepsilon \to 0} \left[\frac{(b-a)^{1-n}}{1-n} - \frac{\varepsilon^{1-n}}{1-n}\right]. \qquad \ldots(1)$$

If $n < 1$, then $1 - n$ is positive and so $\lim_{\varepsilon \to 0} \varepsilon^{1-n} = 0$. Therefore from (1), we have

$$\int_a^b \frac{dx}{(x-a)^n} = \frac{(b-a)^{1-n}}{1-n}, \text{ if } n < 1.$$

Hence the given integral converges when $n < 1$.

If $n > 1$, then $1 - n$ is negative and so $n - 1$ is positive. Therefore in this case, from (1), we have

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0} \left[\frac{(b-a)^{1-n}}{1-n} + \frac{1}{(n-1)\,\varepsilon^{n-1}} \right] = \infty.$$

Hence the given integral diverges when $n > 1$.

When $n = 1$, we have

$$\int_a^b \frac{dx}{(x-a)^n} = \int_a^b \frac{dx}{(x-a)}$$

$$= \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b \frac{dx}{x-a}$$

$$= \lim_{\varepsilon \to 0} \left[\log (x-a) \right]_{a+\varepsilon}^b$$

$$= \lim_{\varepsilon \to 0} \left[\log (b-a) - \log \varepsilon \right]$$

$$= \infty \qquad [\because \log 0 = -\infty]$$

Hence the given integral diverges when $n = 1$.

Illustrative Examples

Example 29: *Show that the integral* $\int_0^1 \frac{dx}{x^{1/3}\,(1+x^2)}$ *is convergent.*

Solution: In the given integral, the integrand $f(x) = \frac{1}{x^{1/3}\,(1+x^2)}$ is unbounded at the lower limit of integration $x = 0$.

Take $g(x) = 1 / x^{1/3}$.

Then $\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{1}{1+x^2} = 1$, which is finite and non-zero.

$\therefore$ by comparison test

$$\int_0^1 f(x)\,dx \text{ and } \int_0^1 g(x)\,dx$$

either both converge or both diverge. But the comparison integral $\int_0^1 \frac{dx}{x^{1/3}}$ is convergent because here $n = 1/3$ which is less than 1. Hence the integral $\int_0^1 \frac{dx}{x^{1/3}\,(1+x^2)}$ is also convergent.

Example 30: *Test the convergence of the integral* $\int_1^2 \frac{dx}{\sqrt{(x^4 - 1)}}$.

Solution: In the given integral the integrand $f(x) = 1/\sqrt{(x^4 - 1)}$ is unbounded at the lower limit of integration $x = 1$.

Take $g(x) = 1/\sqrt{(x^2 - 1)}$.

Then $$\lim_{x \to 1} \frac{f(x)}{g(x)} = \lim_{x \to 1} \left\{ \frac{1}{\sqrt{(x^4 - 1)}} \cdot \sqrt{(x^2 - 1)} \right\} = \lim_{x \to 1} \frac{1}{\sqrt{(x^2 + 1)}}$$

$= 1/\sqrt{2}$, which is finite and non-zero.

Therefore by comparison test,

$$\int_1^2 f(x)\,dx \quad \text{and} \quad \int_1^2 g(x)\,dx$$

are either both convergent or both divergent.

But $$\int_1^2 g(x)\,dx = \int_1^2 \frac{dx}{\sqrt{(x^2 - 1)}} = \lim_{\varepsilon \to 0} \int_{1+\varepsilon}^2 \frac{dx}{\sqrt{(x^2 - 1)}}$$

$$= \lim_{\varepsilon \to 0} \left[\log \{x + \sqrt{(x^2 - 1)}\}\right]_{1+\varepsilon}^2$$

$$= \lim_{\varepsilon \to 0} \left[\log (2 + \sqrt{3}) - \log \{1 + \varepsilon + \sqrt{(\varepsilon^2 + \varepsilon)}\}\right]$$

$= \log (2 + \sqrt{3})$, which is a definite real number.

$\therefore$ $\int_1^2 g(x)\,dx$ is convergent.

Hence $\int_1^2 \frac{1}{\sqrt{(x^4 - 1)}}\,dx$ is also convergent.

Example 31: *Show that the integral* $\int_0^1 \frac{\sec x}{x}\,dx$ *is divergent.*

Solution: In the given integral the integrand $f(x) = \frac{\sec x}{x}$ is unbounded at the lower limit of integration $x = 0$. Take $g(x) = 1/x$.

Then $$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \left\{ \frac{\sec x}{x} \cdot x \right\} = \lim_{x \to 0} \sec x = 1,$$

which is finite and non-zero.

Therefore, by comparison test, $\int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ either both converge or both diverge. But the comparison integral $\int_0^1 \frac{1}{x}\,dx$ is divergent because here $n = 1$.

Hence the given integral $\int_0^1 \frac{\sec x}{x}\,dx$ is also divergent.

Example 32: *Show that* $\int_0^1 x^{n-1} e^{-x}\, dx$ *is convergent if* $n > 0$.

Solution: If $n \geq 1$, then $\int_0^1 x^{n-1} e^{-x}\, dx$ is a proper integral because the integrand $f(x) = x^{n-1} e^{-x}$ is bounded in the interval $(0, 1)$. So the given integral is convergent when $n \geq 1$.

If $0 < n < 1$, the integrand $f(x) = x^{n-1} e^{-x}$ is unbounded at $x = 0$. Take $g(x) = x^{n-1}$.

Then $\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} e^{-x} = 1$, which is finite and non-zero.

$\therefore$ by comparison test, $\int_0^1 f(x)\, dx$ and $\int_0^1 g(x)\, dx$ either both converge or both diverge.

But $\int_0^1 g(x)\, dx = \int_0^1 x^{n-1}\, dx = \lim_{\varepsilon \to 0} \int_\varepsilon^1 x^{n-1}\, dx = \lim_{\varepsilon \to 0} \left[\frac{x^n}{n}\right]_\varepsilon^1$

$$= \lim_{\varepsilon \to 0} \left[\frac{1}{n} - \frac{\varepsilon^n}{n}\right] = \frac{1}{n}, \text{ which is a definite real number.}$$

$\therefore$ $\int_0^1 g(x)\, dx$ is convergent.

Hence $\int_0^1 x^{n-1} e^{-x}\, dx$ is also convergent.

Example 33: *Show that the integral* $\int_0^\infty x^{n-1} e^{-x}\, dx$ *is convergent if* $n > 0$.

Solution: We have

$$\int_0^\infty x^{n-1} e^{-x}\, dx = \int_0^1 x^{n-1} e^{-x}\, dx + \int_1^\infty x^{n-1} e^{-x}\, dx.$$

Let $I_1 = \int_0^1 x^{n-1} e^{-x}\, dx$ and $I_2 = \int_1^\infty x^{n-1} e^{-x}\, dx$.

The integral I_2 is convergent for all values of n.

[For proof see Ex. 17 after article 8.5]

Also the integral I_1 is convergent if $n > 0$. [For proof see Ex. 32 above]

Hence the given integral is convergent if $n > 0$ because then it is the sum of two convergent integrals.

8.11 The μ-Test

Let $f(x)$ *be unbounded at* $x = a$ *and be bounded and integrable in the arbitrary interval* $(a + \varepsilon, b)$, *where* $0 < \varepsilon < b - a$. *If there is a number* μ *between* 0 *and* 1 *such that*

$$\lim_{x \to a + 0} (x - a)^\mu f(x) \text{ exists, then } \int_a^b f(x)\, dx$$

is convergent.

If there is a number $\mu \geq 1$ *such that* $\lim_{x \to a+0} (x-a)^{\mu} f(x)$ *exists and is non-zero, then* $\int_a^b f(x)\,dx$ *is divergent and the same is true if*

$$\lim_{x \to a+0} (x-a)^{\mu} f(x) = +\infty \quad or \quad -\infty.$$

In case $f(x)$ is unbounded at $x = b$, we should find

$$\lim_{x \to b-0} (b-x)^{\mu} \cdot f(x),$$

the other conditions of the test remaining the same.

Illustrative Examples

Example 34: *Prove that the integral* $\int_0^1 \frac{dx}{\sqrt{\{x(1-x)\}}}$ *converges.*

Solution: In the given integral the integrand $f(x) = 1/\sqrt{\{x(1-x)\}}$ is unbounded both at $x = 0$ and at $x = 1$. If $0 < a < 1$, we can write

$$\int_0^1 \frac{dx}{\sqrt{\{x(1-x)\}}} = \int_0^a \frac{dx}{\sqrt{\{x(1-x)\}}} + \int_a^1 \frac{dx}{\sqrt{\{x(1-x)\}}} = I_1 + I_2 \text{ , say.}$$

In the integral I_1 the integrand $f(x)$ is unbounded at the lower limit of integration $x = 0$ and in the integral I_2 the integrand $f(x)$ is unbounded at the upper limit of integration $x = 1$.

To test the convergence of I_1. Take $\mu = \frac{1}{2}$. We have

$$\lim_{x \to 0} x^{\mu} f(x) = \lim_{x \to 0} x^{1/2} \cdot \frac{1}{\sqrt{\{x(1-x)\}}} = \lim_{x \to 0} \frac{1}{\sqrt{(1-x)}}$$

$= 1$ *i.e.*, the limit exists.

Since $0 < \mu < \frac{1}{2}$, therefore by μ-test I_1 is convergent.

To test the convergence of I_2 Take $\mu = \frac{1}{2}$. We have

$$\lim_{x \to 1-0} (1-x)^{\mu} \cdot f(x) = \lim_{x \to 1-0} (1-x)^{1/2} \cdot \frac{1}{\sqrt{\{x(1-x)\}}}$$

$$= \lim_{x \to 1-0} \frac{1}{\sqrt{x}} = \lim_{\varepsilon \to 0} \frac{1}{\sqrt{(1-\varepsilon)}} = 1.$$

Hence by μ-test I_2 is convergent since $0 < \mu < 1$.

Thus the given integral is the sum of two convergent integrals. Hence the given integral itself is convergent.

Example 35: *Test the convergence of* $\int_0^1 \frac{\log x}{\sqrt{(2-x)}}\,dx$. (Agra 2012)

Solution: Let $f(x) = \frac{\log x}{\sqrt{(2-x)}}$. Then $f(x)$ is unbounded both at $x = 0$ and $x = 2$. If $0 < a < 2$, we can write

$$\int_0^2 \frac{\log x}{\sqrt{(2-x)}}\,dx = \int_0^a \frac{\log x}{\sqrt{(2-x)}}\,dx + \int_a^2 \frac{\log x}{\sqrt{(2-x)}}\,dx = I_1 + I_2 \text{ , say.}$$

To test the convergence of I_1. We have

$$\lim_{x \to 0} x^{\mu} f(x) = \lim_{x \to 0} \left\{ x^{\mu} \cdot \frac{\log x}{\sqrt{(x-2)}} \right\} = 0 \text{ if } \mu > 0.$$

Therefore taking μ between 0 and 1, it follows by μ-test that I_1 is convergent.

To test the convergence of I_2.

Take $\mu = \frac{1}{2}$. We have

$$\lim_{x \to 2-0} (2-x)^{\mu} \cdot f(x) = \lim_{x \to 2-0} (2-x)^{1/2} \cdot \frac{\log x}{\sqrt{(2-x)}}$$

$$= \lim_{x \to 2-0} \log x = \lim_{\varepsilon \to 0} \log (2-\varepsilon)$$

$$= \log 2.$$

$\therefore$ by μ-test I_2 is convergent because $0 < \mu < 1$.

Hence the given integral is also convergent, it being the sum of two convergent integrals.

Example 36: *Test the convergence of* $\int_0^1 x^{p-1} e^{-x}\,dx$.

Solution: Let $f(x) = x^{p-1} e^{-x}$

and $$I = \int_0^1 x^{p-1} e^{-x}\,dx.$$

If $p \geq 1$, $f(x)$ is bounded throughout the interval $(0, 1)$ and so I is a proper integral and hence it is convergent if $p \geq 1$.

If $p < 1$, $f(x)$ is unbounded at $x = 0$. In this case, we have

$$\lim_{x \to 0} x^{\mu} f(x) = \lim_{x \to 0} x^{\mu} \cdot x^{p-1} e^{-x} = \lim_{x \to 0} x^{\mu + p - 1} e^{-x}$$

$$= 1 \text{ if } \mu + p - 1 = 0 \text{ } i.e., \mu = 1 - p.$$

So by μ-test when $0 < \mu < 1$ *i.e.*, $0 < p < 1$, the given integral is convergent and when $\mu \geq 1$ *i.e.*, $p \leq 0$, the given integral is divergent.

Hence I is convergent if $p > 0$ and is divergent if $p \leq 0$.

8.12 Abel's Test

If $\int_a^b f(x)\,dx$ *converges and* $\phi(x)$ *is bounded and monotonic for* $a \le x \le b$, *then* $\int_a^b f(x)\,\phi(x)\,dx$ *converges.*

8.13 Dirichlet's Test

If $\int_{a+\varepsilon}^{b} f(x)\,dx$ *be bounded and* $\phi(x)$ *be bounded and monotonic on the interval* $a \le x \le b$, *converging to zero as* x *tends to* a, *then* $\int_a^b f(x)\,\phi(x)\,dx$ *converges.*

Illustrative Examples

Example 37: *Test the convergence of* $\int_0^{\pi/2} \frac{\cos x}{x^n}\,dx$.

Solution: When $n \le 0$, the given integral is a proper integral and hence convergent.

When $n > 0$, the integrand becomes unbounded at $x = 0$.

Let $$f(x) = \frac{\cos x}{x^n}.$$

Then $$\lim_{x \to 0} x^{\mu} f(x) = \lim_{x \to 0} x^{\mu - n} \cos x = 1, \text{ if } \mu = n.$$

Hence by μ-test it follows that the given integral is convergent when $0 < n < 1$, and divergent when $n \ge 1$.

From the above discussion we conclude that the given integral is convergent when $n < 1$, and divergent when $n \ge 1$.

Example 38: *Show that the integral* $\int_0^{\pi/2} \log \sin x\,dx$ *converges.*

(Meerut 2012; Rohilkhand 12)

Solution: The only point of infinite discontinuity of the integrand is $x = 0$.

Now $$\lim_{x \to 0} x^{\mu} \log \sin x, \text{ when } \mu > 0$$

$$= \lim_{x \to 0} \frac{\log \sin x}{x^{-\mu}}, \qquad \left[\text{form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x \to 0} \frac{\cot x}{-\mu\, x^{-\mu - 1}}$$

$$= \lim_{x \to 0} -\frac{1}{\mu} \cdot \frac{x^{\mu + 1}}{\tan x} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} -\frac{1}{\mu} \cdot \frac{(\mu + 1)\, x^{\mu}}{\sec^2 x}, \qquad \text{[by L'Hospital's rule]}$$

$= 0$, if $\mu > 0$.

Taking μ between 0 and 1, it follows from μ-test that the given integral is convergent.

Example 39: *Discuss the convergence of the integral*

$$\int_0^1 x^{n-1} \log x \, dx.$$

(Purvanchal 2009)

Solution: (i) Since $\lim_{x \to 0} x^r \log x = 0$ where $r > 0$, the integral is a proper integral, when $n > 1$.

(ii) When $n = 1$, we have

$$\int_0^1 \log x \, dx = \lim_{\varepsilon \to 0} \int_{\varepsilon}^1 \log x \, dx = \lim_{\varepsilon \to 0} [x \log x - x]_{\varepsilon}^1$$

$$= \lim_{\varepsilon \to 0} [-1 - \varepsilon \log \varepsilon + \varepsilon] = -1.$$

$\therefore$ the integral is convergent if $n = 1$.

(iii) Let $n < 1$ and $f(x) = x^{n-1} \log x$.

Then $\lim_{x \to 0} x^{\mu} f(x) = \lim_{x \to 0} x^{\mu + n - 1} \log x$

$= 0$ if $\mu > 1 - n$ ...(1)

and $= -\infty$ if $\mu \leq 1 - n$. ...(2)

Hence when $0 < n < 1$, we can choose μ between 0 and 1 and satisfying (1). The integral is therefore convergent by μ-test when $0 < n < 1$.

Again when $n \leq 0$, we can take $\mu = 1$ and satisfying (2). Hence by μ-test the integral is divergent when $n \leq 0$.

Therefore from (i), (ii) and (iii), we conclude that the given integral is convergent when $n > 0$ and divergent when $n \leq 0$.

Example 40: *Discuss the convergence or divergence of the integral*

$$\int_0^{\infty} \frac{x^{a-1}}{1+x} \, dx.$$

Solution: Let $f(x) = \frac{x^{a-1}}{1+x}$. If $b > 0$, we can write

$$\int_0^{\infty} \frac{x^{a-1}}{1+x} \, dx = \int_0^b \frac{x^{a-1}}{1+x} \, dx + \int_b^{\infty} \frac{x^{a-1}}{1+x} \, dx$$

$$= I_1 + I_2, \text{ say.}$$

Let $a \geq 1$. Then $f(x)$ is bounded throughout the interval $(0, b)$ and so the integral I_1 is a proper integral and hence it is convergent. To test the convergence of the infinite integral I_2 in this case, we have

$$\lim_{x \to \infty} x^{\mu} f(x) = \lim_{x \to \infty} x^{\mu} \cdot \frac{x^{a-1}}{1+x} = \lim_{x \to \infty} \frac{x^{\mu+a-1}}{x+1}$$

$= 1$, if $\mu + a - 1 = 1$ *i.e.*, if $\mu = 2 - a$

which is ≤ 1 since $a \geq 1$.

Hence by μ-test I_2 is divergent.

$\therefore$ the given integral is divergent if $a \geq 1$.

Let $a < 1$ Then in the interval $(0, b)$, $f(x)$ is unbounded only at $x = 0$. Also $f(x)$ is bounded throughout the interval (b, ∞). Therefore in this case I_1 is an improper integral of the second kind and I_2 is an improper integral of the first kind. To test the convergence of I_1, we have

$$\lim_{x \to 0} x^{\mu} \cdot \frac{x^{a-1}}{x+1} = \lim_{x \to 0} \frac{x^{\mu+a-1}}{x+1} = 1,$$

if $\mu + a - 1 = 0$ *i.e.*, if $\mu = 1 - a$.

If we take $0 < a < 1$, then we have $0 < \mu < 1$ and so by μ-test I_1 is convergent. If we take $a \leq 0$, then $\mu \geq 1$ and so by μ-test I_1 is divergent.

To test the convergence of I_2 when $a < 1$, we have

$$\lim_{x \to \infty} x^{\mu} \cdot \frac{x^{a-1}}{x+1} = \lim_{x \to \infty} \frac{x^{\mu+a+1}}{x+1} = 1, \text{ if } \mu + a - 1 = 1 \text{ } i.e.,$$

if $\mu = 2 - a$ which is > 1 since $a < 1$.

Hence by μ-test I_2 is convergent if $a < 1$.

Thus I_2 is convergent if $a < 1$. But I_1 is convergent if $0 < a < 1$ and is divergent if $a \leq 0$.

$\therefore$ the given integral is convergent if $0 < a < 1$ and is divergent if $a \leq 0$.

Hence the given integral is convergent if $0 < a < 1$ and is divergent if $a \geq 1$ or if $a \leq 0$.

Example 41: *Discuss the convergence of the Beta function*

$$\int_0^1 x^{m-1} (1-x)^{n-1} \, dx.$$

(Purvanchal 2007, 12; Kanpur 12)

Solution: Let $f(x) = x^{m-1} (1-x)^{n-1}$.

The following different cases arise :

(i) When m and n are both ≥ 1, the integrand $f(x)$ is bounded throughout the interval $(0, 1)$ and so the given integral is a proper integral and is convergent.

(ii) When m and n are both < 1, the integrand $f(x)$ becomes infinite both at $x = 0$ and at $x = 1$. In this case we take $0 < a < 1$ and we write

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^a x^{m-1}(1-x)^{n-1}\,dx + \int_a^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= I_1 + I_2, \text{ say.}$$

In the case of the integral I_1, the interval of integration is $(0, a)$ and so the integrand is unbounded at $x = 0$ only. To test the convergence of I_1, we have

$$\lim_{x \to 0} x^{\mu} \cdot f(x) = \lim_{x \to 0} x^{\mu} \cdot x^{m-1}(1-x)^{n-1}$$

$$= \lim_{x \to 0} x^{\mu+m-1}(1-x)^{n-1}$$

$$= 1, \text{ if } \mu + m - 1 = 0 \text{ } i.e., \text{ if } \mu = 1 - m.$$

If we take $0 < m < 1$, we have $0 < \mu < 1$ and so by μ-test I_1 is convergent. If we take $m \leq 0$, we have $\mu \geq 1$ and so by μ-test I_1 is divergent.

Again in the case of the integral I_2, the interval of integration is $(a, 1)$ and so the integrand is unbounded at $x = 1$ only. To test the convergence of I_2, we have

$$\lim_{x \to 1-0} (1-x)^{\mu} \cdot f(x) = \lim_{x \to 1-0} (1-x)^{\mu} x^{m-1}(1-x)^{n-1}$$

$$= \lim_{x \to 1-0} (1-x)^{\mu+n-1} x^{m-1}$$

$$= \lim_{\varepsilon \to 0} \{1-(1-\varepsilon)\}^{\mu+n-1}(1-\varepsilon)^{m-1}$$

$$= \lim_{\varepsilon \to 0} \varepsilon^{\mu+n-1}(1-\varepsilon)^{m-1}$$

$$= 1, \text{ if } \mu + n - 1 = 0 \text{ } i.e., \text{ if } \mu = 1 - n.$$

If we take $0 < n < 1$, we have $0 < \mu < 1$ and so by μ-test I_2 is convergent. If we take $n \leq 0$, we have $\mu \geq 1$ and so by μ-test I_2 is divergent.

Thus if m and n are both < 1, the given integral is convergent only if $0 < m < 1$ and $0 < n < 1$.

(iii) When $m < 1$ and $n \geq 1$, the integrand $f(x)$ is unbounded only at $x = 0$. In this case by μ-test, the given integral is convergent if $0 < m < 1$ and is divergent if $m \leq 0$.

Again if $m \geq 1$ and $n < 1$, the integrand $f(x)$ is unbounded only at $x = 1$. In this case by μ-test, the given integral is convergent if $0 < n < 1$ and is divergent if $n \leq 0$.

Hence from (i), (ii) and (iii) it follows that the given integral is convergent if both m and n are > 0 and divergent otherwise.

Example 42: *Discuss the convergence of the Gamma function*

$$\int_0^{\infty} x^{n-1} e^{-x}\,dx.$$

(Kanpur 2008; Purvganchal 08, 11, 12; Rohilkhand 10,11)

Solution: We can write

$$\int_0^\infty x^{n-1} e^{-x}\, dx = \int_0^1 x^{n-1} e^{-x}\, dx + \int_1^\infty x^{n-1} e^{-x}\, dx$$

$$= I_1 + I_2 \text{, say.}$$

Let us first discuss the convergence of I_1.

Let $f(x) = x^{n-1} e^{-x}$.

If $n \geq 1$, $f(x)$ is bounded throughout the interval $[0, 1]$ and so I_1 is a proper integral and hence it is convergent if $n \geq 1$.

If $n < 1$, $f(x)$ is unbounded at $x = 0$. In this case we have

$$\lim_{x \to 0} x^\mu f(x) = \lim_{x \to 0} x^\mu \cdot x^{n-1} e^{-x} = \lim_{x \to 0} x^{\mu + n - 1} e^{-x}$$

$$= 1, \text{ if } \mu + n - 1 = 0 \text{ i.e., } \mu = 1 - n.$$

So by μ-test when $0 < \mu < 1$ i.e., $0 < n < 1$, the integral I_1 is convergent and when $\mu \geq 1$ i.e. $n \leq 0$, the integral I_1 is divergent.

$\therefore$ I_1 is convergent if $n > 0$ and is divergent if $n \leq 0$.

Now let us discuss the convergence of the integral I_2. The function $f(x) = x^{n-1} e^{-x}$ is bounded for all values of x in the interval $(1, \infty)$. We have

$$\lim_{x \to \infty} x^\mu f(x) = \lim_{x \to \infty} \frac{x^\mu \cdot x^{n-1}}{e^x} = \lim_{x \to \infty} \frac{x^{\mu + n - 1}}{1 + x + \frac{x^2}{2!} + \ldots}$$

$$= 0 \text{ for all values of } \mu \text{ and } n.$$

Taking $\mu > 1$, we see by μ-test that the integral

$I_2 = \int_1^\infty x^{n-1} e^{-x}\, dx$ is convergent for all values of n.

Hence the given integral is convergent if $n > 0$ and is divergent if $n \leq 0$.

Comprehensive Exercise 2

1. Show that the integral $\int_\pi^\infty \frac{\sin^2 x}{x^2}\, dx$ is convergent.

2. Test the convergence of the following integrals.

   (i) $\int_0^\infty \frac{\cos x}{1 + x^2}\, dx$.

   (ii) $\int_\pi^\infty \frac{\sin x}{x^2}\, dx$

   (Purvanchal 2010; Rohilkhand 12; Bundelkhand 11)

   (iii) $\int_0^\infty \frac{\sin mx}{x^2 + a^2}\, dx$

   (iv) $\int_0^\infty \frac{x^3}{(x^2 + a^2)^2}\, dx$.

(v) $\int_1^\infty \frac{dx}{\sqrt{(x^3+1)}}$.

(vi) $\int_0^\infty \frac{1-\cos x}{x^2}\,dx$.

(vii) $\int_2^\infty \frac{dx}{\sqrt{(x^2-x-1)}}$;

(viii) $\int_2^\infty \frac{dx}{\sqrt{(x^2-1)}}$.

(ix) $\int_0^\infty \frac{x^2\,dx}{(1+x)^3}$.

(x) $\int_0^\infty \frac{x^{3/2}}{(b^2x^2+c)}\,dx$

3. Show that the following integrals are convergent.

(i) $\int_0^\infty \frac{x^2}{(a^2+x^2)^2}\,dx$

(ii) $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$.

4. Test the convergence of $\int_b^\infty \frac{x^{3/2}\,dx}{\sqrt{(x^4-a^4)}}$, where $b > a$.

5. Show that the integral $\int_a^\infty x^{n-1}e^{-x}\,dx$ is convergent, where $a > 0$.

6. Test the convergence of the following integrals.

(i) $\int_0^1 \frac{dx}{x^3(1+x^2)}$.

(ii) $\int_0^1 \frac{dx}{(x+1)\sqrt{(1-x^2)}}$.

(iii) $\int_0^{\pi/2} \frac{\cos x}{x^2}\,dx$.

(iv) $\int_0^{\pi/4} \frac{1}{\sqrt{(\tan x)}}\,dx$. (Agra 2012)

(v) $\int_0^{\pi/2} \frac{\sin x}{x^{1+n}}\,dx$.

7. Test the convergence of the following integrals :

(i) $\int_0^\infty \frac{dx}{x^{1/3}(1+x^{1/2})}$

(ii) $\int_0^\infty \frac{dx}{x\sqrt{(1+x^2)}}$

(iii) $\int_0^\infty \frac{x^{1/2}}{x^2+4}\,dx$

(iv) $\int_0^\infty \frac{x}{1+x^2}\sin x\,dx$

8. Examine the convergence of the integral $\int_0^\infty \frac{\sin x}{x^{3/2}}\,dx$

9. Show that the integral $\int_0^\infty e^{-a^2x^2}\cos bx\,dx$ is absolutely convergent.

Answers 2

2. (i) Convergent
(ii) Convergent
(iii) Convergent
(iv) Divergent
(v) Convergent
(vi) Convergent
(vii) Divergent
(viii) Divergent
(ix) Divergent
(x) Divergent.

4. Divergent.

6. (i) Divergent (ii) Convergent
(iii) Divergent (iv) Convergent
(v) Convergent if $n < 1$ and divergent if $n \geq 1$
(vi) Convergent.

7. (i) Divergent (ii) Divergent
(iii) Divergent (iv) Convergent.

8. Convergent.

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The integral $\int_a^\infty \frac{dx}{x^n}$, where $a > 0$, is convergent when

(a) $n = 1$ (b) $n < 1$
(c) $n \leq 1$ (d) $n > 1$. **(Rohilkhand 2011)**

2. The integral $\int_a^b \frac{dx}{(x-a)^n}$ is convergent when

(a) $n < 1$ (b) $n > 1$
(c) $n = 1$ (d) $n \geq 1$.

3. The integral $\int_0^1 x^{m-1} (1-x)^{n-1} \, dx$ is convergent when

(a) $m > 0$ (b) $n > 0$
(c) $m > 0, n > 0$ (d) $m = 0, n > 1$.

4. The integral $\int_0^\infty x^{n-1} e^{-x} \, dx$ is divergent when

(a) $n > 0$ (b) $n > 1$
(c) $n \leq 0$ (d) $n = \frac{1}{2}$.

5. The integral $\int_a^\infty \frac{\sin^2 x}{x^2} \, dx, a > 0$

(a) convergent (b) divergent
(c) uniformly convergent (d) none of these.
(Rohilkhand 2012)

Fill In The Blank(s)

Fill in the blanks "....." so that the following statements are complete and correct.

1. The definite integral $\int_a^b f(x)\,dx$ is said to be a ..... if the range of integration (a, b) is finite and the integrand $f(x)$ is bounded over (a, b).
2. The definite integral $\int_a^b f(x)\,dx$ is said to be an improper integral if the interval (a, b) is finite and $f(x)$ is not ..... over this interval.
3. The definite integral $\int_a^b f(x)\,dx$ is said to be an ..... if the interval (a, b) is not finite and $f(x)$ is bounded over (a, b).
4. A definite integral $\int_a^b f(x)\,dx$ in which the range of integration (a, b) is finite but the integrand $f(x)$ is unbounded at one or more points of the interval $a \le x \le b$, is called an improper integral of the ..... kind.
5. The integral $\int_0^\infty \frac{dx}{1+x^2}$ is an improper integral of the ..... kind.
6. The integral $\int_0^4 \frac{dx}{(x-2)(x-3)}$ is an improper integral of the ..... kind.
7. The integral $\int_a^\infty \frac{dx}{x^n}$, where $a > 0$, is convergent when .....
8. The integral $\int_a^b \frac{dx}{(x-a)^n}$ is divergent when ..... .
9. The integral $\int_0^\infty e^{-x}\, x^{n-1}\,dx$ is convergent if ..... .
10. The integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is convergent if both m and n are > .....

True or False

Write 'T' for true and 'F' for false statement.

1. The integral $\int_3^\infty \frac{dx}{(x-2)^2}$ is divergent.
2. The integral $\int_0^\infty \frac{\cos x}{1+x^2}\,dx$ is convergent.
3. The integral $\int_a^\infty \frac{dx}{x^n}$, where $a > 0$, is convergent when $n \le 1$.
4. The integral $\int_a^b \frac{dx}{(x-a)^n}$ is divergent when $n < 1$.
5. The integral $\int_0^1 \frac{dx}{x^3(1+x^2)}$ is convergent.

6. The integral $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ is convergent.

7. The integral $\int_1^\infty \frac{dx}{\sqrt{(x^3+1)}}$ is convergent.

8. The integral $\int_2^\infty \frac{dx}{\sqrt{(x^2-1)}}$ is convergent.

Answers

Multiple Choice Questions

1. (d) 2. (a) 3. (c) 4. (c) 5. (a)

Fill in the Blank(s)

1. proper integral
2. bounded
3. improper integral of the first kind
4. second
5. first
6. second
7. $n > 1$
8. $n \geq 1$
9. $n > 0$
10. 0

True or False

1. *F*. 2. *T* 3. *F* 4. *F* 5. *F*
6. *T* 7. *T* 8. *F*

□

Chapter

9

Integral as a Function of a Parameter

9.1 Definite Integral as a Function of a Parameter

Consider a continuous function f of two variables x and α. For a fixed $\alpha \in [\alpha_1, \alpha_2]$, the function of x with domain $[a, b]$ is continuous and so, $\int_a^b f(x, \alpha)\, dx$ exists and is a function of a single variable α.

Let ϕ be a function of α such that

$$\phi(\alpha) = \int_a^b f(x, \alpha)\, dx,$$

then ϕ is called an integral function of the function $f(x, \alpha)$ of the parameter α.

9.2 Continuity of Proper Integrals as a Function of a Parameter

Theorem: *The function* $\phi(\alpha) = \int_a^b f(x, \alpha)\, dx$, *where* $f(x, \alpha)$ *is continuous in* $a \le x \le b$, $\alpha_1 \le \alpha \le \alpha_2$; *is a continuous function of* α *on* $[\alpha_1, \alpha_2]$.

Proof: Let $f\ (x, \alpha)$ be continuous in $a \le x \le b$ for a fixed $\alpha \in [\alpha_1, \alpha_2]$ and let α and $\alpha + k$ be any two points of $[\alpha_1, \alpha_2]$.

Then $\phi(\alpha + k) - \phi(\alpha) = \int_a^b [f\ (x, \alpha + k) - f\ (x, \alpha)]\, dx$

Let ε be any positive number.

Since f is continuous in $[a, b] \times [\alpha_1, \alpha_2]$, it is uniformly continuous and so there exists a positive number δ, such that for $|k| \le \delta$ and $x \in [a, b]$,

we have $|f\ (x, \alpha + k) - f\ (x, \alpha)| < \dfrac{\varepsilon}{(b - a)}$.

$$\therefore \quad |\phi(x, \alpha + k) - \phi(\alpha)| = \left| \int_a^b [f\ (x, \alpha + k) - f\ (x, \alpha)]\, dx \right|$$

$$\le \int_a^b |f\ (x, \alpha + k) - f\ (x, \alpha)|\, dx$$

$$\le \left[\frac{\varepsilon}{(b - a)}\right][(b - a)] = \varepsilon, \text{ when } |k| \le \delta.$$

Thus $\phi(\alpha)$ is a continuous function in $[\alpha_1, \alpha_2]$.

9.3 Derivability of Proper Integrals as a Function of a Parameter

Theorem 1: *If $f(x,\alpha)$ and f_α be continuous in $[a, b] \times [\alpha_1, \alpha_2]$, then $\phi(\alpha)$ is differentiable and*

$$\phi'(\alpha) = \int_a^b f_\alpha\ (x, \alpha)\, dx, \ \forall \alpha \in [\alpha_1, \alpha_2]$$

i.e., $$\frac{d}{d\alpha}\left\{\int_a^b f\ (x, \alpha)\, dx\right\} = \int_a^b f_\alpha\ (x, \alpha)\, dx,$$

where f_α denotes the partial derivative of f with respect to α.

Proof: Let $f\ (x, \alpha)$ be continuous in $a \le x \le b$ for a fixed $\alpha \in [\alpha_1, \alpha_2]$ and let α and $\alpha + k$ be any two points of $[\alpha_1, \alpha_2]$.

Then $\phi(\alpha + k) - \phi(\alpha) = \int_a^b [f\ (x, \alpha + k) - f\ (x, \alpha)]\, dx.$

By Lagrange's mean value theorem, we have

$$f\ (x, \alpha + k) - f\ (x, \alpha) = k\ f_\alpha\ (x, \alpha + \theta k)$$

$$= k\ [f_\alpha\ (x, \alpha + \theta k) - f_\alpha\ (x, \alpha) + f_\alpha\ (x, \alpha)]$$

Thus, we have

$$\frac{\phi(\alpha, k) - \phi(\alpha)}{k} - \int_a^b f_\alpha\ (x, \alpha)\, dx = \int_a^b [f_\alpha\ (x, \alpha + \theta k) - f_\alpha\ (x, \alpha)]\, dx$$

Since f_α is continuous in $[a, b] \times [\alpha_1, \alpha_2]$, it is uniformly continuous and so for a given $\varepsilon > 0$, there exists $\delta > 0$ such that for

$|k| \le \delta$ and $x \in [a, b]$, we have

$$|f_\alpha(x, \alpha + k) - f_\alpha(x, \alpha)| < \frac{\varepsilon}{(b-a)}.$$

$$\therefore \quad \left|\frac{\phi(\alpha + k) - \phi(\alpha)}{k} - \int_a^b f_\alpha(x, \alpha)\, dx\right| < \varepsilon, \text{ when } |k| \le \delta.$$

As $\varepsilon \to 0$, we have $\phi'(\alpha) = \int_a^b f_\alpha(x, \alpha)\, dx$

or $$\frac{d}{dx}\left\{\int_a^b f(x, \alpha)\, dx\right\} = \int_a^b f_\alpha(x, \alpha)\, dx.$$

Theorem 2: *If $f(x, \alpha)$ and f_α be continuous in $[a, b] \times [\alpha_1, \alpha_2]$ and $h(\alpha)$, $g(\alpha)$ be two functions differentiable in $[\alpha_1, \alpha_2]$ such that for all $\alpha \in [\alpha_1, \alpha_2]$, the points $(h(\alpha), \alpha)$ and $(g(\alpha), \alpha) \in [a, b] \times [\alpha_1, \alpha_2]$, then the function*

$$\phi(\alpha) = \int_{h(\alpha)}^{g(\alpha)} f(x, \alpha)\, dx \text{ is differentiable in } [\alpha_1, \alpha_2]$$

and $$\phi'(\alpha) = \left[\int_{h(\alpha)}^{g(\alpha)} f_\alpha(x, \alpha)\, dx\right] - h'(\alpha) f(h(\alpha), \alpha) + g'(\alpha) f(g(\alpha), \alpha).$$

Proof: We have $\phi(\alpha) = \int_{h(\alpha)}^{g(\alpha)} f(x, \alpha)\, dx$

$$\Rightarrow \quad \phi(\alpha + k) - \phi(\alpha) = \int_{h(\alpha+k)}^{g(\alpha+k)} f(x, \alpha + k)\, dx - \int_{h(\alpha)}^{g(\alpha)} f(x, \alpha)\, dx$$

$$= \int_{h(\alpha)}^{g(\alpha)} [f(x, \alpha + k) - f(x, \alpha)]\, dx$$

$$- \int_{h(\alpha)}^{h(\alpha+k)} f(x, \alpha + k)\, dx + \int_{g(\alpha)}^{g(\alpha+k)} f(x, \alpha + k)\, dx$$

$$= k\int_{h(\alpha)}^{g(\alpha)} f_\alpha(x, \alpha + \theta k)\, dx - \{h(\alpha + k) - h(\alpha)\} f(\xi, \alpha + k)$$

$$+ \{g(\alpha + k) - g(\alpha)\} f(\eta, \alpha + k),$$

by applying Lagrange's mean value theorem in the first integral and by using the first mean value theorem of integral calculus in the remaining two integrals, which states that : if f is bounded, then there exists η between a and b such that

$$\int_a^b f(x)\, dx = (b - a) f(\eta).$$

Now dividing the above result by k, we get

$$\frac{\phi(\alpha + k) - \phi(\alpha)}{k} = \int_{h(\alpha)}^{g(\alpha)} f_\alpha(x, \alpha + \theta k)\, dx - \frac{h(\alpha + k) - h(\alpha)}{k} f(\xi, \alpha + k)$$

$$+ \frac{g(\alpha + k) - g(\alpha)}{k} f(\eta, \alpha + k) \qquad \ldots(1)$$

where ξ, η are numbers lying in between $h(\alpha)$, $h(\alpha + k)$ and $g(\alpha)$, $g(\alpha + k)$ respectively.

Now as $k \to 0$, we have

$$\lim_{k \to 0} \frac{\phi(\alpha + k) - \phi(\alpha)}{k} = \lim_{k \to 0} \int_{h(\alpha)}^{g(\alpha)} f_\alpha(x, \alpha + \theta k)\, dx$$

$$- \lim_{k \to 0} \frac{h(\alpha + k) - h(\alpha)}{k} \cdot \lim_{k \to 0} f(\xi, \alpha + k)$$

$$+ \lim_{k \to 0} \frac{g(\alpha + k) - g(\alpha)}{k} \cdot \lim_{k \to 0} f(\eta, \alpha + k)$$

Since as $k \to 0$, $\int_{h(\alpha)}^{g(\alpha)} f_\alpha(x, \alpha + \theta k)\, dx \to \int_{h(\alpha)}^{g(\alpha)} f_\alpha(x, \alpha)\, dx$, $\xi \to h(\alpha)$ and $\eta \to g(\alpha)$, we have

$$\phi'(\alpha) = \int_{h(\alpha)}^{g(\alpha)} f_\alpha(x, \alpha)\, dx - h'(\alpha)\, f(h(\alpha), \alpha) + g'(\alpha)\, f(g(\alpha), \alpha).$$

9.4 Integrability of Proper Integrals as a Function of a Parameter

Theorem: If $f(x, \alpha)$ be continuous in $[a, b] \times [\alpha_1, \alpha_2]$, then

$$\int_{\alpha_1}^{\alpha_2} \left\{ \int_a^b f(x, \alpha)\, dx \right\} d\alpha = \int_a^b \left\{ \int_{\alpha_1}^{\alpha_2} f(x, \alpha)\, d\alpha \right\} dx.$$

Proof: Given f is continuous in $[a, b] \times [\alpha_1, \alpha_2]$.

Therefore both the integrals of $f(x, \alpha)$ with respect to x and α exist and are continuous functions of α and x respectively.

Consider two functions ϕ and ψ defined as

$$\phi(t) = \int_{\alpha_1}^{t} \left\{ \int_a^b f(x, \alpha)\, dx \right\} d\alpha \quad \text{and} \quad \psi(t) = \int_a^b \left\{ \int_{\alpha_1}^{t} f(x, \alpha)\, d\alpha \right\} dx$$

where $t \in [\alpha_1, \alpha_2]$, so that $\phi(c) = \psi(c) = 0$.

By the fundamental theorem of Integral Calculus, we have

$$\phi'(t) = \int_a^b f(x, t)\, dx.$$

By previous theorem, we have

$$\psi'(t) = \int_a^b \left\{ \frac{\partial}{\partial t} \int_{\alpha_1}^{t} f(x, \alpha)\, d\alpha \right\} dx$$

Now $\phi'(t) = \psi'(t)$

$\Rightarrow$ $\phi(t) = \psi(t) + c_1$, where c_1 is constant

$\Rightarrow$ $\phi(t) = \psi(t)$ $\quad [\because \phi(c) = \psi(c) = 0]$

Putting $t = \alpha_2$, we get

$$\int_{\alpha_1}^{\alpha_2} \left\{ \int_a^b f(x, \alpha)\, dx \right\} d\alpha = \int_a^b \left\{ \int_{\alpha_1}^{\alpha_2} f(x, \alpha)\, d\alpha \right\} dx.$$

Illustrative Examples

Example 1: *Show that*

$$\int_0^{\pi} \frac{log\ (1 + \alpha\ cos\ x)}{cos\ x}\ dx = \pi\ sin^{-1}\ \alpha,\ \ if\ |\alpha| < 1.$$

Solution: Let $f\ (x, \alpha) = \dfrac{\log\ (1 + \alpha \cos\ x)}{\cos\ x}$.

We have $$\lim_{x \to \frac{\pi}{2}} \frac{\log\ (1 + \alpha \cos\ x)}{\cos\ x} = \alpha$$

and $$f_\alpha\ (x, \alpha) = \frac{1}{1 + \alpha \cos\ \alpha} \qquad \ldots(1)$$

Here, f and f_α are continuous functions in domain $\{(x, \alpha) : x \in [0, \pi] \text{and } |\alpha| < 1\}$.

Also let

$$\phi\ (\alpha) = \int_0^{\pi} f\ (x, \alpha)\ dx \qquad \ldots(2)$$

Then $$\phi'\ (\alpha) = \int_0^{\pi} f_\alpha\ (x, \alpha)\ dx$$

$$= \int_0^{\pi} \frac{dx}{1 + \alpha \cos\ x} \qquad [\text{From } (1)]$$

$$= \int_0^{\pi} \frac{dx}{\sin^2\ \frac{x}{2} + \cos^2\ \frac{x}{2} + \alpha\left(\cos^2\ \frac{x}{2} - \sin^2\ \frac{x}{2}\right)}$$

$$= \int_0^{\pi} \frac{\sec^2\ \frac{x}{2}\ dx}{(1 + \alpha) + (1 - \alpha)\ \tan^2\ \frac{x}{2}}$$

$$= \int_0^{\infty} \frac{2dt}{(1 + \alpha) + (1 - \alpha)\ t^2} \qquad \left[\text{Putting } \tan\ \frac{x}{2} = t\right]$$

$$= \frac{2}{(1 - \alpha)} \cdot \frac{1}{\sqrt{\frac{1 + \alpha}{1 - \alpha}}} \frac{\pi}{2} = \frac{\pi}{\sqrt{1 - \alpha^2}}.$$

Integrating w.r.t. α, we get

$$\phi\ (\alpha) = \pi \int \frac{d\alpha}{\sqrt{1 - \alpha^2}} + c, \text{ where } c \text{ is an arbitrary constant}$$

$$= \pi\ \sin^{-1}\ \alpha + c \qquad \ldots(3)$$

But from (1), we have

$$\phi(0) = \int_0^{\pi} f(x, 0)\, dx = \int_0^{\pi} \frac{\log 1}{\cos x}\, dx$$

$$= 0 .$$

Putting $\alpha = 0$ in (3), we get

$$\phi(0) = \pi \sin^{-1} 0 + c$$

or $0 = 0 + c$ or $c = 0$

Finally, putting $c = 0$ in (3), we get

$$\phi(\alpha) = \pi \sin^{-1} \alpha$$

or $$\int_0^{\pi} \frac{\log(1 + \alpha \cos x)}{\cos x}\, dx = \pi \sin^{-1} \alpha.$$ [From (2)]

Example 2: *Show that*

$$\int_0^{\pi/2} \log(a^2 \cos^2 \theta + b^2 \sin^2 \theta)\, d\theta = \pi \log\left[\frac{1}{2}(a + b)\right],$$

$$[a > 0, a > b, b > 0].$$

Solution : Let $f(\theta, a) = \log(a^2 \cos^2 \theta + b^2 \sin^2 \theta)$.

Then $$f_a(\theta, a) = \frac{2a \cos^2 \theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} .$$ ...(1)

Here f and f_a are continuous in the domain $\{\theta, a : \theta \in [0, \pi/2] \text{ and } a > 0\}$.

Also let

$$\phi(a) = \int_0^{\pi/2} f(\theta, a)\, d\theta$$

$$= \int_0^{\pi/2} \log(a^2 \cos^2 \theta + b^2 \sin^2 \theta)\, d\theta.$$ ...(2)

Then $$\phi'(a) = \int_0^{\pi/2} f_a(\theta, a)\, d\theta$$

$$= \int_0^{\pi/2} \frac{2a \cos^2 \theta\, d\theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta}$$ [From (1)]

$$= \int_0^{\pi/2} \frac{2a\, d\theta}{a^2 + b^2 \tan^2 \theta}$$

$$= \frac{2ab}{(b^2 - a^2)} \int_0^{\infty} \left[\frac{1}{(a^2 + t^2)} - \frac{1}{(b^2 + t^2)}\right] dt$$ [Putting $t = b \tan \theta$]

$$= \frac{\pi}{(a + b)} .$$

Thus, $$\phi'(a) = \frac{\pi}{(a + b)}$$, [when $a > b$ and $b > 0$].

Integrating w.r.t. a, we get

$$\phi(a) = \int \frac{\pi\, da}{(a+b)} + c, \text{ where } c \text{ is an arbitrary constant}$$

$$= \pi \log (a+b) + c. \qquad \ldots(3)$$

From (3) we have

$$\phi(0) = \pi \log b + c \qquad \ldots(4)$$

Putting $a = 0$ in equation (2), we get

$$\phi(0) = \int_0^{\pi/2} \log (b^2 \sin^2 \theta)\, d\theta$$

$$= \pi \log b + \pi \log \frac{1}{2} \qquad \ldots(5)$$

$$\left[\because \int_0^{\pi/2} \log \sin\theta\, d\theta = \frac{\pi}{2} \log \frac{1}{2}\right]$$

From (4) and (5), we get

$$c = \pi \log \frac{1}{2}.$$

Substituting this value of c in (3), we get

$$\phi(a) = \pi \log (a+b) + \pi \log \frac{1}{2} = \pi \log \left(\frac{a+b}{2}\right)$$

or $$\int_0^{\pi/2} \log (a^2 \cos^2 \theta + b^2 \sin^2 \theta)\, d\theta = \pi \log \left(\frac{a+b}{2}\right), \text{ [From (2)].}$$

Example 3: *Show that* $\int_0^{\alpha} \frac{\log (1+\alpha x)}{1+x^2}\, dx = \frac{1}{2} \log (1+\alpha^2)\, \tan^{-1} \alpha.$

Hence or otherwise show that

$$\int_0^1 \frac{\log (1+x)}{1+x^2}\, dx = \frac{\pi \log 2}{8}.$$

Solution: Here the limits of integration are functions of α, therefore we shall apply the special case for differentiation with respect to a parameter.

Let $$\phi(\alpha) = \int_0^{\alpha} \frac{\log (1+\alpha x)}{1+x^2}\, dx = \int_0^{\alpha} f(x, \alpha)\, dx, \qquad \ldots(1)$$

where $$f(x, \alpha) = \frac{\log (1+\alpha x)}{1+x^2} \qquad \ldots(2)$$

Then $$f_\alpha(x, \alpha) = \frac{x}{(1+\alpha x)(1+x^2)} \qquad \ldots(3)$$

Here since f and f_α are continuous in $[0, \alpha]$.

$\therefore$ $$\phi'(\alpha) = \int_0^{\alpha} f_\alpha(x, \alpha)\, dx - \frac{d}{d\alpha}(0)\, f(0, \alpha) + \frac{d}{d\alpha}(\alpha)\, f(\alpha, \alpha)$$

[By theorem 2 of article 9.3]

$$= \int_0^{\alpha} \frac{x\,dx}{(1+\alpha x)(1+x^2)} + \frac{\log(1+\alpha^2)}{1+\alpha^2} \qquad \text{[From (3)]}$$

$$= \frac{1}{(1+\alpha^2)} \int_0^{\alpha} \frac{(x+\alpha)}{(1+x^2)}\,dx - \frac{\alpha}{(1+\alpha^2)} \int_0^{\alpha} \frac{dx}{(1+\alpha x)} + \frac{\log(1+\alpha^2)}{(1+\alpha^2)}$$

$$= \frac{1}{2(1+\alpha^2)} [\log(1+x^2)]_0^{\alpha} + \frac{\alpha}{(1+\alpha^2)} [\tan^{-1} x]_0^{\alpha}$$

$$- \frac{1}{(1+\alpha^2)} [\log(1+\alpha x)]_0^{\alpha} + \frac{\log(1+\alpha^2)}{1+\alpha^2}$$

$$= \frac{\log(1+\alpha^2)}{2(1+\alpha^2)} + \frac{\alpha}{1+\alpha^2} \tan^{-1} \alpha.$$

Integrating w.r.t. α, we get

$$\phi(\alpha) = \int \frac{\log(1+\alpha^2)}{2(1+\alpha^2)}\,d\alpha + \int \frac{\alpha \tan^{-1} \alpha}{1+\alpha^2}\,d\alpha$$

$$= \frac{1}{2} \log(1+\alpha^2) \tan^{-1} \alpha - \int \frac{\alpha \tan^{-1} \alpha}{1+\alpha^2}\,d\alpha + \int \frac{\alpha \tan^{-1} \alpha}{1+\alpha^2}\,d\alpha + c$$

$$= \frac{1}{2} \log(1+\alpha^2) \tan^{-1} \alpha + c. \qquad \ldots(4)$$

where c is an arbitrary constant.

Putting $\alpha = 0$ in (4) and (2), we get

$$\phi(0) = c$$

and $$\phi(0) = 0$$

$\therefore$ $$c = 0.$$

Substituting $c = 0$ in equation (4), we get

$$\phi(\alpha) = \frac{1}{2} \log(1+\alpha^2) \tan^{-1} \alpha$$

or $$\int_0^{\alpha} \frac{\log(1+\alpha x)}{1+x^2}\,dx = \frac{1}{2} \log(1+\alpha^2) \tan^{-1} \alpha \qquad \ldots(5)$$

Putting $\alpha = 1$ in (5), we get

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{1}{2} \log(1+1) \tan^{-1} 1$$

$$= \frac{1}{2} \cdot \frac{\pi}{4} \log(2)$$

$$= \frac{\pi}{8} \log 2.$$

Comprehensive Exercise 1

1. Compute the following integral by differentiating with respect to parameter:

$$\int_0^1 \frac{x^\alpha - 1}{\log x}\,dx \text{ , for } \alpha > -1 .$$

2. Evaluate $\int_0^{\pi/2} \log\left(\frac{1+\alpha \sin x}{1-\alpha \sin x}\right)\frac{dx}{\sin x}$, for $\alpha^2 < 1$.

3. Evaluate $\int_0^{\pi} \frac{dx}{a + b\cos x}$; $a > 0$, $|b| < a$.

Also deduce that $\int_0^{\pi} \frac{dx}{(a+b\cos x)^2} = \frac{\pi a}{(a^2 - b^2)^{3/2}}$

and $\int_0^{\pi} \frac{\cos x\,dx}{(a+b\cos x)^2} = \frac{-\pi b}{(a^2 - b^2)^{3/2}}$.

4. Show that $F(\alpha) = \int_0^{\pi} \frac{\log(1+\cos\alpha \sin x)}{\sin x}\,dx = \frac{\pi^2 - 4\alpha^2}{4}\ \forall\, \alpha \in [-\alpha, \alpha]$

5. Prove that $\int_0^{\pi/2} \frac{\log(1+\cos\alpha\cos x)}{\cos x}\,dx = \frac{\pi^2 - 4\alpha^2}{8}$.

6. Show that for $\alpha^2 \le 1$,

$$F(\alpha) = \int_0^{\pi/2} \log(1 - \alpha^2 \sin^2 x)\,dx = \pi\{\log(1+\sqrt{1-\alpha^2}) - \log 2\}.$$

7. Show that

$$\int_0^{\pi/2} \log(1 - x^2\cos^2\theta)\,d\theta = \pi \log\left[1+\sqrt{(1+x^2)}\right] - \pi\log 2 \text{ , if } x^2 \le 1 .$$

8. If f is continuous in $[a, b] \times [\alpha_1, \alpha_2]$

and $\phi(x, \alpha) = \int_{\alpha_1}^{\alpha_2} \left\{\int_a^x f(x, \alpha)\,dx\right\} d\alpha$,

show that

$$\frac{\partial^2 \phi}{\partial x\,\partial \alpha} = \frac{\partial^2 \phi}{\partial \alpha\,\partial x} = f(x, \alpha)\ \forall\, x \in [a, b] \text{ and } \alpha \in [\alpha_1, \alpha_2] .$$

Deduce that

$$\int_a^b \left\{\int_{\alpha_1}^{\alpha_2} f(x, \alpha)\,d\alpha\right\} dx = \phi(a, \alpha_1) + \phi(b, \alpha_2) - \phi(a, \alpha_2) - \phi(b, \alpha_1)$$

$$= \int_{\alpha_1}^{\alpha_2} \left\{\int_a^b f(x, \alpha)\,dx\right\} d\alpha.$$

9. If $f\ (x, \alpha)$ is continuous in $[a \times b] \times [\alpha_1, \alpha_2]$ and F is a function of x which is bounded and integrable in $[a, b]$, show that

$$\int_a^b f\ (x, \alpha)\ F\ (x)\, dx \text{ is continuous in } [\alpha_1, \alpha_2].$$

Answers 1

1. $\log\ (1 + \alpha)$, 2. $\pi \sin^{-1} \alpha$. 3. $\frac{\pi}{a^2 - b^2}$

9.5 Improper Integrals as a Function of a Parameter

Uniform convergence of improper integrals when range is infinite.

The improper integral $\phi\ (\alpha) = \int_a^\infty f\ (x,\ \alpha)\, dx$

which is an infinite series of functions of parameter α, is said to be uniformly convergent in the interval $[\alpha_1, \alpha_2]$ if for every $\varepsilon > 0$, there exists a positive number δ depending on ε but not on α such that

$$|\ \phi\ (\alpha) - \int_a^t f\ (x, \alpha)\, dx\ | \ = |\ \int_t^\infty f\ (x, \alpha)\, dx\ | < \varepsilon$$

$$\forall\, t \geq \delta \text{ and } \alpha \in [\alpha_1, \alpha_2].$$

Uniform convergence of improper integrals when the integrand is unbounded.

The improper integral $\phi\ (\alpha) = \int_a^b f\ (x, \alpha)\, dx$

such that $f\ (x, \alpha) \to \infty$, when $x \to a$ for some $\alpha \in [\alpha_1, \alpha_2]$ is said to be uniformly convergent in the interval $[\alpha_1, \alpha_2]$ if for every $\varepsilon > 0$,

there exists a positive number δ depending on ε but not on α such that

$$|\ \phi\ (\alpha) - \int_{a+t}^b f\ (x, \alpha)\, dx\ | < \varepsilon, \forall\, 0 \leq t \leq \delta.$$

Note: Weierstrass's 'M' test can be applied to test uniform convergence.

Thus, if there exists a function $M\ (x) > 0$ for all x, such that $\int_a^\infty M\ (x)\, dx$

converges and for all α and x

$$|\ f\ (x, \alpha)\ | \leq M\ (x),$$

then the improper integral

$$\phi\ (\alpha) = \int_a^\infty f\ \ (x, \alpha)\, dx$$

is uniformly convergent.

9.6 Continuity of Improper Integrals as a Function of a Parameter

Theorem: *The improper integral*

$$\phi(\alpha) = \int_a^{\infty} f(x, \alpha)\, dx$$

is continuous in $[\alpha_1, \alpha_2]$ if

(i) $f(x, \alpha)$ *is continuous for* $x \geq \alpha$ *and* $\alpha \in [\alpha_1, \alpha_2]$

and (ii) $\int_a^{\infty} f(x, \alpha)\, dx$ *is uniformly convergent for* $\alpha \in [\alpha_1, \alpha_2]$.

Proof: Let $f(x, \alpha)$ be continuous for $x \geq \alpha$ and $\alpha \in [\alpha_1, \alpha_2]$. Also let

$$\phi(\alpha) = \int_a^{\infty} f(x, \alpha)\, dx \qquad \ldots(1)$$

be uniformly convergent for $\alpha \in [\alpha_1, \alpha_2]$.

Then, for a given $\varepsilon > 0$ there exists a positive number δ which does not depend on α such that

$$|\phi(\alpha) - \int_a^t f(x, \alpha)\, dx| = |\int_t^{\infty} f(x, \alpha)\, dx| < \frac{\varepsilon}{3}, \forall\, t \geq \delta$$

$$\therefore \quad |\phi(\alpha) - \int_a^t f(x, \alpha)\, dx| < \frac{\varepsilon}{3}$$

and $$|\phi(\alpha + h) - \int_a^t f(x, \alpha + h)\, dx| < \frac{\varepsilon}{3}, \forall\, t \geq \delta$$

Thus

$$|\phi(\alpha + h) - \phi(\alpha)| \leq |\int_a^t [f(x, \alpha + h) - f(x, \alpha)]\, dx| + \frac{2\varepsilon}{3}. \qquad \ldots(2)$$

Since f is continuous, therefore we choose a positive number δ_1, such that

$$|f(x, \alpha + h) - f(x, \alpha)| < \frac{\varepsilon}{3(t - a)}, \text{ when } |h| \leq \delta_1,$$

or $$\int_a^b |f(x, \alpha + h) - f(x, \alpha)|\, dx < \frac{\varepsilon}{3}, \text{ when } |h| \leq \delta_1$$

[Using theorem 1 of article 9.3]

Putting this value in equation (2), we get

$$|\phi(\alpha + h) - \phi(\alpha)| < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3},$$

when $|h| \leq \delta_1$

$\Rightarrow$ $|\phi(\alpha + h) - \phi(\alpha)| < \varepsilon$, when $|h| \leq \delta_1$

$\Rightarrow$ $\phi(\alpha)$ is continuous in $[\alpha_1, \alpha_2]$.

9.7 Integrability of Improper Integrals as a Function of a Parameter

Theorem: *If (i)* $f(x, \alpha)$ *is continuous for* $x \geq \alpha$ and $\alpha \in [\alpha_1, \alpha_2]$ *and*

(ii) $\int_a^\infty f(x, \alpha)\, dx$ *is uniformly convergent for* $\alpha \in [\alpha_1, \alpha_2]$,

then the improper integral $\phi(\alpha) = \int_a^\infty f(x, \alpha)\, dx$

can be integrated under the integral sign and we have

$$\int_{\alpha_1}^{\alpha_2} \left\{ \int_a^\infty f(x, \alpha)\, dx \right\} d\alpha = \int_{\alpha_1}^{\alpha_2} \phi(\alpha)\, d\alpha = \int_a^\infty \left\{ \int_{\alpha_1}^{\alpha_2} f(x, \alpha)\, d\alpha \right\} dx$$

Proof: Let $\phi(\alpha) = \int_a^\infty f(x, \alpha)\, dx$ ...(1)

be uniformly convergent.

Then, $\left| \int_t^\infty f(x, \alpha)\, dx \right| < \dfrac{\varepsilon}{(\alpha_2 - \alpha_1)}, \forall\, t \geq n > 0$

$\Rightarrow$ $\int_{\alpha_1}^{\alpha_2} \left| \int_t^\infty f(x, \alpha)\, dx \right| d\alpha < \int_{\alpha_1}^{\alpha_2} \dfrac{\varepsilon}{(\alpha_2 - \alpha_1)}, \forall t \geq n$

$\Rightarrow$ $\int_{\alpha_1}^{\alpha_2} \left| \int_t^\infty f(x, \alpha)\, dx \right| d\alpha < \varepsilon, \forall\, t \geq n$ ...(2)

We have

$$\int_{\alpha_1}^{\alpha_2} \phi(\alpha)\, d\alpha = \int_{\alpha_1}^{\alpha_2} \left\{ \int_a^t f(x, \alpha)\, dx \right\} d\alpha + \int_{\alpha_1}^{\alpha_2} \left\{ \int_t^\infty f(x, \alpha)\, dx \right\} d\alpha$$

$$= \int_a^t \left\{ \int_{\alpha_1}^{\alpha_2} f(x, \alpha)\, d\alpha \right\} dx + \int_{\alpha_1}^{\alpha_2} \left\{ \int_t^\infty f(x, \alpha)\, dx \right\} d\alpha$$

[By theorem of article 9.4 we have

$$\int_{\alpha_1}^{\alpha_2} \left\{ \int_a^t f(x, \alpha)\, dx \right\} d\alpha = \int_a^t \left\{ \int_{\alpha_1}^{\alpha_2} f(x, \alpha)\, d\alpha \right\} dx \Big]$$

$\therefore$ $\left| \int_{\alpha_1}^{\alpha_2} \phi(\alpha)\, d\alpha - \int_a^t \left\{ \int_{\alpha_1}^{\alpha_2} f(x, \alpha)\, d\alpha \right\} dx \right| = \left| \int_{\alpha_1}^{\alpha_2} \left\{ \int_t^\infty f(x, \alpha)\, dx \right\} d\alpha \right|$

$< \varepsilon$, when $t \geq n$ [From 2]

When $t \to \infty$, we have $\varepsilon \to 0$ and so

$$\int_{\alpha_1}^{\alpha_2} \phi(\alpha)\, d\alpha = \int_a^\infty \left\{ \int_{\alpha_1}^{\alpha_2} f(x, \alpha)\, d\alpha \right\} dx$$

From (1), $\int_{\alpha_1}^{\alpha_2} \phi(\alpha)\, d\alpha = \int_{\alpha_1}^{\alpha_2} \left\{ \int_a^\infty f(x, \alpha)\, dx \right\} d\alpha$

Thus $\int_{\alpha_1}^{\alpha_2} \left\{ \int_a^\infty f(x, \alpha)\, dx \right\} d\alpha = \int_{\alpha_1}^{\alpha_2} \phi(\alpha)\, d\alpha = \int_a^\infty \left\{ \int_{\alpha_1}^{\alpha_2} f(x, \alpha)\, d\alpha \right\} dx$

9.8 Derivability of Improper Integrals as a Function of a Parameter

Theorem : *If (i) $f\ (x, \alpha)$ is continuous and has a continuous partial derivative with respect to α for $x \geq a$ and $\alpha \in [\alpha_1, \alpha_2]$ and (ii) $\int_a^\infty f_\alpha\ (x, \alpha)\ dx$ converges uniformly in $\alpha \in [\alpha_1, \alpha_2]$, then $\phi\ (\alpha)$ is differentiable and*

$$\phi'\ (\alpha) = \int_a^\infty f_\alpha\ (x, \alpha)\ dx.$$

Proof : Let $f\ (x, \alpha)$ is continuous and has a continuous partial derivative with respect to α for $x \geq a$ and $\alpha \in [\alpha_1, \alpha_2]$. Also let $\int_a^\infty f_\alpha\ (x, \alpha)\ dx$ converges uniformly in $\alpha \in [\alpha_1, \alpha_2]$.

Take $\psi\ (\alpha) = \int_a^\infty f_\alpha\ (x, \alpha)\ dx.$

Then $$\int_{\alpha_1}^{\alpha} \psi\ (\alpha)\ d\alpha = \int_{\alpha_1}^{\alpha} \left\{\int_a^\infty f_\alpha\ (x, \alpha)\ dx\right\} d\alpha$$

$$= \int_a^\infty \left\{\int_{\alpha_1}^{\alpha} f_\alpha\ (x, \alpha)\ d\alpha\right\} dx$$

[By theorem of article 9.7, changing the order of integration]

$$= \int_a^\infty [f\ (x, \alpha) - f\ (x, \alpha_1)]\ dx$$

$$= \phi\ (\alpha) - \phi\ (\alpha_1).$$

Differentiating both sides with respect to α, we get

$$\psi\ (\alpha) = \phi'\ (\alpha) \quad \text{or} \quad \phi'\ (\alpha) = \int_a^\infty f_\alpha\ (x, \alpha)\ dx.$$

Note : The results for improper integrals hold in similar manner when the range of integration is finite but the integrand is unbounded.

Illustrative Examples

Example 1: *Evaluate $\int_0^\infty e^{-x^2} \cos \alpha x\ dx$ with differentiation w.r.t. parameter.*

Solution: Let $\phi\ (\alpha) = \int_0^\infty f\ (x, \alpha)\ dx$ ...(1)

where $f\ (x, \alpha) = e^{-x^2} \cos \alpha x$

Then $f_\alpha\ (x, \alpha) = -x\, e^{-x^2} \sin \alpha x$

We have $|x e^{-x^2} \sin \alpha x| \leq x e^{-x^2}$

$\Rightarrow \quad |e^{-x^2} \cos \alpha x| \leq e^{-x^2}$ $\quad [\because |\cos \alpha x| \leq 1]$

Both the integrals $\int_0^\infty e^{-x^2} dx$ and $\int_0^\infty x e^{-x^2} dx$

are convergent, therefore by Weierstrass M test

$\int_0^\infty f(x, \alpha)\, dx$ and $\int_0^\infty f_\alpha(x, \alpha)\, dx$ are uniformly convergent,

$\Rightarrow \quad \phi'(\alpha)$ exists.

Now $\quad \phi'(\alpha) = \int_0^\infty f_\alpha(x, \alpha)\, dx = \int_0^\infty -x e^{-x^2} \sin \alpha x\, dx$

$$= \left[\frac{1}{2} e^{-x^2} \sin x\right]_0^\infty - \frac{\alpha}{2} \int_0^\infty e^{-x^2} \cos \alpha x\, dx, \quad \text{[Integrating by parts]}$$

$$= -\frac{\alpha}{2} \phi(\alpha) \quad \text{or} \quad \frac{\phi'(\alpha)}{\phi(\alpha)} = -\frac{\alpha}{2}.$$

Integrating w.r.t. α, we get

$$\log \phi(\alpha) = -\frac{\alpha^2}{4} + c_1,$$

where c_1 is an arbitrary constant.

or $\quad \phi(\alpha) = c\, e^{-\alpha^2/4}, \quad \ldots(2)$

where c is another constant

Putting $\alpha = 0$ in (1) and (2), we get

$$\phi(0) = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \quad \text{and} \quad \phi(0) = c\, e^0 = c \;\therefore\; c = \frac{\sqrt{\pi}}{2}.$$

Substituting this value of c in (2), we get

$$\phi(\alpha) = \frac{\sqrt{\pi}}{2} e^{-\alpha^2/4}$$

Hence $\quad \int_0^\infty e^{-x^2} \cos \alpha x\, dx = \frac{\sqrt{\pi}}{2} e^{-\alpha^2/4}$ $\quad$ [From (1)]

Example 2: *Evaluate* $f(\alpha, \beta) = \int_0^\infty e^{-\alpha x} \frac{\sin \beta x}{x} dx$, *where* $\alpha \geq 0$. *Hence deduce that*

$$\int_0^\infty \frac{\sin \beta x}{x} dx = \begin{cases} \pi/2, & \text{if } \beta > 0 \\ 0, & \text{if } \beta = 0. \\ -\pi/2, & \text{if } \beta < 0 \end{cases}$$

Solution: We have

$$\left| e^{-\alpha x} \frac{\sin \beta x}{x} \right| \leq \frac{e^{-\alpha x}}{x} \quad \text{for } x > 0.$$

$\int_0^\infty \frac{e^{-\alpha x}}{x} dx$ is convergent at ∞ if $\alpha > 0$;

therefore by Weierstrass M test,

$\int_0^\infty e^{-\alpha x} \frac{\sin \beta x}{x} dx$ is uniformly convergent.

Now the derivative of $\frac{e^{-\alpha x} \sin \beta x}{x}$ with respect to β is $e^{-\alpha x} \cos \beta x$. For a fixed $\alpha > 0$, we have

$$|e^{-\alpha x} \cos \beta x| \leq e^{-\alpha x} \qquad [\because |\cos \beta x| \leq 1]$$

But $\int_0^\infty e^{-\alpha x} dx$ is convergent over $[0, \infty)$ therefore for a fixed α we have

$\int_0^\infty e^{-\alpha x} \cos \beta x \, dx$ is uniformly convergent

$\therefore \quad f_\beta (\alpha, \beta)$ exists.

Thus $f_\beta (\alpha, \beta) = \int_0^\infty e^{-\alpha x} \cos \beta x \, dx$

$$= [e^{-\alpha x} \{\beta \sin \beta x - \alpha \cos \beta x\}]_0^\infty = \frac{\alpha}{(\alpha^2 + \beta^2)}.$$

Integrating with respect to β, we get

$$f (\alpha, \beta) = \tan^{-1} \frac{\beta}{\alpha} + c, \qquad \ldots(1)$$

where c is an arbitrary constant.

Putting $\beta = 0$ in (1), we get

$$0 = f (\alpha, 0) = 0 + c \quad \text{or} \quad c = 0 \quad \therefore \quad f (\alpha, \beta) = \tan^{-1} \frac{\beta}{\alpha}.$$

Deduction: For $\alpha > 0$ and for a fixed β, we have

$$\left| \int_{t_1}^{t} e^{-\alpha x} \frac{\sin \beta x}{x} dx \right| = \left| e^{-\alpha t} \int_{t_1}^{\xi} \frac{\sin \beta x}{x} dx \right| \leq \left| \int_{t_1}^{\xi} \frac{\sin \beta x}{x} dx \right|,$$

[Applying the Bonnett's form of second mean value theorem]

Since $\int_0^\infty \frac{\sin \beta x}{x} dx$ is convergent, we have $\int_0^\infty e^{-\alpha x} \frac{\sin \beta x}{x} dx$ is uniformly convergent with α as a parameter.

$$\therefore \quad \lim_{x \to 0} f (\alpha, \beta) = f (0, \beta) \qquad \ldots(2)$$

But when $\alpha \to 0$ from positive side and $\beta > 0$, we have

$$\lim_{\alpha \to 0} f (\alpha, \beta) = \lim_{\alpha \to 0} \int_0^\infty e^{-\alpha x} \frac{\sin \beta x}{x} dx = \frac{\pi}{2} \qquad \ldots(3)$$

From (2) and (3), we get

$$f (0, \beta) = \frac{\pi}{2}, \text{ if } \beta > 0$$

or $$\int_0^\infty \frac{\sin \beta x}{x}\, dx = \frac{\pi}{2}, \text{ if } \beta > 0.$$

Similarly $$\int_0^\infty \frac{\sin \beta x}{x}\, dx = 0, \text{ if } \beta = 0 \quad \text{and} \quad \int_0^\infty \frac{\sin \beta x}{x}\, dx = -\frac{\pi}{2}, \text{ if } \beta < 0.$$

Example 3: *Evaluate* $\int_0^\infty \frac{1 - e^{-\alpha x}}{x\, e^x}\, dx$ *if* $a > -1$, *with the help of differentiation w.r.t. parameter.*

Solution: Let $\phi(\alpha) = \int_0^\infty \frac{1 - e^{-\alpha x}}{x\, e^x}\, dx = \int_0^\infty f(x, \alpha)\, dx$ ...(1)

Then $f_\alpha(x, \alpha) = \frac{e^{-\alpha x}}{e^x} = e^{-(\alpha + 1)x}$ ...(2)

Now $\int_0^\infty f(x, \alpha)\, dx$ and $\int_0^\infty f_\alpha(x, \alpha)\, dx$ are uniformly convergent.

$\Rightarrow$ $\phi'(\alpha)$ exists.

$\therefore$ $$\phi'(\alpha) = \int_0^\infty f_\alpha(x, \alpha)\, dx = \int_0^\infty e^{-(\alpha + 1)x}\, dx$$

$$= \left[\frac{e^{-(\alpha + 1)x}}{-(\alpha + 1)}\right]_0^\infty = \frac{1}{\alpha + 1}, \text{ if } \alpha > -1.$$

Integrating w.r.t. α, we get

$$\phi(\alpha) = \log(1 + \alpha) + c, \quad \text{where } c \text{ is an arbitrary constant} \qquad ....(3)$$

$\Rightarrow$ $$\phi(0) = c \qquad ...(4)$$

Putting $\alpha = 0$ in (1), we have

$$\phi(0) = 0 \qquad \therefore \qquad c = 0$$

Hence $\phi(\alpha) = \log(1 + \alpha)$

or $$\int_0^\infty \frac{1 - e^{-\alpha x}}{x\, e^x}\, dx = \log(1 + \alpha); \text{ if } \alpha > -1.$$

Example 4: *Evaluate* $\int_0^1 \frac{\log(1 - \alpha^2 x^2)}{x^2 \sqrt{1 - x^2}}\, dx$ *by using differentiation with respect to parameter.*

Solution : Let $\phi(\alpha) = \int_0^1 \frac{\log(1 - \alpha^2 x^2)}{\sqrt{1 - x^2}}\, dx = \int_0^1 f(x, \alpha)\, dx$ ...(1)

where $f(x, \alpha) = \frac{\log(1 - \alpha^2 x^2)}{\sqrt{1 - x^2}}$. Here $f(x, \alpha)$ is unbounded in $[0, 1]$.

We have $f_\alpha = \frac{-2\alpha x^2}{x^2 (1 - \alpha^2 x^2)\sqrt{1 - x^2}} = \frac{-2x}{(1 - \alpha^2 x^2)\sqrt{1 - x^2}}$ ...(2)

Both f and f_α are uniformly convergent, therefore $\phi'(\alpha)$ exists

Now $$\phi'(\alpha) = \int_0^1 \frac{-2\alpha x^2\, dx}{x^2(1-\alpha^2x^2)\sqrt{(1-x^2)}}$$

$$= \int_0^{\pi/2} \frac{-2\alpha \sec^2\theta\, d\theta}{(1-\alpha^2)\tan^2\theta + 1} \qquad [\text{Putting } x = \sin\theta]$$

$$= \int_0^{\infty} \frac{-2\alpha\, dt}{\sqrt{1-\alpha^2}\,(t^2+1)} \qquad [\text{Putting } \sqrt{1-\alpha^2}\tan\theta = t]$$

$$= \frac{-\pi\alpha}{\sqrt{1-\alpha^2}}.$$

Integrating with respect to α, we get

$$\phi(\alpha) = \int \frac{-\pi\alpha}{\sqrt{1-\alpha^2}}\, d\alpha + c = \pi\sqrt{1-\alpha^2} + c \qquad \ldots(3)$$

where c is an arbitrary constant.

Putting $\alpha = 0$ in (3), we get $0 = \phi(0) = \pi + c$ or $c = -\pi$.

Substituting the value of c in equation (3), we get

$$\phi(\alpha) = \pi\sqrt{1-\alpha^2} - \pi = \pi\{\sqrt{1-\alpha^2} - 1\}$$

or $$\int_0^1 \frac{\log(1-\alpha^2x^2)}{x^2\sqrt{1-x^2}}\, dx = \pi\{\sqrt{1-\alpha^2} - 1\}.$$

Comprehensive Exercise 2

1. Evaluate $\int_0^{\infty} \frac{\cos xy}{1+x^2}\, dx$ with the help of differentiation w.r.t., parameter.

2. Show that $\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x}\, dx = \log\frac{b}{a}$, $(a > 0, b > 0)$;

   if $\frac{1}{y} = \int_0^{\infty} e^{-xy}\, dx$, $y > 0$.

3. Evaluate $\int_0^1 \frac{\tan^{-1}\alpha x}{x\sqrt{1-x^2}}\, dx$ by using differentiation with respect to parameter.

4. Prove that the differential equation

   $y'' + y = \frac{1}{x}$ is satisfied by the function $y = \int_0^{\infty} \frac{e^{-xz}}{1+z^2}\, dz$.

5. Applying the differentiation with respect to parameter, evaluate $\int_0^\infty x^3 e^{-\alpha x^2} dx$.

6. Evaluate $\int_0^\infty e^{-x^2} \cos \alpha x \, dx$ using differentiation with respect to parameter.

7. Show that $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \cos mx \, dx = \frac{1}{2} \log \frac{b^2 + m^2}{a^2 + m^2}$, where $a, b > 0$.

8. If $\beta > \alpha > 0$, and $p > 0$, show that

$$\int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x} \sin px \, dx = \tan^{-1} \frac{\beta}{p} - \tan^{-1} \frac{\alpha}{p}.$$

Hence deduce that

(i) $\int_0^\infty \frac{\sin px}{x} dx = \frac{\pi}{2}$, if $p > 0$,

(ii) $\int_0^\infty \frac{1 - e^{-\alpha x}}{x} \sin px \, dx = \tan^{-1} \frac{\alpha}{p}$, if $\alpha > 0$, $p \neq 0$.

9. Show that $I = \int_0^\infty e^{-xt} \left(\frac{\cos ax - \cos bx}{x} \right) dx$ converges uniformly on $t \geq 0$.

Hence show that $I = \frac{1}{2} \log \frac{b^2 + t^2}{a^2 + t^2}$ when $t > 0$; and $ab \neq 0$ when $t = 0$.

Answers 2

1. $\frac{\pi}{2} e^{-\alpha}$ 3. $\frac{\pi}{2} \log [\alpha + \sqrt{1 + \alpha^2}]$ 5. $\frac{1}{2\alpha^2}$ 6. $\frac{\sqrt{\pi}}{2} e^{-\alpha^2/4}$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. Let $f(x, \alpha)$ be continuous for $x \geq \alpha$ and $\alpha \in [\alpha_1, \alpha_2]$. Also let $\int_a^\infty f(x, \alpha) dx$, be uniformly convergent for $\alpha \in [\alpha_1, \alpha_2]$, then $\phi(\alpha)$ can be integrated under the integral sign. Which of the following statements is correct in this case

(a) $\int_{\alpha_1}^{\alpha_2} \left\{ \int_a^\infty f(x, \alpha) dx \right\} d\alpha = \int_a^\infty \left\{ \int_{\alpha_1}^{\alpha_2} f(x, \alpha) d\alpha \right\} dx$

(b) $\int_{\alpha_1}^{\alpha_2}\left\{\int_a^{\infty} f(x,\alpha)\,dx\right\}d\alpha = \int_{\alpha_1}^{\alpha_2}\phi(\alpha)\,dx$

(c) both (a) and (b) are correct

(d) none of these.

2. If f is continuous in $[a,b]\times[\alpha_1,\alpha_2]$, then we have

(a) $\int_{\alpha_1}^{\alpha_2}\left\{\int_a^b f(x,\alpha)\,d\alpha\right\}dx = \int_a^b\left|\int_{\alpha_1}^{\alpha_2} f(x,\alpha)\,d\alpha\right|dx$

(b) $\int_{\alpha_1}^{\alpha_2}\left\{\int_a^b f(x,\alpha)\,dx\right\}d\alpha = \int_a^b\left\{\int_{\alpha_1}^{\alpha_2} f(x,\alpha)\,d\alpha\right\}dx$

(c) both (a) and (b) are correct

(d) none of these.

3. If $f(x,\alpha)=\dfrac{\partial^2\phi}{\partial x\,\partial\alpha}$, then we have

(a) $\int_{\alpha_1}^{\alpha_2} f(x,\alpha)\,d\alpha = \phi_x(x,\alpha_1)-\phi_x(x,\alpha_2)$

(b) $\int_{\alpha_1}^{\alpha_2} f(x,\alpha)\,d\alpha = \phi_x(x,\alpha_2)-\phi_x(x,\alpha_1)$

(c) $\int_{\alpha_1}^{\alpha_2} f(x,\alpha)\,dx = \phi_x(x,\alpha_1)-\phi_x(x,\alpha_2)$

(d) none of these.

4. If F is a function of x which is bounded and integrable in $[a,b]$ and $f(x,\alpha)$ is continuous in $[a,b]\times[\alpha_1,\alpha_2]$, then

(a) $\int_a^b f(x,\alpha)\,dx$ is continuous in $[\alpha_1,\alpha_2]$

(b) $\int_a^b f(x,\alpha)\,dx$ is continuous in (α_1,α_2)

(c) $\int_a^b f(x,\alpha)\,F(x)\,dx$ is continuous in $[\alpha_1,\alpha_2]$

(d) $\int_a^b f(x,\alpha)\,F(x)\,dx$ is continuous in (α_1,α_2).

5. $\int_0^{\pi}\dfrac{\log(1+\alpha\cos x)}{\cos x}\,dx = \pi\sin^{-1}\alpha$

(a) If $-1<\alpha<1$ (b) If $-1\le\alpha\le1$

(c) If $-\infty<\alpha<\infty$ (d) If $-\infty\le\alpha\le\infty$.

Fill in the Blank(s)

Fill in the blanks "....." so that the following statements are complete and correct.

1. If $\dfrac{1}{y}=\int_0^{\infty} e^{-xy}\,dx$, $y>0$ then $\int_0^{\infty}\dfrac{e^{-ax}-e^{-bx}}{x}\,dx =$ .....

2. If $\alpha>-1$, then $\int_0^1\dfrac{x^{\alpha}-1}{\log x}\,dx =$ .....

3. If $\alpha^2 < 1$, then $\int_0^{\pi/2} \log\left(\frac{1+\alpha \sin x}{1-\alpha \sin x}\right) \frac{dx}{\sin x} =$ .....

4. $\int_0^1 \frac{\tan^{-1} \alpha x}{x\sqrt{1-x^2}} =$ .....

True or False

Write 'T' for true and 'F' for false statement.

1. Let $f(x, \alpha)$ and $f_\alpha(x, \alpha)$ be continuous functions in $[a, b] \times [\alpha_1, \alpha_2]$, then ϕ is derivable and $\phi'(\alpha) = \int_a^b f_\alpha(x, \alpha)\, dx, \forall \alpha \in [\alpha_1, \alpha_2]$.

2. Let $f(x, \alpha)$ be continuous in $[a, b] \times [\alpha_1, \alpha_2]$ and

$$\phi(x, \alpha) = \int_{\alpha_1}^{\alpha_2} \left\{\int_a^x f(x, \alpha)\, dx\right\} d\alpha, \text{ then } \frac{\partial^2 \phi}{\partial x\, \partial \alpha} = \frac{\partial^2 \phi}{\partial \alpha\, \partial x} = f(x, \alpha)$$

$\forall x \in [a, b], \alpha \in [\alpha_1, \alpha_2]$.

3. If f is continuous in $[a, b] \times [\alpha_1, \alpha_2]$, then we have

$$\int_{\alpha_1}^{\alpha_2} \left\{\int_a^b f(x, \alpha)\, d\alpha\right\} dx = \int_a^b \left|\int_{\alpha_1}^{\alpha_2} f(x, \alpha)\, d\alpha\right| dx.$$

4. Let $f(x, \alpha)$ be continuous for $x \geq \alpha$ and $\alpha \in [\alpha_1, \alpha_2]$. Also if $\int_a^\infty f(x, \alpha)\, dx$ is uniformly convergent for $\alpha \in [\alpha_1, \alpha_2]$, then $\phi(\alpha)$ cannot be integrated as under the integral sign.

Answers

Multiple Choice Questions

1. (c) 2. (b) 3. (b) 4. (c) 5. (a)

Fill in the Blank(s)

1. $\log \frac{b}{a}$ 2. $\log(1+\alpha)$
3. $\pi \sin^{-1} \alpha$ 4. $\frac{\pi}{2} \log[\alpha + \sqrt{1+\alpha^2}]$

True or False

1. *T* 2. *T* 3. *F* 4. *F*

□

Chapter

Metric Spaces

10.1 Metric

Definition: *Let X be a non-empty set. A mapping d of $X \times X$ into $\mathbf{R}$ (the set of reals) i.e., $d : X \times X \to \mathbf{R}$ is said to be a* **metric (or distance function)** *on X iff d satisfies the following axioms :*

[m 1] : $d(x, y) \geq 0 \quad \forall\ x, y \in X$.

[m 2] : $d(x, y) = 0$ *if and only if* $x = y$.

[m 3] : $d(x, y) = d(y, x) \quad \forall\ x, y \in X$. **(symmetry)**

[m 4] : $d(x, y) \leq d(x, z) + d(z, y) \quad \forall\ x, y, z \in X$. **(triangle inequality)**

If d is a metric for X, then the ordered pair (X, d) is called a **metric space** *and $d(x, y)$ is called the distance between x and y.*

Thus a *metric space* is a non-empty set X equipped with a metric d on X and is denoted by the pair (X, d) or simply by X. Different metrics can be defined on a single non-empty set and this gives rise to distinct metric spaces.

Note that if $x \neq y$, then **[m 1]** and **[m 2]** imply that $d(x, y) > 0$ so that the distance between two distinct points is positive.

The axiom **[m 1]** states that the distance between any two points of X is a non-negative real number. The axiom **[m 2]** means that if the two points coincide then the distance is zero and if the distance is zero, the two points are the same. The axiom **[m 3]** states that the distance does not depend on the order of the points x and y. The axiom **[m 4]**, known as the **triangle inequality**, states figuratively speaking, that the sum of the lengths of two sides of a triangle is greater than or equal to the length of the third side, the sign of equality occurring when the three points are in a straight line.

Pseudo-metric. Definition: *A mapping d of $X \times X$ into* **R** *is called a* **pseudo-metric** *(or* **semi-metric***) for X iff d satisfies the axioms* **[m 1], [m 3], [m 4]** *of the above definition of a metric on* **X** *and the axiom*

[m′ 2] : $d(x, x) = 0 \ \forall \ x \in X$.

Note that for a pseudo-metric, we have

$d(x, y) = 0$ even if $x \neq y$.

Thus for a pseudo-metric $x = y \Rightarrow d(x, y) = 0$ but not conversely. It follows that every metric is a pesudo-metric but a pseudo-metric is not necessarily a metric.

Theorem I: *Let (X, d) be a metric space and let x, y, z be any three points of X. Then*

$$d(x, y) \geq |\, d(x, z) - d(z, y) \,|.$$

Proof: We have $d(x, z) \leq d(x, y) + d(y, z)$, by **[m 4]**

$$= d(x, y) + d(z, y), \text{ by } \textbf{[m 3]}.$$

Thus, $d(x, y) \geq d(x, z) - d(z, y)$. ...(1)

Again $d(z, y) \leq d(z, x) + d(x, y)$, by **[m 4]**

$$= d(x, z) + d(x, y), \text{ by } \textbf{[m 3]}.$$

Thus, $d(x, y) \geq d(z, y) - d(x, z)$

i.e., $d(x, y) \geq -[d(x, z) - d(z, y)]$. ...(2)

It follows from (1) and (2) that

$$d(x, y) \geq |\, d(x, z) - d(z, y) \,|. \quad \ldots(3)$$

Speaking figuratively, the inequality (3) states that the difference of the lengths of any two sides of a triangle is less than or equal to the third side, the sign of equality occurring when the three points are in a straight line.

Theorem II: *Let (X, d) be a metric space and let $x, x', y, y' \in X$. Then*

$$|\, d(x, y) - d(x', y') \,| \leq d(x, x') + d(y, y').$$

Proof: We have

$$d(x, y) \leq d(x, x') + d(x', y), \text{ by } \textbf{[m 4]}$$

$$\leq d(x, x') + d(x', y') + d(y', y), \text{ by } \textbf{[m 4]}$$

$= d(x, x') + d(x', y') + d(y, y')$, by **[m 3]**.

$\therefore \quad d(x, y) - d(x', y') \leq d(x, x') + d(y, y')$. ...(1)

Again, $d(x', y') \leq d(x', x) + d(x, y')$, by **[m 4]**

$\leq d(x', x) + d(x, y) + d(y, y')$, by **[m 4]**

$= d(x, x') + d(x, y) + d(y, y')$, by **[m 3]**.

$\therefore \quad d(x', y') - d(x, y) \leq d(x, x') + d(y, y')$. ...(2)

From (1) and (2), we have

$$|d(x, y) - d(x', y')| \leq d(x, x') + d(y, y').$$

An alternative set of postulates for a metric

Theorem III: *Let X be a non-empty set. Then a mapping d of $X \times X$ into $\mathbf{R}$ is a metric on X if and only if the following conditions are satisfied :*

[m* 1] : $d(x, y) = 0$ *iff* $x = y \quad \forall\, x, y \in X$.

[m* 2] : $d(x, y) \leq d(x, z) + d(y, z) \quad \forall\, x, y, z \in X$.

Proof: We have to show that the postulates **[m 1], [m 2], [m 3]** and **[m 4]** of the definition of a metric imply **[m* 1]** and **[m* 2]** and conversely. Let the postulates **[m 1]** to **[m 4]** hold. Then by **[m 4]**, we have

$$d(x, y) \leq d(x, z) + d(z, y). \quad \ldots(1)$$

Also by **[m 3]**, $d(z, y) = d(y, z)$ ...(2)

From (1) and (2), we have

$$d(x, y) \leq d(x, z) + d(y, z) \quad \forall\, x, y, z \in X$$

which is **[m* 2]**. Of course **[m* 1]** is the same as **[m 2]**.

Now suppose the conditions **[m* 1]** and **[m* 2]** hold.

If x, y are any two elements of X, then applying **[m* 2]** for x, x and y, we get

$$d(x, x) \leq d(x, y) + d(x, y)$$

i.e., $2\, d(x, y) \geq d(x, x)$. ...(3)

But $d(x, x) = 0$ by **[m* 1]**. Hence (3) gives $d(x, y) \geq 0$ which is **[m 1]**.

Also **[m 2]** is the same as **[m* 1]**.

Again applying **[m* 2]** to the points x, y, x we obtain

$$d(x, y) \leq d(x, x) + d(y, x)$$

$= 0 + d(y, x)$, by **[m* 1]**.

Thus $d(x, y) \leq d(y, x)$. ...(4)

Similarly applying **[m* 2]** to the points y, x, y we get

$$d(y, x) \leq d(y, y) + d(x, y)$$

$= 0 + d(x, y)$, by **[m* 1]**.

Thus $d(x, y) \geq d(y, x)$. ...(5)

If follows from (4) and (5) that

$$d(x, y) = d(y, x) \quad \ldots(6)$$

and so **[m 3]** is satisfied.

Finally for any x, y, z we have by **[m* 2]**,

$$d(x, y) \leq d(x, z) + d(y, z) = d(x, z) + d(z, y), \text{ by } (6)$$

which is **[m 4]**.

10.2 Some Examples of Metric Spaces

Illustrative Examples

Example 1: *Let* $\mathbf{R}$ *denote the set of real numbers. The function*

$$d : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$$

defined by $d(x, y) = |x - y|, \forall x, y \in \mathbf{R}$

is a metric on $\mathbf{R}$.

This metric on $\mathbf{R}$ *is referred to as the* **usual metric** *on* $\mathbf{R}$.

Solution: We shall show that the function d satisfies all the four axioms of a metric on $\mathbf{R}$.

[m 1]: We have $|x - y| \geq 0 \quad \forall x, y \in \mathbf{R}$.

$\therefore \quad d(x, y) \geq 0 \quad \forall x, y \in \mathbf{R}$.

[m 2]: We have $|x - y| = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y$.

$\therefore \quad d(x, y) = 0$ iff $x = y$.

[m 3]: We have $|x - y| = |y - x| \quad \forall x, y \in \mathbf{R}$. $[\because |x| = |-x|]$

$\therefore \quad d(x, y) = d(y, x) \quad \forall x, y \in \mathbf{R}$.

[m 4]: Let $x, y, z \in \mathbf{R}$. We have

$$|x - y| = |(x - z) + (z - y)|$$
$$\leq |x - z| + |z - y|.$$

[By triangle inequality $|x + y| \leq |x| + |y|$]

$\therefore \quad d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in \mathbf{R}$.

Hence, d is a metric on $\mathbf{R}$.

Remark: The number $|x - y|$ is the usual '*distance*' between the points x, y on the real line. Therefore $d(x, y) = |x - y|$ is sometimes referred to as the *usual metric* on $\mathbf{R}$. The metric space $\mathbf{R}$ with respect to the usual metric on $\mathbf{R}$ is called the *usual metric space* $\mathbf{R}$.

Example 2: *Let* $\mathbf{C}$ *be the set of all complex numbers. The mapping*

$$d : \mathbf{C} \times \mathbf{C} \to \mathbf{R}$$

defined by $d(z_1, z_2) = |z_1 - z_2|, \forall z_1, z_2 \in \mathbf{C}$

Solution: We shall show that the mapping d satisfies all the four axioms of a metric on $\mathbf{C}$.

[m 1]: We have $|z_1 - z_2| \geq 0 \quad \forall\, z_1, z_2 \in \mathbf{C}$.

$\therefore \quad d(z_1, z_2) \geq 0 \quad \forall\, z_1, z_2 \in \mathbf{C}$.

[m 2]: We have $|z_1 - z_2| = 0 \Leftrightarrow z_1 - z_2 = 0 \Leftrightarrow z_1 = z_2$.

$\therefore \quad d(z_1, z_2) = 0$ iff $z_1 = z_2$.

[m 3]: We have $|z_1 - z_2| = |z_2 - z_1| \quad \forall\, z_1, z_2 \in \mathbf{C}$.

$\therefore \quad d(z_1, z_2) = d(z_2, z_1) \; \forall\, z_1, z_2 \in \mathbf{C}$.

[m 4]: Let $u, v, w \in \mathbf{C}$. We have

$$|u - v| = |(u - w) + (w - v)|$$

$$\leq |u - w| + |w - v|. \qquad [\because |z_1 + z_2| \leq |z_1| + |z_2|]$$

$\therefore \quad d(u, v) \leq d(u, w) + d(w, v) \quad \forall\, u, v, w \in \mathbf{C}$.

Hence, d is a metric on $\mathbf{C}$.

Example 3: Discrete metric space.

Let X be any non-empty set. The mapping $d : \mathbf{X} \times \mathbf{X} \to \mathbf{R}$ defined by

$$d(x, y) = \begin{cases} 0 & \text{if} \quad x = y \\ 1 & \text{if} \quad x \neq y \end{cases}$$

is a metric on X called the discrete metric on X.

Solution: We shall show that the mapping d satisfies all the four axioms of a metric on X.

[m 1]: By the definition of the mapping d, we have either $d(x, y) = 0$ or $d(x, y) = 1$.

$\therefore \quad d(x, y) \geq 0 \; \forall\, x, y \in X$.

[m 2]: If $x = y$, then by the definition of d, we have $d(x, y) = 0$.

Conversely, if $d(x, y) = 0$, then we must have $x = y$. For if $x \neq y$, then by the definition of d, we have $d(x, y) = 1$ which contradicts our assumption that $d(x, y) = 0$.

Hence, $d(x, y) = 0$ iff $x = y$.

[m 3]: If $x = y$, then $d(x, y) = 0 = d(y, x)$.

Again if $x \neq y$, then $d(x, y) = 1 = d(y, x)$.

Hence, $\quad d(x, y) = d(y, x) \; \forall\, x, y \in X$.

[m 4]: Let x, y, z be any three elements of X.

If $x = y$, then $d(x, y) = 0$.

Also $d(x, z) \geq 0$ and $d(z, y) \geq 0$.

$\therefore \quad d(x, z) + d(z, y) \geq 0$.

Hence, $\quad d(x, y) \leq d(x, z) + d(z, y)$.

If $x \neq y$, then $d(x, y) = 1$. In this case z is different from at least one of x and y and hence at least one of $d(x, z)$ and $d(z, y)$ must be equal to 1. It follows that –

$$d(x, y) \leq d(x, z) + d(z, y).$$

Thus, $d(x, y) \leq d(x, z) + d(z, y) \quad \forall\, x, y, z \in X$.

Hence, d is a metric on X.

Example 4: *Example of a Pseudo-metric which is not a metric.*

Let $\mathbf{R}$ *denote the set of real numbers. The mapping* $d : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ *defined by*

$$d(x, y) = |x^2 - y^2| \quad \forall\, x, y \in \mathbf{R}$$

is a Pseudo-metric on $\mathbf{R}$ *which is not a metric on* $\mathbf{R}$.

Solution: We shall show that the mapping d satisfies all the four axioms of a Pseudo-metric on $\mathbf{R}$.

[m 1]: We have $|x^2 - y^2| \geq 0 \quad \forall\, x, y \in \mathbf{R}$.

$\therefore \quad d(x, y) \geq 0 \quad \forall\, x, y \in \mathbf{R}$.

[m 2]: We have $d(x, x) = |x^2 - x^2| = 0 \quad \forall\, x \in \mathbf{R}$.

[m 3]: We have $|x^2 - y^2| = |y^2 - x^2| \quad \forall\, x, y \in \mathbf{R}$.

$\therefore \quad d(x, y) = d(y, x) \quad \forall\, x, y \in \mathbf{R}$.

[m 4]: Let $x, y, z \in \mathbf{R}$. We have

$$|x^2 - y^2| = |(x^2 - z^2) + (z^2 - y^2)|$$
$$\leq |x^2 - z^2| + |z^2 - y^2|.$$

$\therefore \quad d(x, y) \leq d(x, z) + d(z, y) \quad \forall\, x, y, z \in \mathbf{R}$.

Hence, d is a Pseudo-metric on $\mathbf{R}$.

However, the mapping d is not a metric on $\mathbf{R}$. For the mapping d to be a metric on $\mathbf{R}$ we must have $d(x, y) = 0$ iff $x = y$.

But here, $d(x, y) = |x^2 - y^2| = 0$

$\Rightarrow \quad x^2 - y^2 = 0 \Rightarrow x = \pm y$.

Thus $d(x, y) = 0$ does not necessarily imply that $x = y$. For example,

$$d(2, -2) = |2^2 - (-2)^2| = 0 \text{ while } 2 \neq -2.$$

Hence, the mapping d is not a metric on $\mathbf{R}$.

10.3 Some Important Inequalities

(i) Triangle Inequality. If x, y are any two real numbers, then

$$|x + y| \leq |x| + |y|.$$

If u, v are any two complex numbers, then

$$|u + v| \leq |u| + |v|.$$

(ii) If u, v are any two complex numbers, then

$$\frac{|u + v|}{1 + |u + v|} \leq \frac{|u|}{1 + |u|} + \frac{|v|}{1 + |v|}.$$

(iii) Cauchy-Schwarz inequality. *Let* $u = (u_1, u_2, \ldots, u_n)$ *and* $v = (v_1, v_2, \ldots, v_n)$ *be any two n-tuples of complex numbers, then*

$$\sum_{i=1}^{n} |u_i \, v_i| \leq \left(\sum_{i=1}^{n} |u_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} |v_i|^2 \right)^{1/2}.$$

Also, $$\left| \sum_{i=1}^{n} u_i \, v_i \right| \leq \sum_{i=1}^{n} |u_i \, v_i|.$$

$$\therefore \quad \left| \sum_{i=1}^{n} u_i \, v_i \right| \leq \left(\sum_{i=1}^{n} |u_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} |v_i|^2 \right)^{1/2}.$$

Let $x = (x_1, x_2, \ldots, x_n)$ *and* $y = (y_1, y_2, \ldots, y_n)$ *be any two n-tuples of real numbers, then*

$$(x_1 \, y_1 + x_2 \, y_2 + \ldots + x_n \, y_n)^2 \leq (x_1^2 + x_2^2 + \ldots + x_n^2)(y_1^2 + y_2^2 + \ldots + y_n^2)$$

or $$|x_1 \, y_1 + x_2 \, y_2 + \ldots + x_n \, y_n| \leq \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} \sqrt{y_1^2 + y_2^2 + \ldots + y_n^2}.$$

Also, $$x_1 \, y_1 + x_2 \, y_2 + \ldots + x_n \, y_n \leq \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} \sqrt{y_1^2 + y_2^2 + \ldots + y_n^2}.$$

(iv) Minkowski's Inequality:

If $u_1, u_2, \ldots, u_n$ *and* $v_1, v_2, \ldots, v_n$ *be any complex numbers, then*

$$\left(\sum_{i=1}^{n} |u_i + v_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^{n} |u_i|^p \right)^{1/p} + \left(\sum_{i=1}^{n} |v_i|^p \right)^{1/p}, \quad \text{if} \quad p \geq 1$$

and $$\sum_{i=1}^{n} |u_i + v_i|^p \leq \sum_{i=1}^{n} |u_i|^p + \sum_{i=1}^{n} |v_i|^p, \quad \text{if} \quad 0 < p \leq 1.$$

If $x_1, x_2, \ldots, x_n$ *and* $y_1, y_2, \ldots, y_n$ *are any real numbers, then*

$$\left[\sum_{i=1}^{n} (x_i + y_i)^2 \right]^{1/2} \leq \left[\sum_{i=1}^{n} x_i^2 \right]^{1/2} + \left[\sum_{i=1}^{n} y_i^2 \right]^{1/2}.$$

If a_i, b_i $(i = 1, 2, \ldots, n)$ *are any non-negative real numbers, then*

$$\left(\sum_{i=1}^{n} (a_i + b_i)^p \right)^{1/p} \leq \left(\sum_{i=1}^{n} a_i^p \right)^{1/p} + \left(\sum_{i=1}^{n} b_i^p \right)^{1/p}, \quad \text{if} \quad p \geq 1$$

and $$\sum_{i=1}^{n} (a_i + b_i)^p \leq \sum_{i=1}^{n} a_i^p + \sum_{i=1}^{n} b_i^p, \quad \text{if} \quad 0 < p \leq 1.$$

10.4 More Examples of Metric Spaces

Illustrative Examples

Example 5: *Let* $\mathbf{R}$ *denote the set of all real numbers and let* $\mathbf{R}^2$ *denote the set of all ordered pairs of real numbers. The function*

$$d : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$$

defined by $d(x, y) = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2}$, *where*

$$x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbf{R}^2$$

is a metric on $\mathbf{R}^2$.

This metric on $\mathbf{R}^2$ *is called the* **usual metric** *on* $\mathbf{R}^2$ *because*

$$d(x, y) = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2}$$

is the usual distance formula for two points in the cartesian plane.

Solution: We shall show that the function d satisfies all the four axioms of a metric on $\mathbf{R}^2$.

Let $x = (x_1, x_2),\ y = (y_1, y_2),\ z = (z_1, z_2)$ be any arbitrary elements of $\mathbf{R}^2$.

[m 1]: We have $(x_1 - y_1)^2 \geq 0$ and $(x_2 - y_2)^2 \geq 0$

$\Rightarrow$ $(x_1 - y_1)^2 + (x_2 - y_2)^2 \geq 0$

$\Rightarrow$ $[(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2} \geq 0 \quad \Rightarrow \quad d(x, y) \geq 0.$

$\therefore$ $d(x, y) \geq 0 \quad \forall\ x, y \in \mathbf{R}^2.$

[m 2]: We have $d(x, y) = 0 \Leftrightarrow [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2} = 0$

$\Leftrightarrow$ $(x_1 - y_1)^2 + (x_2 - y_2)^2 = 0$

$\Leftrightarrow$ $(x_1 - y_1)^2 = 0$ and $(x_2 - y_2)^2 = 0$

$\Leftrightarrow$ $x_1 - y_1 = 0$ and $x_2 - y_2 = 0 \quad \Rightarrow \quad x_1 = y_1$ and $x_2 = y_2$

$\Leftrightarrow$ $(x_1, x_2) = (y_1, y_2) \Leftrightarrow x = y.$

$\therefore$ $d(x, y) = 0 \Leftrightarrow x = y.$

[m 3]: We have $d(x, y) = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2}$

$= [(y_1 - x_1)^2 + (y_2 - x_2)^2]^{1/2} = d(y, x).$

[m 4]: We have $d(x, y) = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2}$

$= [\{(x_1 - z_1) + (z_1 - y_1)\}^2 + \{(x_2 - z_2) + (z_2 - y_2)\}^2]^{1/2}$

$\leq [(x_1 - z_1)^2 + (x_2 - z_2)^2]^{1/2} + [(z_1 - y_1)^2 + (z_2 - y_2)^2]^{1/2}$

[By Minkowski's inequality]

$= d(x, z) + d(z, y).$

$\therefore \quad d(x, y) \le d(x, z) + d(z, y) \quad \forall\, x, y, z \in \mathbf{R}^2$.

Hence, d is a metric on $\mathbf{R}^2$.

Example 6: *Let $X = \mathbf{R}^n$ denote the set of all ordered n-tuples of real numbers for a fixed $n \in \mathbf{N}$. Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbf{R}^n$.*

The mapping $d : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ defined by

$$d(x, y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2 \right]^{1/2}$$

is a metric on $\mathbf{R}^n$.

Solution: We shall show that the mapping d satisfies all the four axioms of a metric on $\mathbf{R}^n$.

Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ and $z = (z_1, z_2, \ldots, z_n)$ be any arbitrary elements of $\mathbf{R}^n$.

[m 1]: We have $(x_i - y_i)^2 \ge 0$ for all $i = 1, 2, \ldots, n$

$$\Rightarrow \quad \sum_{i=1}^{n} (x_i - y_i)^2 \ge 0 \quad \Rightarrow \quad \left[\sum_{i=1}^{n} (x_i - y_i)^2 \right]^{1/2} \ge 0$$

$$\Rightarrow \quad d(x, y) \ge 0.$$

$$\therefore \quad d(x, y) \ge 0 \quad \forall\, x, y \in \mathbf{R}^n.$$

[m 2]: We have $d(x, y) = 0 \Leftrightarrow \left[\sum_{i=1}^{n} (x_i - y_i)^2 \right]^{1/2} = 0$

$$\Leftrightarrow \quad (x_i - y_i)^2 = 0 \text{ for each } i = 1, 2, \ldots, n$$

$$\Leftrightarrow \quad x_i = y_i \text{ for each } i = 1, 2, \ldots, n \Leftrightarrow x = y.$$

$$\therefore \quad d(x, y) = 0 \Leftrightarrow x = y.$$

[m 3]: We have $d(x, y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2 \right]^{1/2}$

$$= \left[\sum_{i=1}^{n} (y_i - x_i)^2 \right]^{1/2} = d(y, x).$$

[m 4]: We have to show that $d(x, y) \le d(x, z) + d(z, y)$.

Set $\alpha_i = x_i - z_i$, $\beta_i = z_i - y_i$, $i = 1, 2, \ldots, n$. Then

$$d(x, z) = \left[\sum_{i=1}^{n} (x_i - z_i)^2 \right]^{1/2} = \left(\sum_{i=1}^{n} \alpha_i^2 \right)^{1/2},$$

$$d(z, y) = \left[\sum_{i=1}^{n} (z_i - y_i)^2 \right]^{1/2} = \left(\sum_{i=1}^{n} \beta_i^2 \right)^{1/2}$$

and $$d(x, y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2 \right]^{1/2} = \left[\sum_{i=1}^{n} (x_i - z_i + z_i - y_i)^2 \right]^{1/2}$$

$$= \left[\sum_{i=1}^{n} (\alpha_i + \beta_i)^2 \right]^{1/2}.$$

Using Minkowski's inequality, we have

$$\left[\sum_{i=1}^{n}(\alpha_i+\beta_i)^2\right]^{1/2} \le \left[\sum_{i=1}^{n}\alpha_i^2\right]^{1/2} + \left[\sum_{i=1}^{n}\beta_i^2\right]^{1/2}$$

$\therefore \quad d(x, y) \le d(x, z) + d(z, y) \quad \forall\, x, y, z \in \mathbf{R}^n.$

Hence, d is a metric on $\mathbf{R}^n$.

This metric d on $\mathbf{R}^n$ is called the **usual metric** on $\mathbf{R}^n$. The metric space $(\mathbf{R}^n, d)$ with the usual metric d is called the **real Euclidean n-space.**

Example 7: *Let $X = \mathbf{R}^n$ denote the set of all ordered n-tuples of real numbers for a fixed $n \in \mathbf{N}$. Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbf{R}^n$.*

The mapping $d : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ defined by $d(x, y) = \sum_{i=1}^{n} |x_i - y_i|$ is a metric on $\mathbf{R}^n$.

Solution: We shall show that the mapping d satisfies all the four axioms of a metric on $\mathbf{R}^n$.

Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ and $z = (z_1, z_2, \ldots, z_n)$ be any arbitrary elements of $\mathbf{R}^n$.

[m 1]: We have $|x_i - y_i| \ge 0$ for all $i = 1, 2, \ldots, n$

$$\Rightarrow \quad \sum_{i=1}^{n}|x_i - y_i| \ge 0 \Rightarrow d(x, y) \ge 0.$$

$$\therefore \quad d(x, y) \ge 0 \quad \forall\, x, y \in \mathbf{R}^n.$$

[m 2]: We have $d(x, y) = 0 \Leftrightarrow \sum_{i=1}^{n}|x_i - y_i| = 0$

$$\Leftrightarrow \quad |x_i - y_i| = 0 \text{ for each } i = 1, 2, \ldots, n$$
$$\Leftrightarrow \quad x_i = y_i \text{ for each } i = 1, 2, \ldots, n$$
$$\Leftrightarrow \quad (x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \Leftrightarrow x = y.$$
$$\therefore \quad d(x, y) = 0 \Leftrightarrow x = y.$$

[m 3]: We have $d(x, y) = \sum_{i=1}^{n}|x_i - y_i| = \sum_{i=1}^{n}|y_i - x_i| = d(y, x)$.

[m 4]: We have to show that $d(x, y) \le d(x, z) + d(z, y)$.

Set $\alpha_i = x_i - z_i$, $\beta_i = z_i - y_i$, $i = 1, 2, \ldots, n$.

Then $$d(x, z) = \sum_{i=1}^{n}|x_i - z_i| = \sum_{i=1}^{n}|\alpha_i|,$$

$$d(z, y) = \sum_{i=1}^{n}|z_i - y_i| = \sum_{i=1}^{n}|\beta_i|$$

and $$d(x, y) = \sum_{i=1}^{n}|x_i - y_i| = \sum_{i=1}^{n}|x_i - z_i + z_i - y_i| = \sum_{i=1}^{n}|\alpha_i + \beta_i|.$$

Since $|\alpha_i + \beta_i| \le |\alpha_i| + |\beta_i|$, $i = 1, 2, \ldots, n$,

therefore $$\sum_{i=1}^{n}|\alpha_i + \beta_i| \le \sum_{i=1}^{n}|\alpha_i| + \sum_{i=1}^{n}|\beta_i|$$

i.e., $$d(x, y) \le d(x, z) + d(z, y).$$

Thus, $d(x, y) \le d(x, z) + d(z, y) \quad \forall\, x, y, z \in \mathbf{R}^n$.

Hence, d is a metric on $\mathbf{R}^n$.

Example 8: *Let $X = \mathbf{R}^n$ denote the set of all ordered n-tuples of real numbers for a fixed $n \in \mathbf{N}$. Let*

$$x = (x_1, x_2, \ldots, x_n),\ y = (y_1, y_2, \ldots, y_n) \in \mathbf{R}^n.$$

The mapping $d : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ defined by

$$d(x, y) = \text{maximum } \{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}$$

is a metric on $\mathbf{R}^n$.

Solution: We shall show that the mapping d satisfies all the four axioms of a metric on $\mathbf{R}^n$.

Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ and $z = (z_1, z_2, \ldots, z_n)$ be any arbitrary elements of $\mathbf{R}^n$.

[m 1]: We have $|x_i - y_i| \ge 0$ for all $i = 1, 2, \ldots, n$

$\Rightarrow$ $\max. \{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\} \ge 0 \quad \Rightarrow \quad d(x, y) \ge 0.$

$\therefore$ $d(x, y) \ge 0 \quad \forall\, x, y \in \mathbf{R}^n.$

[m 2]: We have $d(x, y) = 0 \Leftrightarrow \max. \{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\} = 0$

$\Leftrightarrow$ $|x_i - y_i| = 0$ for each $i = 1, 2, \ldots, n$

$\Leftrightarrow$ $x_i = y_i$ for each $i = 1, 2, \ldots, n \quad \Leftrightarrow \quad x = y.$

$\therefore$ $d(x, y) = 0$ iff $x = y$.

[m 3]: We have $d(x, y) = \max. \{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}$

$= \max. \{|y_1 - x_1|, |y_2 - x_2|, \ldots, |y_n - x_n|\} = d(y, x).$

[m 4]: We have to show that $d(x, y) \le d(x, z) + d(z, y)$.

Set $\alpha_i = x_i - z_i$, $\beta_i = z_i - y_i$, $i = 1, 2, \ldots, n$. Then

$$d(x, z) = \max. \{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_n|\},$$

$$d(z, y) = \max. \{|\beta_1|, |\beta_2|, \ldots, |\beta_n|\}$$

and $d(x, y) = \max. \{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}$

$= \max. \{|x_1 - z_1 + z_1 - y_1|, |x_2 - z_2 + z_2 - y_2|, \ldots, |x_n - z_n + z_n - y_n|\}$

$= \max. \{|\alpha_1 + \beta_1|, |\alpha_2 + \beta_2|, \ldots, |\alpha_n + \beta_n|\}.$

Since $|\alpha_i + \beta_i| \le |\alpha_i| + |\beta_i|$, $i = 1, 2, \ldots, n$,

therefore $|\alpha_i + \beta_i| \le \max. \{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_n|\} + \max. \{|\beta_1|, |\beta_2|, \ldots, |\beta_n|\}$, $i = 1, 2, \ldots, n$.

Hence, $\max. \{|\alpha_1 + \beta_1|, |\alpha_2 + \beta_2|, \ldots, |\alpha_n + \beta_n|\}$

$\le \max. \{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_n|\} + \max. \{|\beta_1|, |\beta_2|, \ldots, |\beta_n|\}$

i.e., $d(x, y) \le d(x, z) + d(z, y).$

Thus, $d(x, y) \le d(x, z) + d(z, y) \quad \forall\, x, y, z \in \mathbf{R}^n.$

Hence, d is a metric on $\mathbf{R}^n$.

Example 9: *Let X be the set of all real valued bounded continuous functions defined on the closed interval $[0, 1]$. We define the norm of a function $f \in X$ by*

$$\| f \| = \int_0^1 | f(x) | \, dx.$$

The symbol $\| f \|$ is read as norm of f.

Let d be a mapping of $X \times X$ into $\mathbf{R}$ defined by

$$d(f, g) = \| f - g \| = \int_0^1 | f(x) - g(x) | \, dx \; \forall \; f, g \in X.$$

Show that d is a metric for X.

Solution: Since all the functions are bounded and continuous, therefore Riemann's integrals exist.

We shall show that the mapping d satisfies all the four axioms of a metric on X.

Let f, g, h be any arbitrary elements of X.

[m 1]: We have $| f(x) - g(x) | \geq 0 \; \forall \; x \in [0, 1]$

$$\Rightarrow \quad \int_0^1 | f(x) - g(x) | \, dx \geq 0 \quad \Rightarrow \quad \| f - g \| \geq 0 \quad \Rightarrow \quad d(f, g) \geq 0.$$

$$\therefore \quad d(f, g) \geq 0 \quad \forall \; f, g \in X.$$

[m 2]: We have to show that $d(f, g) = 0$ iff $f = g$.

If $f = g$, then $f(x) = g(x) \; \forall \; x \in [0, 1]$

$$\Rightarrow \quad | f(x) - g(x) | = 0 \quad \forall \; x \in [0, 1]$$

$$\Rightarrow \quad \int_0^1 | f(x) - g(x) | \, dx = 0 \quad \Rightarrow \quad d(f, g) = 0.$$

Conversely, let $d(f, g) = 0$. Then to show that $f = g$.

We have $d(f, g) = 0 \quad \Rightarrow \quad \int_0^1 | f(x) - g(x) | \, dx = 0$

$$\Rightarrow \quad \lim_{x \to a} | f(x) - g(x) | = 0 \quad \forall \; a \in [0, 1]$$

$$\Rightarrow \quad \lim_{x \to a} [f(x) - g(x)] = 0 \quad \forall \; a \in [0, 1]$$

$$\Rightarrow \quad \lim_{x \to a} f(x) = \lim_{x \to a} g(x) \; \forall \; a \in [0, 1]$$

$$\Rightarrow \quad f(x) = g(x) \quad \forall \; x \in [0, 1]$$

$[\because \; f$ and g are continuous functions on $[0, 1]]$

$$\Rightarrow \quad f = g.$$

Hence, $d(f, g) = 0$ iff $f = g$.

[m 3]: We have

$$d(f, g) = \int_0^1 | f(x) - g(x) | \, dx = \int_0^1 | g(x) - f(x) | \, dx = d(g, f).$$

[m 4]: We have to show that $d(f, g) \leq d(f, h) + d(h, g)$.

We have $d(f, g) = \| f - g \| = \int_0^1 |f(x) - g(x)|\, dx$

$$= \int_0^1 |f(x) - h(x) + h(x) - g(x)|\, dx$$

$$\leq \int_0^1 [|f(x) - h(x)| + |h(x) - g(x)|]\, dx$$

$$= \int_0^1 |f(x) - h(x)|\, dx + \int_0^1 |h(x) - g(x)|\, dx$$

$$= \| f - h \| + \| h - g \| = d(f, h) + d(h, g).$$

Thus, $d(f, g) \leq d(f, h) + d(h, g)\ \forall\ f, g, h \in X$.

Hence, d is a metric on X.

Example 10: *Let $C[0, 1]$ denote the collection of all real valued bounded continuous functions defined on the closed interval $[0, 1]$. We define the norm of $f \in C[0, 1]$ by*

$$\| f \| = sup\{|f(x)| : x \in [0, 1]\}.$$

Let d be defined by

$$d(f, g) = \| f - g \| = sup\{|f(x) - g(x)| : x \in [0, 1]\}$$

Show that d is a metric for $C[0, 1]$.

Solution: We shall show that the mapping d satisfies all the four axioms of a metric on $X = C[0, 1]$.

Let f, g, h be any arbitrary elements of X *i.e.*, of $C[0, 1]$.

[m 1]: We have $|f(x) - g(x)| \geq 0\ \forall\ x \in [0, 1]$

$\Rightarrow$ $\sup\{|f(x) - g(x)| : x \in [0, 1]\} \geq 0$

$\Rightarrow$ $d(f, g) \geq 0$.

[m 2]: We have to show that $d(f, g) = 0$ iff $f = g$.

If $f = g$, then $f(x) = g(x)\ \forall\ x \in [0, 1]$

$\Rightarrow$ $|f(x) - g(x)| = 0\ \forall\ x \in [0, 1]$

$\Rightarrow$ $\sup\{|f(x) - g(x)| : x \in [0, 1]\} = 0 \Rightarrow d(f, g) = 0$

Conversely, let $d(f, g) = 0$. Then to show that $f = g$.

We have $d(f, g) = 0 \Rightarrow \sup\{|f(x) - g(x)| : x \in [0, 1]\} = 0$

$\Rightarrow$ $|f(x) - g(x)| = 0\ \ \forall\ x \in [0, 1]$

$[\because f$ and g are continuous functions on $[0, 1]]$

$\Rightarrow$ $f(x) = g(x)\ \ \forall\ x \in [0, 1] \ \Rightarrow\ f = g$.

Hence, $d(f, g) = 0$ iff $f = g$.

[m 3]: We have $d(f, g) = \sup\{|f(x) - g(x)| : x \in [0, 1]\}$

$$= \sup\{|g(x) - f(x)| : x \in [0, 1])\} = d(g, f)$$

[m 4]: We have to show that $d(f, g) \leq d(f, h) + d(h, g)$.

For all $x \in [0, 1]$, we have

$$|f(x) - g(x)| = |f(x) - h(x) + h(x) - g(x)|$$

$$\leq |f(x) - h(x)| + |h(x) - g(x)|.$$

$\therefore \sup \{ | f(x) - g(x) | : x \in [0, 1] \}$

$\leq \sup \{ | f(x) - h(x) | : x \in [0, 1] \} + \sup \{ | h(x) - g(x) | : x \in [0, 1] \}$

i.e., $d(f, g) \leq d(f, h) + d(h, g)$.

Hence, d is a metric on $C[0, 1]$.

Example 11: *Show that the set* **C** *of all complex numbers is a metric space under*

$$d(z_1, z_2) = \frac{| z_1 - z_2 |}{(1 + | z_1 |^2)^{1/2} (1 + | z_2 |^2)^{1/2}}.$$

Solution: We shall show that the mapping d satisfies all the four axioms of a metric on **C**.

[m 1]: We have $| z_1 - z_2 | \geq 0 \ \forall \ z_1, z_2 \in \mathbf{C}$

$$\Rightarrow \quad \frac{| z_1 - z_2 |}{(1 + | z_1 |^2)^{1/2} (1 + | z_2 |^2)^{1/2}} \geq 0 \quad \forall \ z_1, z_2 \in \mathbf{C}$$

$$\Rightarrow \quad d(z_1, z_2) \geq 0 \quad \forall \ z_1, z_2 \in \mathbf{C}.$$

[m 2]: We have $d(z_1, z_2) = 0$

$$\frac{| z_1 - z_2 |}{(1 + | z_1 |^2)^{1/2} (1 + | z_2 |^2)^{1/2}} = 0$$

$$| z_1 - z_2 | = 0 \Leftrightarrow z_1 - z_2 = 0 \Leftrightarrow z_1 = z_2 .$$

[m 3]: We have $d(z_1, z_2) = \dfrac{| z_1 - z_2 |}{(1 + | z_1 |^2)^{1/2} (1 + | z_2 |^2)^{1/2}}$

$$= \frac{| z_2 - z_1 |}{(1 + | z_2 |^2)^{1/2} (1 + | z_1 |^2)^{1/2}} = d(z_2, z_1).$$

[m 4]: If $z_1, z_2, z_3 \in \mathbf{C}$, then we have to show that

$$d(z_1, z_3) \leq d(z_1, z_2) + d(z_2, z_3).$$

If z_1, z_2, z_3 are any complex numbers, then from the identity

$$(z_1 - z_3)(1 + z_2 \bar{z}_2) = (z_1 - z_2)(1 + \bar{z}_2 z_3) + (z_2 - z_3)(1 + z_1 \bar{z}_2),$$

we obtain

$$| z_1 - z_3 | (1 + | z_2 |^2) = | (z_1 - z_2)(1 + \bar{z}_2 z_3) + (z_2 - z_3)(1 + z_1 \bar{z}_2) |$$

$$\leq | z_1 - z_2 | | 1 + \bar{z}_2 z_3 | + | z_2 - z_3 | | 1 + z_1 \bar{z}_2 | \quad \ldots(1)$$

Now by Cauchy-Schwarz inequality, we have

$$| 1 + \bar{z}_2 z_3 | \leq (1 + | z_2 |^2)^{1/2} (1 + | z_3 |^2)^{1/2} \quad \ldots(2)$$

and $$| 1 + z_1 \bar{z}_2 | \leq (1 + | z_1 |^2)^{1/2} (1 + | z_2 |^2)^{1/2}. \quad \ldots(3)$$

From (1), (2) and (3), we obtain

$$|z_1 - z_3|(1+|z_2|^2) \le |z_1 - z_2|(1+|z_2|^2)^{1/2}(1+|z_3|^2)^{1/2} + |z_2 - z_3|(1+|z_1|^2)^{1/2}(1+|z_2|^2)^{1/2}.$$

Dividing by $(1+|z_2|^2)(1+|z_1|^2)^{1/2}(1+|z_3|^2)^{1/2}$, we obtain

$$\frac{|z_1 - z_3|}{[(1+|z_1|^2)(1+|z_3|^2)]^{1/2}} \le \frac{|z_1 - z_2|}{[(1+|z_1|^2)(1+|z_2|^2)]^{1/2}} + \frac{|z_2 - z_3|}{[(1+|z_2|^2)(1+|z_3|^2)]^{1/2}}.$$

$\therefore \quad d(z_1, z_3) \le d(z_1, z_2) + d(z_2, z_3).$

Hence, d is a metric on $\mathbf{C}$ and consequently $(\mathbf{C}, d)$ is a metric space.

10.5 Sequence Spaces

A sequence in a set X is any mapping from $\mathbf{N}$ (the set of natural numbers) into X. Here we are assuming that the reader is familiar with convergent sequences of real or complex numbers. An important property of convergent sequences of numbers is that they are bounded. We shall make use of this property in examples that follow.

We now give some examples of metric sequence spaces. In other words, we shall consider sets of infinite sequences $x = < x_n >$, x_n complex, which are made into metric spaces in a natural way. All the sequences considered in the following examples are sequences of complex numbers.

Illustrative Examples

Example 12: **Frechet space:** *Let F denote the set of all possible sequences $x = < x_n >$. Define*

$$d(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n (1+|x_n - y_n|)}, (x, y \in F).$$

Show that d is a metric on F.

Solution: First we see that the mapping d is well-defined since the nth term of the above series defining $d(x, y)$ is less than $\frac{1}{2^n}$ and the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent, hence, the series

$$\sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n (1+|x_n - y_n|)}$$

is also convergent.

We shall show that the mapping d satisfies all the four axioms of a metric on F.

Let $x = < x_n >, y = < y_n >, z = < z_n >$ be any arbitrary elements of F.

[m 1]: We have $d(x, y) \geq 0 \quad \forall\, x, y \in F$ because

$$|x_n - y_n| \geq 0 \ \forall\, n \in \mathbf{N}.$$

[m 2]: We have

$$d(x, y) = 0 \Leftrightarrow \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n (1 + |x_n - y_n|)} = 0$$

$$\frac{|x_n - y_n|}{2^n (1 + |x_n - y_n|)} = 0 \ \forall\, n \in \mathbf{N} \Leftrightarrow |x_n - y_n| = 0 \ \forall\, n \in \mathbf{N}$$

$$x_n = y_n \ \forall\, n \in \mathbf{N} \Leftrightarrow < x_n > = < y_n > \Leftrightarrow x = y.$$

Thus, $d(x, y) = 0$ iff $x = y$.

[m 3]: We have

$$d(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n (1 + |x_n - y_n|)}$$

$$= \sum_{n=1}^{\infty} \frac{|y_n - x_n|}{2^n (1 + |y_n - x_n|)} = d(y, x).$$

[m 4]: We have to show that

$$d(x, y) \leq d(x, z) + d(z, y).$$

We have $d(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n (1 + |x_n - y_n|)}$

$$= \sum_{n=1}^{\infty} \frac{|x_n - z_n + z_n - y_n|}{2^n (1 + |x_n - z_n + z_n - y_n|)}$$

$$\leq \sum_{n=1}^{\infty} \frac{|x_n - z_n|}{2^n (1 + |x_n - z_n|)} + \sum_{n=1}^{\infty} \frac{|z_n - y_n|}{2^n (1 + |z_n - y_n|)}$$

$$\left[\because \text{ if } u, v \text{ are any complex numbers, then } \frac{|u + v|}{1 + |u + v|} \leq \frac{|u|}{1 + |u|} + \frac{|v|}{1 + |v|}\right]$$

$$= d(x, z) + d(z, y).$$

Thus, $d(x, y) \leq d(x, z) + d(z, y) \quad \forall\, x, y, z \in F.$

Hence, d is a metric on F and consequently (F, d) is a metric space which is called the **Frechet space.**

Example 13: **(The space l_∞).** *Let l_∞ denote the set of all bounded sequences. If $x = < x_n >$ and $y = < y_n >$ are any two elements of l_∞, we define*

$$d(x, y) = \sup \{|x_n - y_n| : n \in \mathbf{N}\}.$$

Show that l_∞ is a metric space under d.

Solution: We first observe that the function d is well-defined since

$$\sup |x_n - y_n| \leq \sup |x_n| + \sup |y_n|,$$

which is finite because the sequences are bounded.

Now we shall show that the function d satisfies all the four axioms of a metric on l_∞.

Let $x = < x_n >, y = < y_n >, z = < z_n >$ be any arbitrary elements of l_∞.

[m 1]: We have $| x_n - y_n | \geq 0 \ \forall \ n \in \mathbf{N}$

$\Rightarrow \quad \sup \{ | x_n - y_n | : n \in \mathbf{N} \} \geq 0$

$\Rightarrow \quad d(x, y) \geq 0.$

[m 2]: We have $d(x, y) = 0 \Leftrightarrow \sup \{ | x_n - y_n | : n \in \mathbf{N} \} = 0$

$\Leftrightarrow \quad | x_n - y_n | = 0 \ \forall \ n \in \mathbf{N} \Leftrightarrow x_n = y_n \ \forall \ n \in \mathbf{N}$

$\Leftrightarrow \quad < x_n > = < y_n > \ \Leftrightarrow x = y.$

[m 3]: We have $d(x, y) = \sup \{ | x_n - y_n | : n \in \mathbf{N} \}$

$= \sup \{ | y_n - x_n | : n \in \mathbf{N} \} = d(y, x)$

[m 4]: We have to show that

$$d(x, y) \leq d(x, z) + d(z, y).$$

For any positive integer n, we have

$$\begin{aligned} | x_n - y_n | &= | x_n - z_n + z_n - y_n | \\ &\leq | x_n - z_n | + | z_n - y_n | \\ &\leq \sup \{ | x_n - z_n | : n \in \mathbf{N} \} + \sup \{ | z_n - y_n | : n \in \mathbf{N} \} \\ &= d(x, z) + d(z, y). \qquad \ldots (1) \end{aligned}$$

Since (1) holds for every $n \in \mathbf{N}$, therefore we have

$$\sup \{ | x_n - y_n | : n \in \mathbf{N} \} \leq d(x, z) + d(z, y)$$

i.e., $\quad d(x, y) \leq d(x, z) + d(z, y).$

Hence, d is a metric on l_∞ and consequently (l_∞, d) is a metric space.

Example 14: **(The spaces c and c_0):** *Let c denote the set of all convergent sequences and c_0 the set of all null sequences (that is, the set of all sequences which converge to 0). Show that both c and c_0 are metric spaces under d defined by*

$$d(x, y) = \sup \{ | x_n - y_n | : n \in \mathbf{N} \}$$

where $x = < x_n >$ and $y = < y_n >$ are any two elements of c or c_0.

Solution: Since convergent sequences are bounded, the solution is exactly the same as that of the previous example.

Example 15: **(The space l_p).** *Let l_p be the set of all sequences $x = < x_n >$ such that $\sum_{n=1}^{\infty} | x_n |^p < \infty$ where p is a positive constant.*

Define (i) $d(x, y) = \left(\Sigma | x_n - y_n |^p \right)^{1/p}, p \geq 1$

(ii) $d^*(x, y) = \Sigma | x_n - y_n |^p, 0 < p < 1$

Show that d and d^* are both metrics on l_p.

Solution: (i) First we see that the mapping d is well-defined. For this we must show that $d(x, y) < \infty$.

We have $d(x, y) = (\Sigma \mid x_n - y_n \mid^p)^{1/p}$

$\leq (\Sigma \mid x_n \mid^p)^{1/p} + (\Sigma \mid -y_n \mid^p)^{1/p}$ [By Minkowski's inequality]

$< \infty$ [$\because \Sigma \mid x_n \mid^p < \infty$ and $\Sigma \mid y_n \mid^p < \infty$]

$\therefore$ the mapping d is well-defined.

Now we shall show that the mapping d satisfies all the four axioms of a metric on l_p.

The axioms **[m 1]**, **[m 2]** and **[m 3]** are self-evident.

We shall establish here only the axiom **[m 4]** of triangle inequality.

Let $x = < x_n >, y = < y_n >, z = < z_n >$ be any arbitrary elements of l_p.

We have $d(x, y) = (\Sigma \mid x_n - y_n \mid^p)^{1/p} = (\Sigma \mid x_n - z_n + z_n - y_n \mid^p)^{1/p}$

$\leq (\Sigma \mid x_n - z_n \mid^p)^{1/p} + (\Sigma \mid z_n - y_n \mid^p)^{1/p}$

[By Minkowski's inequality]

$= d(x, z) + d(z, y)$.

Hence, d is a metric on l_p and consequently (l_p, d) is a metric space.

(ii) First we see that the mapping d^* is well-defined. For this we must show that $d^*(x, y) < \infty$.

We have $d^*(x, y) = \Sigma \mid x_n - y_n \mid^p$

$\leq \Sigma \mid x_n \mid^p + \Sigma \mid -y_n \mid^p$

$< \infty$ [$\because \Sigma \mid x_n \mid^p < \infty$ and $\Sigma \mid y_n \mid^p < \infty$]

$\therefore$ the mapping d^* is well-defined.

Now we shall show that the mapping d^* satisfies all the four axioms of a metric on l_p. The axioms **[m 1]**, **[m 2]** and **[m 3]** are self-evident.

We shall establish here only the axiom **[m 4]** of triangle inequality.

Let $x = < x_n >, y = < y_n >, z = < z_n >$ be any arbitrary elements of l_p.

We have $d^*(x, y) = \Sigma \mid x_n - y_n \mid^p = \Sigma \mid x_n - z_n + z_n - y_n \mid^p$

$\leq \Sigma \mid x_n - z_n \mid^p + \Sigma \mid z_n - y_n \mid^p$

$= d^*(x, z) + d^*(z, y)$.

Hence, d^* is a metric on l_p.

Example 16: **(Hilbert space or l_2-space):** *Let l_2 or H be the set of all real sequences $x = < x_n >$ such that $\sum_{n=1}^{\infty} x_n^2 < \infty$.*

Let $x = < x_n >$ and $y = < y_n >$ be any arbitrary elements of H.

Define a mapping $d : H \times H \to \mathbf{R}$ such that

$$d(x, y) = \left[\sum_{n=1}^{\infty} (x_n - y_n)^2\right]^{1/2}.$$

Show that d is a metric on H.

Solution: First we see that the mapping d is well-defined. For this we must show that $d(x, y) < \infty$.

We have $d(x, y) = \left(\sum_{n=1}^{\infty} \{x_n + (-y_n)\}^2 \right)^{1/2}$

$$\leq \left(\sum_{n=1}^{\infty} x_n^2 \right)^{1/2} + \left[\sum_{n=1}^{\infty} (-y_n)^2 \right]^{1/2} \text{ [By Minkowski's inequality]}$$

$$< \infty$$

$$\left[\because \sum_{n=1}^{\infty} x_n^2 < \infty \text{ and } \sum_{n=1}^{\infty} y_n^2 < \infty \right]$$

$\therefore$ the mapping d is well-defined.

Now we shall show that the mapping d satisfies all the four axioms of a metric on H.

Let $x = < x_n >, y = < y_n >, z = < z_n >$ be any arbitrary elements of H.

[m 1]: We have $(x_n - y_n)^2 \geq 0 \quad \forall\, n \in \mathbf{N}$

$$\Rightarrow \quad \left[\sum_{n=1}^{\infty} (x_n - y_n)^2 \right]^{1/2} \geq 0$$

$$\Rightarrow \quad d(x, y) \geq 0.$$

[m 2]: We have $d(x, y) = 0 \Leftrightarrow \left[\sum_{n=1}^{\infty} (x_n - y_n)^2 \right]^{1/2} = 0$

$$\Leftrightarrow \quad \sum_{n=1}^{\infty} (x_n - y_n)^2 = 0 \Leftrightarrow x_n - y_n = 0 \quad \forall\, n \in \mathbf{N}$$

$$x_n = y_n \ \forall\, n \in \mathbf{N} \Leftrightarrow < x_n > = < y_n > \Leftrightarrow x = y.$$

[m 3]: We have $d(x, y) = \left[\sum_{n=1}^{\infty} (x_n - y_n)^2 \right]^{1/2} = \left[\sum_{n=1}^{\infty} (y_n - x_n)^2 \right]^{1/2} = d(y, x)$

[m 4]: We have $d(x, y) = \left[\sum_{n=1}^{\infty} (x_n - y_n)^2 \right]^{1/2}$

$$= \left[\sum_{n=1}^{\infty} (x_n - z_n + z_n - y_2)^2 \right]^{1/2}$$

$$\leq \left[\sum_{n=1}^{\infty} (x_n - z_n)^2 \right]^{1/2} + \left[\sum_{n=1}^{\infty} (z_n - y_n)^2 \right]^{1/2}$$

[By Minkowski's inequality]

$$= d(x, z) + d(z, y).$$

Hence, d is a metric on H and consequently (H, d) is a metric space called *Hilbert space* or l_2-space.

10.6 Distance of a Point from a Given Set

Definition: *Let (X, d) be a metric space and let A be any non-empty subset of X. The distance between a point $p \in X$ and A is denoted by $d(p, A)$ and defined by*

$$d(p, A) = \inf \{d(p, x) : x \in A\}.$$

It is evident that $d(p, A) \geq 0$. $[\because d(p, x) \geq 0]$

Also, it is evident that $d(p, A) = 0$ if $p \in A$.

However, if $d(p, A) = 0$, then it does not necessarily imply that $p \in A$ as is clear from the following example.

Consider the metric space $(\mathbf{R}, d)$ where d is the usual metric on $\mathbf{R}$.

Let $A = \{x \in \mathbf{R} : 0 < x \leq 1\}$.

We have $d(0, A) = 0$, although $0 \notin A$.

Theorem 1: *If A is any non-empty subset of a metric space (X, d), then*

$$|d(x, A) - d(y, A)| \leq d(x, y)$$

for any points $x, y \in X$.

Proof: Since $d(x, y) = d(y, x)$, we may assume $d(x, A) \geq d(y, A)$.

Let $\varepsilon > 0$ be given.

Since $d(y, A) = \inf \{d(y, z) : z \in A\}$, we can choose $a \in A$ such that

$$d(y, a) < d(y, A) + \varepsilon.$$

Then $|d(x, A) - d(y, A)| = d(x, A) - d(y, A)$

$$\leq d(x, a) - d(y, A)$$
$$< d(x, a) - d(y, a) + \varepsilon$$
$$\leq d(x, y) + \varepsilon.$$

$[\because d(x, y) + d(y, a) \geq d(x, a) \Rightarrow d(x, a) - d(y, a) \leq d(x, y)]$

Since ε is arbitrary, we have

$$|d(x, A) - d(y, A)| \leq d(x, y).$$

Aliter: We have $d(x, A) = \inf \{d(x, a) : x \in A\}$.

By triangle inequality

$$d(x, a) \leq d(x, y) + d(y, a), \forall\, a \in A \subseteq X,\ x, y \in X.$$

On taking infimum, we have

$$d(x, A) = \inf_{a \in A} d(x, a) \leq d(x, y) + \inf_{a \in A} d(y, a)$$

$[\because d(x, y)$ is independent of $a]$

$$= d(x, y) + d(y, A).$$

$$\therefore \quad d(x, A) - d(y, A) \leq d(x, y). \qquad \ldots(1)$$

Since (1) is true for each $x, y \in X$, therefore interchanging x and y, we have

$$d(y, A) - d(x, A) \leq d(x, y)$$

i.e., $$-[d(x, A) - d(y, A)] \leq d(x, y). \qquad \ldots(2)$$

From (1) and (2), we conclude that

$$|d(x, A) - d(y, A)| \leq d(x, y).$$

10.7 Distance between two Subsets of a Metric Space

Definition: *Let (X, d) be a metric space and let A and B be any two non-empty subsets of X. The distance between the sets A and B is denoted by $d(A, B)$ and is defined as*

$$d(A, B) = \inf \{d(x, y) : x \in A, y \in B\}.$$

It is evident that $d(A, B) \geq 0$.

Also, it is evident that $d(A, B) = 0$ if $A \cap B \neq \emptyset$.

However, if $d(A, B) = 0$, then it does not necessarily imply that $A \cap B \neq \emptyset$ as is clear from the following example.

Consider the metric space $(\mathbf{R}, d)$ where d is the usual metric on $\mathbf{R}$.

Let $A = [0, 1[$ and $B =]1, 2]$.

Then $d(A, B) = 0$ although $A \cap B = \emptyset$.

Similarly if $A = \left\{1, \frac{1}{3}, \frac{1}{5}, \ldots, \frac{1}{2n-1}, \ldots\right\}$ and $B = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots, \frac{1}{2n}, \ldots\right\}$, then $d(A, B) = 0$ although $A \cap B = \emptyset$.

10.8 Diameter of a Subset of a Metric Space

Definition: *Let (X, d) be a metric space and let A be any non-empty subset of X. Then the* **diameter** *of A, denoted by $\delta(A)$, is defined by*

$$\delta(A) = \sup \{d(x, y) : x, y \in A\}$$

i.e., the diameter of A is the supremum of the set of all distances between the points of A.

The diameter of a set A is also denoted by $d(A)$.

It is clear from definition that $d(A) \geq 0$.

If $d(A) < \infty$, then the diameter of A is said to be finite otherwise infinite.

If $d(A) < \infty$, then A is said to be bounded.

By convention we take $d(\emptyset) = -\infty$.

Example: Consider the usual metric d for $\mathbf{R}$ defined by

$$d(x, y) = |x - y|.$$

Let $A = [1, 2[$, $B =]2, 4]$. Then

$$\delta(A) = 1, \ \delta(B) = 2, \ d(A, B) = 0,$$

$$d\left(1\frac{1}{2}, B\right) = \frac{1}{2}, \ d\left(\frac{5}{4}, A\right) = 0, \ d\left(\frac{9}{4}, B\right) = 0, \ d(5, B) = 1 \text{ etc.}$$

Theorem 1: *Let A, B be non-empty subsets of a metric space (X, d).*

Then $\delta(A \cup B) \leq \delta(A) + \delta(B) + d(A, B)$.

Proof: Let x, y be any pair of points of $A \cup B$.

Then either (i) $x, y \in A$ or (ii) $x, y \in B$ or (iii) $x \in A$ and $y \in B$ or (iv) $x \in B$ and $y \in A$.

In case (i), $d(x, y) \leq \delta(A)$. $[\because \delta(A) = \sup \{ d(x, y) : x, y \in A \}]$

In case (ii), $d(x, y) \leq \delta(B)$.

In case (iii), if $a \in A$ and $b \in B$ then $d(x, y) \leq d(x, a) + d(a, y)$, by **[m 4]**

$$\leq d(x, a) + d(a, b) + d(b, y), \text{ by } \textbf{[m 4]}$$

Similarly in case (iv), we shall have

$$d(x, y) \leq d(x, b) + d(b, a) + d(a, y).$$

It follows that

$$\delta(A \cup B) \leq \delta(A) + d(a, b) + \delta(B).$$

Since this holds for any $a \in A, b \in B$, we obtain

$$\delta(A \cup B) \leq \delta(A) + d(A, B) + \delta(B).$$

Cor: If $A \cap B \neq \emptyset$, we shall have

$$\delta(A \cup B) \leq \delta(A) + \delta(B),$$

since $d(A, B) = 0$ in this case.

10.9 Bounded and Unbounded Metric Spaces

A metric d on a non-empty set X is said to be *bounded* if there exists a real number $k > 0$ such that

$$d(x, y) \leq k, \forall\, x, y \in X \quad i.e., \quad d(X) \leq k.$$

Also, (X, d) is then called a *bounded metric space*, otherwise unbounded.

Thus a bounded metric space may be defined as follows :

Definition: *Let (X, d) be a metric space. We say that X is* **bounded** *if there exists a positive number M such that $d(x, y) \leq M$ for every pair of points x and y of X. A metric space which is not bounded is said to be* **unbounded.**

Thus a metric space X is bounded if its diameter is finite. Similarly a subset A of X is said to be bounded if $\delta(A)$ is finite.

Example: Let $X = \mathbf{R}$ and $d(x, y) = |x - y|$.

This metric space X is unbounded since the diameter of $\mathbf{R}$ is infinite.

A discrete metric space (X, d) where

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

is bounded since $\delta(X) = 1$.

Theorem 1: *Let (X, d) be any metric space and let M be a positive number, then there exists a metric d^* on X such that the metric space (X, d^*) is bounded with $\delta(X) \leq M$.*

Proof: Define d^* by $d^*(x, y) = \dfrac{Md(x, y)}{1 + d(x, y)}$ whenever $x, y \in X$.

Then d^* is a metric for X as shown below :

[m 1]: $d(x, y) \geq 0, M > 0 \Rightarrow \dfrac{Md(x, y)}{1 + d(x, y)} \geq 0$

$\Rightarrow$ $d^*(x, y) \geq 0$, for every pair x, y of points of X.

[m 2]: $d^*(x, y) = 0 \Leftrightarrow \dfrac{Md(x, y)}{1 + d(x, y)} = 0$

$d(x, y) = 0$

$x = y$. [$\because$ d is a metric on X and so $d(x, y) = 0 \Leftrightarrow x = y$]

[m 3]: $d^*(x, y) = \dfrac{Md(x, y)}{1 + d(x, y)} = \dfrac{Md(y, x)}{1 + d(y, x)} = d^*(y, x)$.

[m 4]: Let $x, y, z \in X$. Then

$$d^*(x, y) = \frac{Md(x, y)}{1 + d(x, y)} = M - \frac{M}{1 + d(x, y)}$$

$$\leq M - \frac{M}{1 + d(x, z) + d(z, y)} \quad [\because d(x, y) \leq d(x, z) + d(z, y)]$$

$$= \frac{M[d(x, z) + d(z, y)]}{1 + d(x, z) + d(z, y)}$$

$$= \frac{Md(x, z)}{1 + d(x, z) + d(z, y)} + \frac{Md(z, y)}{1 + d(x, z) + d(z, y)}$$

$$\leq \frac{Md(x, z)}{1 + d(x, z)} + \frac{Md(z, y)}{1 + d(z, y)}$$

[$\because$ $1 + d(x, z) \leq 1 + d(x, z) + d(z, y)$ etc.]

$$= d^*(x, z) + d^*(z, y).$$

Thus $d^*(x, y) \leq d^*(x, z) + d^*(z, y)$.

Also $d^*(x, y) = \dfrac{Md(x, y)}{1 + d(x, y)} < M$,

$\left[\because 0 \leq \dfrac{d(x, y)}{1 + d(x, y)} < 1\right]$

for every pair of points x, y of X.

Hence d^* is a bounded metric for X with $\delta(X) \leq M$.

Cor: *If (X, d) is any metric space then d^* defined by*

$$d^*(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric for X with $\delta(X, d^*) \leq 1$.

Proof: Put $M = 1$ in the above theorem and proceed.

Theorem 2: *Let (X, d) be a metric space and let*

$$d^*(x, y) = \min\{1, d(x, y)\}.$$

Then d is a metric for X.

Proof: **[m 1]**: We have

$$d^*(x, y) = 1 \quad \text{or} \quad d^*(x, y) = d(x, y).$$

Since d is a metric, therefore $d(x, y) \geq 0$.

Hence, $d^*(x, y) \geq 0 \quad \forall\, x, y \in X$.

[m 2]: If $d^*(x, y) = 0$, then $d^*(x, y) = d(x, y) = 0$. Since d is a metric, therefore $d(x, y) = 0 \Rightarrow x = y$.

Again if $x = y$, then $d(x, y) = 0$ and so

$$d^*(x, y) = d(x, y) = 0.$$

Hence, $d^*(x, y) = 0$ iff $x = y$.

[m 3]: We have either

$$d^*(x, y) = d(x, y) \quad \text{or} \quad d^*(x, y) = 1.$$

If $d^*(x, y) = d(x, y)$, then $d(x, y) < 1$.

Hence $d(y, x) = d(x, y) < 1$.

But $d(y, x) < 1$ implies that

$$d^*(y, x) = d(y, x) = d(x, y) = d^*(x, y).$$

And if $d^*(x, y) = 1$, then $d(x, y) \geq 1$ and consequently $d(y, x) \geq 1$.

But $d(y, x) \geq 1$ implies that $d^*(y, x) = 1$.

Hence, $d^*(x, y) = d^*(y, x)$.

Thus in either case, we have

$$d^*(x, y) = d^*(y, x).$$

[m 4]: We have to prove that

$$d^*(x, y) \leq d^*(x, z) + d^*(z, y). \qquad \ldots(1)$$

Now $d^*(x, y) \leq 1$. Hence if either $d^*(x, z) = 1$ or $d^*(z, y) = 1$ then the inequality (1) holds. And if both $d^*(x, z) \neq 1$ and $d^*(z, y) \neq 1$, then

$$d^*(x, z) = d(x, z) \quad \text{and} \quad d^*(z, y) = d(z, y).$$

We then have

$$d(x, y) \leq d(x, z) + d(z, y) = d^*(x, z) + d^*(z, y). \qquad \ldots(2)$$

But $d^*(x, y) = \min\{1, d(x, y)\} \leq d(x, y)$. $\qquad \ldots(3)$

From (2) and (3), we have

$$d^*(x, y) \leq d^*(x, y) + d^*(z, y).$$

Hence d^* is a metric for X.

Since $d^*(x, y) \leq 1$ for every pair of points x, y of X, therefore d^* is a bounded metric for X with $\delta(X) \leq 1$.

The metric d^* is called the standard bounded metric corresponding to the metric d.

Comprehensive Exercise 1

1. (i) Define a metric space and given one example.
   (ii) Prove that the mapping $d : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ defined by
   $$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$ is a metric on $\mathbf{R}$.
2. Let $\mathbf{R}^2$ denote the set of all ordered pairs of real numbers. Show that the mapping
   $$d : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R} \text{ defined by}$$
   $$d(x, y) = |x_1 - y_1| + |x_2 - y_2|,$$
   where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbf{R}^2$ is a metric on $\mathbf{R}^2$.
3. Let $\mathbf{R}^2$ denote the set of all ordered pairs of real numbers. Show that the mapping
   $$d : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R} \text{ defined by}$$
   $$d(x, y) = \max. \{|x_1 - y_1|, |x_2 - y_2|\},$$
   where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbf{R}^2$ is a metric on $\mathbf{R}^2$.
4. Give two different metrics for the set $\mathbf{R}$ of real numbers.
5. Give an example of a pseudo-metric which is not a metric. Is every metric a pseudo-metric ?
6. Let $C[0, 1]$ denote the family of all Riemann integrable functions f from $[0, 1]$ into $\mathbf{R}$. Show that the mapping
   $$d : C[0, 1] \times C[0, 1] \to \mathbf{R}$$
   defined by $d(f, g) = \int_0^1 |f - g|(x)\, dx = \int_0^1 |f(x) - g(x)|\, dx$,
   where $f, g \in C[0, 1]$ is a pseudo-metric on $C[0, 1]$ but not a metric on $C[0, 1]$.
7. If $d(x, y) = 2|x - y| \ \forall\, x, y \in \mathbf{R}$, show that d is a metric for $\mathbf{R}$.
8. (i) Let d be a metric for a non-empty set X. Show that d^* defined as $d^*(x, y) = 2\, d(x, y)$ is also a metric for X.
   (ii) Let (X, d) be any metric space. Show that the function d^* defined by
   $$d^*(x, y) = \min \{2, d(x, y)\}, \forall\, x, y \in X$$

is a metric on X.

9. Let d_1, d_2 be two metrics for a non-empty set X. Show that the mapping d defined by

$$d(x, y) = d_1(x, y) + d_2(x, y) \ \forall\ x, y \in X$$

is also a metric for X.

10. Let X be a non-empty set and let d be a real valued function of ordered pairs of elements of X which satisfies the following conditions :

(i) $d(x, y) = 0 \Leftrightarrow x = y$

(ii) $d(x, y) \leq d(x, z) + d(y, z) \ \forall\ x, y, z \in X$.

Show that d is a metric on X.

11. Let (X, d) be any metric space. Show that the function d_1 defined by

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \ \forall\ x, y \in X$$

is a metric on X.

12. Let $\mathbf{R}$ be the set of all real numbers and let

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}, \forall\ x, y \in \mathbf{R}.$$

Show that d is a metric for $\mathbf{R}$.

13. Show that the function

$$d(x, y) = \min\{2, |x - y|\}, \forall\ x, y \in \mathbf{R}$$

is a metric for $\mathbf{R}$.

14. Let C denote the set of complex numbers. Show that the mapping $d : \boldsymbol{C} \times \boldsymbol{C} \to \boldsymbol{R}$ defined by

$$d(u, v) = \begin{cases} |u| + |v|, & \text{if } u \neq v \\ 0, & \text{if } u = v \end{cases}$$

is a metric on C

15. Let (X_1, d_1) and (X_2, d_2) be two metric spaces. For any pair of points $x = (x_1, x_2), y = (y_1, y_2)$ in $X = X_1 \times X_2$, define d by setting

$$d(x, y) = \sqrt{\{d_1^2(x_1, y_1) + d_2^2(x_2, y_2)\}}.$$

Prove that d is a metric for $X = X_1 \times X_2$.

The metric space (X, d) is called the product space of the metric spaces (X_1, d_1) and (X_2, d_2).

16. Let (X_1, d_1) and (X_2, d_2) be two metric spaces. For any pair of points $x = (x_1, x_2), y = (y_1, y_2)$ in $X = X_1 \times X_2$, define d by setting

$$d(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2).$$

Prove that d is a metric for $X = X_1 \times X_2$.

17. Show that the function d defined by $d(p, q) = \| p - q \|$, where p, q are vectors in a normed vector space V, is a metric on V.
18. Define the diameter of a subset A of a metric space X. What is the diameter of the empty set ? What do you mean by the distance between two non-empty subsets A and B of the metric space X ? If x is a point of X and A is a subset of X, write the distance of x from A.
19. Let A and B be any two non-empty subsets of a metric space X. Prove that if $A \subseteq B$, then $\delta(A) \leq \delta(B)$.

Open and Closed Sets in a Metric Space

10.10 Spheres (or Balls)

Definition: *Let (X, d) be a metric space, and let $x_0 \in X$. If r is any positive real number, then the set*

$$\{x \in X : d(x, x_0) < r\}$$

is called an **open sphere** *(or open ball). The point x_0 is called the* **centre** *and r the* **radius** *of the sphere.*

We shall denote an open sphere by $S(x_0, r)$ or by $B(x_0, r)$. The notations $S_r(x_0)$ or $B_r(x_0)$ are also used. Thus $S(x_0, r) = \{x \in X : d(x, x_0) < r\}$.

Similarly a **closed sphere** *(or* **closed ball***) is denoted and defined by*

$$S[x_0, r] = \{x \in X : d(x, x_0) \leq r\}.$$

Note: A sphere (or a ball) is always non-empty since it contains its centre.

Illustrative Examples

Example 17: *Describe open and closed spheres for the usual metrics for* $\mathbf{R}$ *and* $\mathbf{R}^2$.

Solution: The usual metric for $\mathbf{R}$ is defined by

$$d(x, y) = |x - y|, \text{ where } x, y \in \mathbf{R}.$$

Hence for the usual metric space $\mathbf{R}$ if $x_0 \in \mathbf{R}$, the open sphere with centre x_0 and radius r is given by

$$\begin{aligned} S(x_0, r) &= \{x \in \mathbf{R} : |x - x_0| < r\} \\ &= \{x \in \mathbf{R} : x_0 - r < x < x_0 + r\} \\ &=]x_0 - r, x_0 + r[. \end{aligned}$$

Hence, the open spheres on the real line are open intervals.

Thus $$S\left(1, \frac{3}{2}\right) = \left]1 - \frac{3}{2}, 1 + \frac{3}{2}\right[= \left]-\frac{1}{2}, \frac{5}{2}\right[,$$

$$S(-1, 1) = |-1 - 1, -1 + 1| = |-2, 0| \text{ etc.}$$

Again, $S\left(\frac{1}{4}, \frac{1}{4}\right) = \left\{ x \in [0, 1] : \frac{1}{4} - \frac{1}{4} < x < \frac{1}{4} + \frac{1}{4} \right\}$

$= \left\{ x \in [0, 1] : 0 < x < \frac{1}{2} \right\} = \left] 0, \frac{1}{2} \right[$.

Similarly, $S\left[\frac{1}{4}, \frac{1}{4}\right] = \left\{ x \in [0, 1] : \frac{1}{4} - \frac{1}{4} \le x \le \frac{1}{4} + \frac{1}{4} \right\}$

$= \left\{ x \in [0, 1] : 0 \le x \le \frac{1}{2} \right\} = \left[0, \frac{1}{2}\right]$.

Further, $S\left(0, \frac{1}{8}\right) = \left\{ x \in [0, 1] : 0 - \frac{1}{8} < x < 0 + \frac{1}{8} \right\}$

$= \left\{ x \in [0, 1] : -\frac{1}{8} < x < \frac{1}{8} \right\}$

$= \left[0, \frac{1}{8}\right[$.

[Note that no number less than 0 can belong to $S\left(0, \frac{1}{8}\right)$ since it does not belong to the metric space $X = [0, 1]$].

Also, $S\left(\frac{1}{16}, \frac{1}{16}\right) = \left\{ x \in [0, 1] : \frac{1}{16} - \frac{1}{16} < x < \frac{1}{16} + \frac{1}{16} \right\}$

$= \left\{ x \in [0, 1] : 0 < x < \frac{1}{8} \right\} = \left] 0, \frac{1}{8} \right[$.

Finally, $S\left(\frac{1}{32}, \frac{3}{32}\right) = \left\{ x \in [0, 1] : \frac{1}{32} - \frac{3}{32} < x < \frac{1}{32} + \frac{3}{32} \right\}$

$= \left\{ x \in [0, 1] : -\frac{1}{16} < x < \frac{1}{8} \right\} = \left[0, \frac{1}{8}\right[$.

Observe that $S\left(0, \frac{1}{8}\right)$ and $S\left(\frac{1}{32}, \frac{3}{32}\right)$ are the same sets.

Example 19: *Describe open spheres of unit radius about* $(0, 0)$ *for each of the following metrics for* $\mathbf{R}^2$:

(i) $d_1(z_1, z_2) = \sqrt{\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}}$

(ii) $d_2(z_1, z_2) = |x_1 - x_2| + |y_1 - y_2|$

(iii) $d_3(z_1, z_2) = max\{|x_1 - x_2|, |y_1 - y_2|\}$

where $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$ *are any two points of* $\mathbf{R}^2$.

Solution: (i) As shown in example 17, the open unit sphere about $(0, 0)$ in this case is the interior of the circle

$$(x - 0)^2 + (y - 0)^2 = 1 \text{ i.e., } x^2 + y^2 = 1.$$ [See ... 1.]

(ii) In this case, we have

$$S\{(0,0),1\} = \{(x, y) \in \mathbf{R}^2 : |x-0| + |y-0| < 1$$

$$i.e., \quad |x| + |y| < 1\}.$$

Thus the open sphere in this case is the interior of the square bounded by the following four straight lines :

$$x + y = 1, x - y = 1, -x + y = 1, -x - y = 1. \qquad \text{[See Fig. 2.]}$$

(iii) In this case, we have

$$S\{(0,0),1\} = \{(x, y) \in \mathbf{R}^2 : \max\{|x-0|, |y-0|\} < 1\}$$

$$= \{(x, y) \in \mathbf{R}^2 : \max\{|x|, |y|\} < 1\}.$$

Thus the open sphere in this case is the interior of the square bounded by the following four straight lines :

$$x = 1, x = -1, y = 1 \text{ and } y = -1. \qquad \text{[See Fig. 3.]}$$

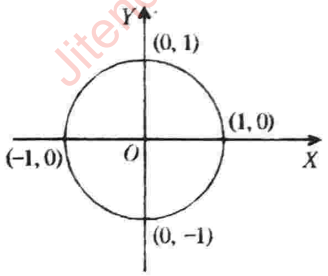

Fig. 1

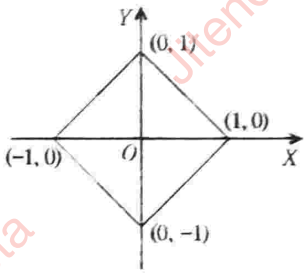

Fig. 2

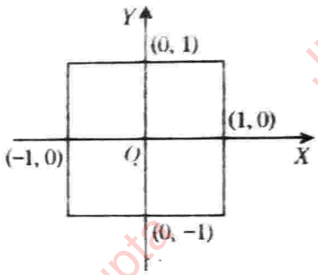

Fig. 3

Example 20: *Let (X, d) be any discrete metric space. Describe open spheres for d.*

Solution: The discrete metric d on X is defined by

$$d(x, y) = \begin{cases} 0 & \text{when } x = y \\ 1 & \text{when } x \neq y. \end{cases}$$

Let $x_0 \in X$. If r is any positive real number greater than 1, we have

$$S(x_0, r) = \{x \in X : d(x, x_0) < r\}$$

$$= X. \quad [\because d(x, x_0) = 0 \text{ or } 1 \text{ each of which is less than } r \text{ so that } x \in X \Rightarrow x \in S(x_0, r)]$$

And if $0 < r \leq 1$, we have

$$S(x_0, r) = \{x \in X : d(x, x_0) < r\}$$

$$= \{x_0\}.$$

$$[\because d(x_0, x_0) = 0 < r \text{ and } d(x, x_0) = 1 \not< r \text{ if } x \neq x_0]$$

For example, $S\left(x_0, \frac{1}{2}\right) = \{x_0\}$, $S(x_0, 1) = \{x_0\}$, $S\left(x_0, \frac{3}{2}\right) = X$ etc.

Example 21: *Let (X, d) be a metric space and let $p \notin S(x_0, r)$ where $x_0 \in X$ and $r > 0$. Then show that $d(p, S(x_0, r)) \geq d(x_0, p) - r$.*

Solution: Let x be an arbitrary point of $S(x_0, r)$. Then

$$d(x_0, p) \le d(x_0, x) + d(x, p)$$
$$[\because \text{ in a metric space, } d(x, y) \le d(x, z) + d(z, y)]$$

or $d(x, p) \ge d(x_0, p) - d(x_0, x)$ ...(1)

But $x \in S(x_0, r) \Rightarrow d(x_0, x) < r$ ...(2)

Hence, from (1) and (2), we have

$$d(x, p) \ge d(x_0, p) - r. \quad ...(3)$$

Since (3) holds for all $x \in S(x_0, r)$ it follows that

$$d(p, S(x_0, r)) \ge d(x_0, p) - r.$$

10.11 Open Sets

Definition: *Let (X, d) be a metric space. A subset A of X is said to be* ***d-open*** *(or simply open) iff to each $x \in A$, there exists $r > 0$, such that $S(x, r) \subseteq A$.*

Thus a subset A of a metric space X is an open set iff each point of A is the centre of some open sphere which is contained in A.

Illustrative Examples

Example 22: *Consider the set* **R** *of all real numbers with usual metric $d(x, y) = |x - y|$ and find whether or not the following sets are open :*

$$A = [0, 1[,\ B =]0, 1[,\ C =]0, 1],\ D = [0, 1],\ E =]0, 1[\cup]2, 3[,$$
$$F = \{1\},\ G = \{1, 2, 3\}.$$

Solution: Recall that an open sphere about $x_0 \in \mathbf{R}$ and radius r is in this case the open interval $]x_0 - r, x_0 + r[$. Thus in order to see whether a set is open or not, we have to see whether or not about each point of a given set, an open interval of the type described above exists which is contained in that set.

Here A is not an open set, since however small we choose a positive number r, the open interval $]0 - r, 0 + r[=]-r, r[$ is not contained in A. Thus there exists no open sphere with 0 as centre and contained in A.

B is an open set. For let x be any point of B. We take $r = \min\{x - 0, 1 - x\}$. Then it is evident that $]x - r, x + r[\subseteq B$. For example, consider $\frac{1}{4} \in]0, 1[$. Then

$$r = \min\left\{\frac{1}{4} - 0, 1 - \frac{1}{4}\right\} = \min\left\{\frac{1}{4}, \frac{3}{4}\right\} = \frac{1}{4}.$$

Thus $$\left]\frac{1}{4} - \frac{1}{4}, \frac{1}{4} + \frac{1}{4}\right[= \left]0, \frac{1}{2}\right[\subseteq]0, 1[= B.$$

C and D are not open sets (why ?). However E is an open set (why ?).

F is not open since it consists of a single point 1 and so it is not possible to find $r > 0$ such that $]1 - r, 1 + r[\subseteq F$ for $]1 - r, 1 + r[$ contains infinitely many points besides 1.

Similarly G is not open.

Example 23: *Let $X = [0, 1]$ and let d be the metric on X defined by*

$$d(x, y) = |x - y|, \forall x, y \in X.$$

Show that $A = [0, 1[$ is an open set in the metric space (X, d).

Solution: If $x \in]0, 1[$, then taking $r = \min\{x - 0, 1 - x\}$, we see that

$$S(x, r) =]x - r, x + r[\subseteq [0, 1[\quad i.e., \quad S(x, r) \subseteq A.$$

If $x = 0$, then taking $r = \frac{1}{2}$ we see that

$$S\left(0, \frac{1}{2}\right) = \left\{x \in [0, 1] : |x - 0| < \frac{1}{2}\right\} = \left\{x \in [0, 1] : -\frac{1}{2} < x < \frac{1}{2}\right\}$$

$$= \left[0, \frac{1}{2}\right[\subseteq [0, 1[\quad i.e., S\left(0, \frac{1}{2}\right) \subseteq A.$$

Thus to each $x \in A = [0, 1[$, there exists a positive real number r such that $S(x, r) \subseteq A$.

Hence, A is an open set in the metric space (X, d).

Note: We have seen in example 22 that $[0, 1[$ is not an open set for the metric space $(\mathbf{R}, d)$ where d is the usual metric on $\mathbf{R}$.

Example 24: *On the real line every open interval is an open set.*

Solution: Let $I =]a, b[$ be an open interval on $\mathbf{R}$ and let $x \in I$. Let $\varepsilon = \min\{x - a, b - x\}$. Then $I_x =]x - \varepsilon, x + \varepsilon[$ is an open interval (open sphere) with x as centre such that $I_x \subseteq I$. Hence, I is open.

10.12 Properties of Open Sets

Theorem 1: *In a metric space (X, d), the empty set $\varnothing$ and the whole space X are open sets.*

Proof: From the definition of an open set of a metric space (X, d) we know that a subset A of X is open iff to each $x \in A$ there exists $r > 0$ such that

$$S(x, r) \subseteq A.$$

Since the empty set $\varnothing$ has no points, therefore for the empty set $\varnothing$ this condition is vaccuously satisfied. Hence, the empty set $\varnothing$ is an open set.

Again by the definition of an open sphere we know that every open sphere is a subset of X.

So to each $x \in X$ and any $r > 0$, we have $S(x, r) \subseteq X$. Hence, X is an open set.

Theorem 2: *In a metric space, every open sphere is an open set.*

Proof: Let (X, d) be a metric space and let $S(x_0, r)$ be an open sphere in it. We have to show that to each point $p \in S(x_0, r)$ there exists an open sphere centred at p and contained in $S(x_0, r)$. So let p be an arbitrary point of $S(x_0, r)$.

Now $\quad p \in S(x_0, r) \Rightarrow d(p, x_0) < r \Rightarrow r - d(p, x_0) > 0.$

Let $\quad \rho = r - d(p, x_0)$, so that $\rho > 0$.

We claim that the open sphere $S(p, \rho)$ is contained in $S(x_0, r)$.

For, let $x \in S(p, \rho)$. Then

$$d(x, p) < \rho = r - d(p, x_0). \qquad \ldots(1)$$

Now $\quad d(x, x_0) \le d(x, p) + d(p, x_0)$

$[\because$ in a metric space, $d(x, y) \le d(x, z) + d(z, y)]$

$< r - d(p, x_0) + d(p, x_0)$ [By (1)]

$= r.$

Thus, $\quad d(x, x_0) < r$ and so $x \in S(x_0, r)$.

Since $x \in S(p, \rho) \Rightarrow x \in S(x_0, r)$, therefore $S(p, \rho) \subseteq S(x_0, r)$.

Thus to each point $p \in S(x_0, r)$ there exists an open sphere $S(p, \rho)$ centred at p and contained in $S(x_0, r)$. Hence, $S(x_0, r)$ is an open set.

Theorem 3: *In a metric space, the union of an arbitrary collection of open sets is open.*

Proof: Let (X, d) be a metric space and let $\{G_\lambda : \lambda \in \Lambda\}$ be an arbitrary collection of open subsets of X. Here Λ is an index set and is such that for $\lambda \in \Lambda$, G_λ is an open set.

To show that $G = \cup \{G_\lambda : \lambda \in \Lambda\}$ is open.

Let $x \in G$. Then by definition of union, $x \in G_\lambda$ for some $\lambda \in \Lambda$.

Since G_λ is open, there exists $r > 0$, such that

$$S(x, r) \subseteq G_\lambda.$$

But $G_\lambda \subseteq G$. Hence $S(x, r) \subseteq G$.

Thus we have shown that to each $x \in G$, there exists a positive number r such that $S(x, r) \subseteq G$. Hence G is open.

Theorem 4: *In a metric space, the intersection of a finite number of open sets is open.*

Proof: Let (X, d) be a metric space and let $\{G_i : i = 1, 2, \ldots n\}$ be a finite collection of open subsets of X. To show that

$$H = \cap \{G_i = i = 1, 2, \ldots, n\} \text{ is also open.}$$

Let $x \in H$. Then $x \in G_i$ for every $i = 1, 2, \ldots n$.

Since each G_i is open, there exists $r_i > 0$ such that

$$S(x, r_i) \subseteq G_i, \; i = 1, 2, \ldots, n.$$

Let $\quad r = \min \{r_1, r_2, \ldots, r_n\}$.

Then $S(x, r) \subseteq S(x, r_i) \; \forall \; i = 1, 2, \ldots, n.$

It follows that

$$S(x, r) \subseteq G_i \ \forall\ i = 1, 2, \ldots, n.$$

This implies that

$$S(x, r) \subseteq \cap \{G_i : i = 1, 2, \ldots, n\} = H.$$

Thus it is shown that to each $x \in H$, there exists $r > 0$, such that $S(x, r) \subseteq H$.

Hence, H is open.

Characterization of open sets in terms of open spheres:

Theorem 5: *A subset of a metric space is open if and only if it is the union of a family of open spheres.*

Proof: The 'only if part'. Let (X, d) be a metric space and $A \subseteq X$. Let A be open. If A is empty, then it is the union of empty family of open spheres.

Now let A be non-empty and let x be any point of A. Since A is open, there exists an open sphere $S(x, r)$, $r > 0$ such that $S(x, r) \subseteq A$. It follows that

$$A \subseteq \cup \{S(x, r) : x \in A\} \subseteq A.$$

Hence $A = \cup \{S(x, r) : x \in A\}$. Thus A is the union of a family of open spheres.

The 'if part': Let A be the union of a family of open spheres. Then to show that A is open.

We know that in a metric space every open sphere is an open set. Also we know that in a metric space the union of an arbitrary collection of open sets is open.

Hence, if A is the union of a family of open spheres, then A is an open set.

Theorem 6: *Every non-empty open set on the real line is the union of a countable collection of pairwise disjoint open intervals.*

Proof: Let G be a non-empty open subset of $\mathbf{R}$ and let x be any point of G. Since G is open there exists an open interval with x as centre and contained in G. Define I_x to be the union of all open intervals which contain x and are contained in G. Then it is easy to see that (i) I_x is an open interval containing x and contained in G (ii) I_x contains every open interval which contains x and is contained in G (iii) if y is another point in I_x, then $I_x = I_y$. Further if x and y are any two distinct points of G, then either $I_x = I_y$ or $I_x \cap I_y = \varnothing$. For if $z \in I_x \cap I_y$, then $z \in I_x$ and $z \in I_y$ so that $I_x = I_z$ and $I_y = I_z$ and hence $I_x = I_y$. Now consider the collection F of all distinct sets of the form I_x for points x belonging to G. Then clearly F is a disjoint collection of open intervals and G is the union of this collection. It remains to prove that F is countable. If G_r denotes the set of all rational points in G, then G_r is clearly non-empty. Define a mapping f of G_r onto F as follows : for each $r \in G_r$, let $f(r)$ be that unique interval in F to which r belongs. Now G_r is countable, being a non-empty subset of the countable set $\mathbf{Q}$ of all rational numbers. It follows that F is countable and the proof is complete.

Illustrative Examples

Example 25: *Show that in a discrete metric space, every set is open.*

Solution: Let A be any subset of a discrete metric space (X, d). If $A = \varnothing$, then A is open. If $A \neq \varnothing$, let x be an arbitrary point of A. Since $S\left(x, \frac{1}{2}\right) = \{x\}$, we have $S\left(x, \frac{1}{2}\right) \subseteq A$. Hence, A is open.

Example 26: *Show that in a metric space, the complement of every singleton set is open. More generally, the complement of a finite set is open.*

Solution: Let $\{x\}$ be a singleton subset of a metric space (X, d). To show that its complement $\{x\}'$ is open.

If $\{x\}' = \varnothing$, then it is open. So let $\{x\}' \neq \varnothing$ and let $y \in \{x\}'$. Then evidently $x \neq y$ so that $d(x, y) = r > 0$. If $r_1 \leq r$, then $d(x, y) = r \not\leq r_1$ and so the open sphere $S(y, r_1)$ does not contain x and is therefore contained in $\{x\}'$. Hence $\{x\}'$ is open.

Again, let $A = \{x_1, x_2, \ldots, x_n\}$ be a finite subset of X. We may regard A as the union of singleton sets. Thus we may write

$$A = \{x_1\} \cup \{x_2\} \cup \ldots \cup \{x_n\}.$$

Then $$A' = [\{x_1\} \cup \{x_2\} \cup \ldots \cup \{x_n\}]' = \{x_1\}' \cap \{x_2\}' \cap \ldots \cap \{x_n\}'.$$

Since each $\{x_i\}'$, $i = 1, 2, \ldots, n$, is open and the intersection of a finite collection of open sets is open, therefore A' is open.

Example 27: *Give an example to show that the intersection of an infinite number of open sets is not necessarily open.*

Solution: Consider the collection $\{]-1/n, 1/n[: n \in \mathbf{N}\}$ of open intervals in $\mathbf{R}$ with the usual metric $d(x, y) = |x - y|$. Then $\cap \{]-1/n, 1/n[: n \in \mathbf{N}\} = \{0\}$ which is not open since there exists no $r > 0$ such that $]-r, r[\subseteq \{0\}$.

Hence, in a metric space the intersection of an infinite collection of open sets is not necessarily an open set.

Equivalent Metrics

Definition: *Two metrics d and d^* on the same set X are said to be equivalent iff every d-open set is d^*-open and every d^*-open set is d-open.*

Example 28: *Let (X, d) be a metric space. Then the mapping*

$$d^* : X \times X \to \mathbf{R}$$

defined by $d^*(x, y) = \dfrac{Md(x, y)}{1 + d(x, y)}$, *$M > 0$ is also a metric for X. Also show that d and d^* are equivalent.*

Solution: We have already shown that d^* is a metric for X. We now show that d and d^* are equivalent. It suffices to show that every d-open sphere centred at $x \in X$ contains a d^*-open sphere centred at x and conversely.

Let $S(x, r)$, $r > 0$ be any d-open sphere centred at $x \in X$. Consider the d^*-open sphere $S^*(x, \rho)$ centred at x and having radius $\rho = \dfrac{Mr}{1+r}$. We shall show that

$$S^*(x, \rho) \subseteq S(x, r).$$

Now $y \in S^*(x, \rho) \Rightarrow d^*(x, y) < \rho \Rightarrow \dfrac{Md(x, y)}{1 + d(x, y)} < \dfrac{Mr}{1+r}$

$\Rightarrow \quad d(x, y) + rd(x, y) < r + rd(x, y)$

$\Rightarrow \quad d(x, y) < r$

$\Rightarrow \quad y \in S(x, r).$

Hence $\quad S^*(x, \rho) \subseteq S(x, r).$

Conversely, let $S^*(x, \rho)$, $\rho > 0$, be any d^*-open sphere centred at $x \in X$. Since $d^*(x, y) \leq M$ for every pair of points x, y of X, we may assume, without loss of generality that $0 < \rho \leq M$. Now consider the d-open sphere $S(x, r)$ where $r = \dfrac{\rho}{M - \rho}$. We shall show that $S(x, r) \subseteq S^*(x, \rho)$. We have

$$y \in S(x, r) \Rightarrow d(x, y) < r$$

$\Rightarrow \quad \dfrac{d^*(x, y)}{M - d^*(x, y)} < \dfrac{\rho}{M - \rho}$

$$\left[\because d^*(x, y) = \frac{Md(x, y)}{1 + d(x, y)} \Rightarrow d(x, y) = \frac{d^*(x, y)}{M - d^*(x, y)}\right]$$

$\Rightarrow \quad Md^*(x, y) - \rho d^*(x, y) < \rho M - \rho d^*(x, y)$

$\Rightarrow \quad d^*(x, y) < \rho \Rightarrow y \in S^*(x, \rho).$

Hence $\quad S(x, r) \subseteq S^*(x, \rho).$

Thus we have shown that every d-open sphere centred at x contains a d^*-open sphere centred at x and conversely. It implies that every d-open set is d^*-open and conversely. Hence d and d^* are equivalent metrics on X.

Example 29: *Let (X, d) be a metric space and let $d^*(x, y) = \min\{1, d(x, y)\}$. Show that d^* is a metric for X. Also show that the two metrics d and d^* are equivalent.*

Solution: We have already shown that d^* is a metric for X. We now show that d and d^* are equivalent. It suffices to show that every d-open sphere about a point $p \in X$ contains a d^*-open sphere about p and conversely.

Since $d^*(x, y) \le d(x, y)$, we have

$$\{x \in X : d(p, x) < r\} \subseteq \{x \in X : d^*(p, x) < r\},$$

for every $p \in X$ and every $r > 0$.

On the other hand, if r is any positive number, let

$$\rho = \min(r, 1).$$

Then $\{x \in X : d^*(p, x) < \rho\} \subseteq \{x \in X : d(p, x) < r\}$,

for every $p \in X$.

It follows that every d-open set is d^*-open and every d^*-open set is d-open.

Example 30: *Let d be a metric for a non-empty set X and let*

$$\rho(x, y) = \min\{1, d(x, y)\} \text{ and } \sigma(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that ρ and σ are also metrics for X and that the three metrics d, ρ and σ are equivalent.

Solution: The proof follows from Comprehensive Exercise-1 Q. No. 17 and Ex. 29.

10.13 Closed Sets

Definition: *Let (X, d) be a metric space. A subset A of X is said to be d-**closed** (or simply closed) if the complement of A is open.*

Illustrative Examples

Example 31: *Show that every singleton set in $\mathbf{R}$ is closed for the usual metric d for $\mathbf{R}$.*

Solution: Let a be any point of $\mathbf{R}$. To show that $\{a\}$ is closed. We have

$$\mathbf{R} - \{a\} =]-\infty, a[\cup]a, \infty[.$$

But $]-\infty, a[$ and $]a, \infty[$ are open sets. Hence their union is also open. It follows that $\mathbf{R} - \{a\}$ is open. Hence, the complement of $\mathbf{R} - \{a\}$ *i.e.*, $\{a\}$ is closed.

Example 32: *Show that every closed interval is a closed set for the usual metric for $\mathbf{R}$.*

Solution: Let $a, b \in \mathbf{R}$ where $a < b$. To show that $[a, b]$ is closed. We have

$$\mathbf{R} - [a, b] = \{x \in \mathbf{R} : x < a \text{ or } x > b\} = \{x \in \mathbf{R} : x < a\} \cup \{x \in \mathbf{R} : x > b\}$$
$$=]-\infty, a[\cup]b, \infty[,$$

which is open, being a union of two open sets. Hence the complement of $\mathbf{R} - [a, b]$ *i.e.*, $[a, b]$ is closed.

Example 33: *Show that the right half open interval $[a, b[$ is neither closed nor open for the usual metric on $\mathbf{R}$.*

Solution: We have $a \in [a, b[$. Since there exists no open interval which contains a and is contained in $[a, b[$, it follows that $[a, b[$ is not open.

Again $\mathbf{R} - [a, b[=]-\infty, a[\in]b, \infty[$.

We have $b \in \mathbf{R} - [a, b[$. Since there is no open interval containing b and contained in $\mathbf{R} - [a, b[$, therefore $\mathbf{R} - [a, b[$ is not open.

Hence, the complement of $\mathbf{R} - [a, b[$ *i.e.*, $[a, b[$ is not closed.

Remark: Similarly we can show that the left half open interval $]a, b]$ is neither closed nor open for the usual metric on $\mathbf{R}$.

Example 34: *Show that in a metric space every singleton set is closed.*

Solution: Let $\{x\}$ be a singleton subset of a metric space (X, d). In order to show that $\{x\}$ is closed, we have to show that its complement $\{x\}'$ is open. If $\{x\}' = \varnothing$, then it is open. So let $\{x\}' \neq \varnothing$ and let $y \in \{x\}'$. Then obviously $x \neq y$ so that $d(x, y) = r > 0$. If $r_1 \leq r$, then $d(x, y) = r \not< r_1$ and so the open sphere $S(y, r_1)$ does not contain x and is therefore contained in $\{x\}'$.

Thus for every $y \in \{x\}'$, there exists an open sphere $S(y, r_1)$ such that $S(y, r_1) \subseteq \{x\}'$. Hence, $\{x\}'$ is open and consequently its complement $\{x\}$ is closed.

10.14 Properties of Closed Sets

Theorem 1: *In a metric space, every closed sphere is a closed set.*

Proof: Let (X, d) be a metric space and let $S[x_0, r]$ be a closed sphere in X. From the definition of a closed sphere, we have

$$S[x_0, r] = \{x \in X : d(x, x_0) \leq r\}.$$

In order to show that $S[x_0, r]$ is closed we shall show that its complement $S[x_0, r]'$ is open.

Now $S[x_0, r]'$ is open if it is empty and so we may assume that it is non-empty. Let $x \in S[x_0, r]'$. Then $x \notin S[x_0, r]$ and so $d(x, x_0) > r$. Thus $d(x, x_0) - r = \rho$, say, is a positive real number. We take an open sphere of radius ρ centred at x and show that

$$S(x, \rho) \subseteq S[x_0, r]'.$$

Let $y \in S(x, \rho)$. Then $d(x, y) < \rho$.

Now $d(x, x_0) \leq d(x, y) + d(y, x_0)$.

$\therefore$ $d(y, x_0) \geq d(x, x_0) - d(x, y)$

$> d(x, x_0) - \rho$ $\quad [\because\ d(x, y) < \rho]$

$= d(x, x_0) - [d(x, x_0) - r] = r.$

Since $d(y, x_0) > r$, therefore $y \notin S[x_0, r]$ and so $y \in S[x_0, r]'$.

Thus we have shown that $y \in S(x, \rho) \Rightarrow y \in S[x_0, r]'$.

$\therefore$ $S(x, \rho) \subseteq S[x_0, r]'$.

Since for every $x \subseteq S[x_0, r]'$, there exists an open sphere $S(x, \rho)$ such that $S(x, \rho) \subseteq S[x_0, r]'$, therefore $S[x_0, r]'$ is open. Accordingly its complement $S[x_0, r]$ is closed.

Cor: *Let (X, d) be a metric space and let S be a subset of X defined by*

$$S = \{x \in X : d(x, x_0) = r\}, (r > 0)$$

where x_0 is a point of X. Then S is a closed set.

Proof: We have

$$S' = \{x \in X : d(x, x_0) < r\} \cup \{x \in X : d(x, x_0) > r\} = S(x_0, r) \cup S'[x_0, r]$$

where $S'[x_0, r]$ is the complement of $S[x_0, r]$.

But the open sphere $S(x_0, r)$ is an open set and by the above theorem $S'[x_0, r]$ is open. Hence their union S' is also open. It follows that S is a closed set.

Theorem 2: *In a metric space (X, d), the empty set $\varnothing$ and the whole space X are closed sets.*

Proof: We know that in a metric space (X, d), the null set $\varnothing$ and X are open sets. But $\varnothing' = X$ and $X' = \varnothing$. Since a set is closed iff its complement is open, therefore X and $\varnothing$ are closed sets as well.

The intersection and union of closed sets:

Theorem 3: *Let (X, d) be a metric space and let $\{H_\lambda : \lambda \in \Lambda\}$ be an arbitrary collection of closed subsets of X. Then $\cap \{H_\lambda : \lambda \in \Lambda\}$ is also a closed set. In other words, the intersection of an arbitrary collection of closed sets is closed.*

Proof: We have H_λ is closed $\forall \lambda \in \Lambda$

$\Rightarrow$ $X - H_\lambda$ is open $\forall \lambda \in \Lambda$

$\Rightarrow$ $\cup \{X - H_\lambda : \lambda \in \Lambda\}$ is open

[$\because$ in a metric space the union of an arbitrary collection of open sets is open]

$\Rightarrow$ $X - \cap \{H_\lambda : \lambda \in \Lambda\}$ is open

[$\because$ By De-Morgan's law, $A' \cup B' = (A \cap B)'$]

$\Rightarrow$ the complement of $X - \cap \{H_\lambda : \lambda \in \Lambda\}$ is closed

[By definition of a closed set]

$\Rightarrow$ $\cap \{H_\lambda : \lambda \in \Lambda\}$ is closed.

Theorem 4: *Let (X, d) be a metric space and let $\{H_i : i = 1, 2, \ldots, n\}$ be a finite collection of closed subsets of X. Then $\cup \{H_i : i = 1, 2, \ldots, n\}$ is also a closed set. In other words, the union of a finite number of closed sets is closed.*

Proof: We have H_i is closed for $i = 1, 2, \ldots, n$

$\Rightarrow$ $X - H_i$ is open for $i = 1, 2, \ldots, n$ [By definition of a closed set]

$\Rightarrow$ $\cap \{X - H_i : i = 1, 2, \ldots, n\}$ is open [$\because$ in a metric space the intersection of a finite number of open sets is open]

$\Rightarrow$ $X - \cup \{H_i : i = 1, 2, \ldots, n\}$ is open

[$\because$ By De-Morgan's law, $A' \cap B' = (A \cup B)'$]

$\Rightarrow \quad \cup \{H_i : i = 1, 2, \ldots, n\}$ is closed. [By definition of a closed set]

Cor: *Every finite subset of a metric space is always closed.*

Proof: Let $A = \{x_1, x_2, \ldots, x_n\}$ be a finite subset of the metric space (X, d).

We know that in a metric space every singleton is a closed set. Also we know that in a metric space the union of a finite number of closed sets is closed.

Hence, $A = \{x_1, x_2, \ldots, x_n\} = \{x_1\} \cup \{x_2\} \cup \ldots \cup \{x_n\}$ is a closed set.

Illustrative Examples

Example 35: *Give an example to show that in a metric space the union of an infinite collection of closed sets is not necessarily closed.*

Solution: For the usual metric space $(\mathbf{R}, d)$ consider the infinite collection

$$F_n = \left[\frac{1}{n}, 1\right], n \in \mathbf{N}$$

of closed intervals.

We know that in the usual metric space $(\mathbf{R}, d)$ every closed interval is a closed set. So each F_n is a closed set in $(\mathbf{R}, d)$.

We have $\cup \{F_n : n \in \mathbf{N}\} = \{1\} \cup \left[\frac{1}{2}, 1\right] \cup \left[\frac{1}{3}, 1\right] \cup \left[\frac{1}{4}, 1\right] \cup \ldots =]0, 1]$.

Since $]0, 1]$ is not a closed set in $(\mathbf{R}, d)$, therefore it follows that in a metric space the union of an infinite collection of closed sets is not necessarily closed.

Example 36: *Give an example of a set in a metric space which (i) is both open and closed (ii) is neither open nor closed.*

Solution: Consider the metric space $(\mathbf{R}, d)$ where $d(x, y) = |x - y|, \forall\, x, y \in \mathbf{R}$ is the usual metric on $\mathbf{R}$.

(i) The null set $\varnothing$ and $\mathbf{R}$ are the examples of the sets which are both open and closed.

(ii) The right half open interval $[2, 4[$ is an example of a set which is neither open nor closed.

10.15 Neighbourhoods

Definition: *Let (X, d) be a metric space and let $x \in X$. A subset N of X is said to be a neighbourhood of x if there exists an open set G such that*

$$x \in G \subseteq N. \qquad \ldots(1)$$

In case N is an open set, it is called an **open neighbourhood** *of x.*

Since $x \in G$ and $G \subseteq N$, therefore if N is a neighbourhood of x, we must have $x \in N$.

Similarly N is said to be a neighbourhood of a subset A of X if there exists an open set G such that $A \subseteq G \subseteq N$.

Note 1: Since G in (1) is open, there exists $r > 0$ such that $S(x, r) \subseteq G \subseteq N$. *Thus N will be a neighbourhood of x iff there exists an open sphere centred at x and contained in N i.e., if there exists $r > 0$ such that $S(x, r) \subseteq N$.*

Further since every open sphere is an open set, it follows that every open sphere containing x is an open neighbourhood of x. In particular, an open sphere $S(x, r)$ centred at x is an open neighbourhood of x.

If $p \subseteq X$, then X is a neighbourhood of p. Note that if $r > 0$, then $S(p, r)$ is an open set and we have $p \in S(p, r) \subseteq X$. So, X is a neighbourhood of p.

Note 2: It is easy to see if N is a neighbourhood of x and $M \supseteq N$, then M is also a neighbourhood of x.

Note 3: It is also easy to see that the intersection of two neighbourhoods of x is also a neighbourhood of x.

Note 4: A set is open if and only if it contains a neighbourhood of each of its points.

Note 5: We shall denote the collection of all neighbourhoods of x by $\mathbf{N}(x)$.

Note 6: We shall use the abbreviated form 'nbd' or 'nhd' for the word neighbourhood.

Illustrative Examples

Example 37: *Consider the usual metric $d(x, y) = |x - y|$ for $\mathbf{R}$ and find whether or not the following subsets of $\mathbf{R}$ are nhds of 1.*

(i) $]0, 2[$; *(ii)* $]0, 2]$; *(iii)* $[1, 2]$; *(iv)* $]1, 2]$;

(v) $[1, 2[$; *(vi)* $[0, 2] - 1\frac{1}{2}$; *(vii)* $[0, 2] - \frac{1}{2}$; *(viii)* $\mathbf{R}$.

Solution: (i) Yes, since $]0, 2[$ is an open set containing 1. Here $]0, 2[$ is an open nhd of 1.

(ii) Yes, since $]0, 2[$ is an open set such that

$$1 \in]0, 2[\subseteq]0, 2].$$

(iii) No, since there exists no open set contained in $[1, 2]$ and containing 1. Observe that however small we take $\varepsilon > 0$, the open interval $]1 - \varepsilon, 1 + \varepsilon[$ contains points to the left of 1 and consequently $]1 - \varepsilon, 1 + \varepsilon[$ cannot be a subset of $[1, 2]$.

(iv) No, since 1 is not a point of $]1, 2]$.

(v) No, since there exists no open set contained in $[1, 2[$ and containing 1.

(vi) Yes, since $\left]0, 1\frac{1}{4}\right[$ is an open set such that

$$1 \in \left]0, 1\frac{1}{4}\right[\subseteq [0, 2] - 1\frac{1}{2}.$$

(vii) Yes, since $\left]\frac{3}{4}, 2\right[$ is an open set such that

$$1 \in \left]\frac{3}{4}, 2\right[\subseteq [0, 2] \sim \frac{1}{2}.$$

(viii) Yes, since **R** is open and contains 1.

Example 38: *Let (X, d) be a discrete space and let $x \in X$. Show that every subset of X, containing x is a neighbourhood of x.*

Solution: Let A be any set containing x. Recall that d is defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

We have $S\left(x, \frac{1}{2}\right) = \left\{y : y \in X \text{ and } d(y, x) < \frac{1}{2}\right\}$

$$= \{x\} \subseteq A.$$

But $S\left(x, \frac{1}{2}\right)$ is an open set, being an open sphere. Hence A is a neighbourhood of x.

10.16 Properties of Neighbourhoods

Theorem 1: *Let (X, d) be a metric space and let $A \subseteq X$. If N is a neighbourhood of A and $M \supseteq N$, then M is also a neighbourhood of A, that is, every superset of a nhd of A is also a nhd of A.*

Proof: Since N is a nhd of A, there exists an open set G such that

$$A \subseteq G \subseteq N. \qquad \ldots(1)$$

Also $$N \subseteq M. \qquad \ldots(2)$$

From (1) and (2), we have

$$A \subseteq G \subseteq M.$$

Hence, M is also a nhd of A.

Theorem 2: *Let (X, d) be a metric space and let $A \subseteq X$. If $N_1, N_2, \ldots, N_k$ are a finite number of neighbourhoods of A, then*

$$\cap \{N_i : i = 1, 2, \ldots, k\}$$

is also a neighbourhood of A, that is, the intersection of a finite number of neighbourhoods of A is also a neighbourhood of A.

Proof: Since each N_i $(i = 1, 2, \ldots, k)$ is a nhd of A, there exist open sets G_i $(i = 1, 2, \ldots, k)$ such that

$$A \subseteq G_i \subseteq N_i, i = 1, 2, \ldots, k.$$

Now $$A \subseteq G_i \ \forall\ i = 1, 2, \ldots, k$$

$\Rightarrow$ $$A \subseteq \cap \{G_i : i = 1, 2, \ldots, k\}.$$

Also $G_i \subseteq N_i \ \forall\, i = 1, 2, \ldots, k$

$\Rightarrow \quad \cap \{G_i : i = 1, 2, \ldots, k\} \subseteq \cap \{N_i : i = 1, 2, \ldots, k\}.$

Hence $A \subseteq \cap \{G_i : i = 1, 2, \ldots, k\} \subseteq \cap \{N_i : i = 1, 2, \ldots, k\}.$

But $\cap \{G_i : i = 1, 2, \ldots, k\}$ is an open set, being the intersection of a finite number of open sets.

It follows that $\cap \{N_i : i = 1, 2, \ldots, k\}$ is also a nhd of A.

Remark: The intersection of an infinite number of neighbourhoods of a set need not be a neighbourhood of that set. For example, in the metric space $(\mathbf{R}, d)$ where $d(x, y) = |x - y|$, $\left]-\frac{1}{n}, \frac{1}{n}\right[$ is a nhd of $\{0\}$ for each $n \in \mathbf{N}$, but

$$\cap \left\{\left]-\frac{1}{n}, \frac{1}{n}\right[: n \in \mathbf{N}\right\} = \{0\},$$ which is not a nhd of $\{0\}$.

Note that there exists no open set G such that $\{0\} \subseteq G \subseteq \{0\}$.

Theorem 3: *Let (X, d) be a metric space and let $A \subseteq X$. Then the set*

$$N_r(A) = \{x \in X : d(x, A) < r\}, r > 0$$

is an open neighbourhood of A.

Proof: First observe that $A \subseteq N_r(A)$. For if $y \in A$, then $d(y, A) = 0 < r$ and so by definition of $N_r(A)$, we have $y \in N_r(A)$. We now show that $N_r(A)$ is open. Let x be an arbitrary point of $N_r(A)$. Then $d(x, A) < r$ so that $r - d(x, A) > 0$.

Choose $\delta > 0$ such that

$$\delta < r - d(x, A). \qquad \ldots(1)$$

We shall show that

$$S(x, \delta) \subseteq N_r(A).$$

Let $y \in S(x, \delta)$. Then $d(x, y) < \delta$. $\ldots(2)$

But $d(y, A) - d(x, A) \le d(x, y)$.

$\therefore \quad d(y, A) \le d(x, A) + d(x, y)$

$< d(x, A) + \delta$ [By (2)]

$< d(x, A) + r - d(x, A)$ [By (1)]

$= r.$

Thus $d(y, A) < r$ and consequently $y \in N_r(A)$.

It follows that $S(x, \delta) \subseteq N_r(A)$. Thus we have shown that to each $x \subseteq N_r(A)$, there exists an open sphere $S(x, \delta)$ contained in $N_r(A)$ and consequently $N_r(A)$ is open. Since $N_r(A)$ is open and contains A, it is an open nhd of A.

Characterization of open sets in terms of Neighbourhoods

Theorem 4: *A subset in a metric space is open if and only if it is a neighbourhood of each of its points.*

Proof: Let (X, d) be a metric space and let $A \subseteq X$.

First suppose that A is open and let x be an arbitrary point of A. Then A is a nhd of x since A is an open set such that $x \in A \subseteq A$. Thus A is a nhd of each of its points.

Conversely let A be a nhd of each of its points. To show that A is open. Let x be any point of A. Since A is a nhd of x, there exists an open set G_x such that

$$x \in G_x \subseteq A.$$

Let $G = \cup \{G_x : x \in A\}$.

We shall show that $G = A$.

If $x \in A$, then by definition of G_x, $x \in G_x$ and so $x \in \cup \{G_x : x \in A\} = G$.

$\therefore$ $A \subseteq G$. ...(1)

Again if $y \in G$, then $y \in G_x$ for some $x \in A$. But $G_x \subseteq A$, and hence $y \in A$.

$\therefore$ $G \subseteq A$. ...(2)

From (1) and (2), we have

$$A = G.$$

But G is open, being the union of a collection of open sets. It follows that A is open.

Remark: Theorems similar to theorems 1, 2, 3 can be proved for neighbourhoods of a point x instead of the neighbourhoods of the set A. Thus the theorem corresponding to theorem 2 can be stated and proved as follows :

Let (X, d) be a metric space and let $x \in X$. If N_i, $i = 1, 2, \ldots, k$ are finite number of nhds of x, then $\cap \{N_i : i = 1, 2, \ldots, k\}$ is also a nhd of x.

Proof: Since each N_i $(i = 1, 2, \ldots, k)$ is a nhd of x, there exist open sets G_i $(i = 1, 2, \ldots, k)$ such that

$$x \in G_i \subseteq N_i, \; i = 1, 2, \ldots, k.$$

This implies that

$$x \in \cap \{G_i : i = 1, 2, \ldots, k\} \subseteq \cap \{N_i : i = 1, 2, \ldots, k\}.$$

But $\cap \{G_i : i = 1, 2, \ldots, k\}$ is an open set, being the intersection of a finite number of open sets. It follows that $\cap \{N_i : i = 1, 2, \ldots, k\}$ is also nhd of x.

10.17 Limit Points and Adherent Points of a Set

Limit point of a set: **Definition:** *Let (X, d) be a metric space and let A be a subset of X. A point $x \in X$ is called a* **limit point** *(or an* **accumulation point***) of A if every neighbourhood of x contains a point of A other than x.*

The set of all limit points of A is called the derived set of A and shall be denoted by $D(A)$.

Thus x will be a limit point of A iff every neighbhourhood of x intersects A in a point different from x.

Obviously x will be a limit point of A iff for every $r > 0$, the open sphere $S(x, r)$ contains a point of A other than x.

Observe that x will not be a limit point of A if there exists a neighbourhood N of x such that

$$N \cap A = \emptyset \quad \text{or} \quad N \cap A = \{x\}.$$

A point $x \in X$ is said to be a **point of condensation** of A if every neighbourhood of x contains uncountably many points of A. Note that every point of condensation is a limit point but not conversely. Also note that a limit point of A need not be a point of A. It may or may not belong to A.

Adherent point of a set: Definition: *Let A be a subset of a metric space X and let $x \in X$. Then x is called an* **adherent point** *of A if every neighbourhood of x contains a point of A (not necessarily distinct from x). The set of all adherent points of A is called the* **adherence** *of A and will be denoted by* **Adh** A.

From the above definition of an adherent point of a set it is obvious that every point of A is an adherent point of A *i.e.*, $A \subseteq \text{Adh } A$.

Also from the definitions of a limit point and an adherent point of a set A, it is obvious that every limit point of A is also an adherent point of A but the converse is not always true.

Further, we observe that $\text{Adh } A = A \cup D(A)$.

For, $x \in \text{Adh } A \Leftrightarrow x \in A$ or x is a limit point of A

$$\Leftrightarrow x \in A \text{ or } x \in D(A)$$

$$\Leftrightarrow x \in A \cup D(A).$$

Hence, $\text{Adh } A = A \cup D(A)$.

Isolated point of a set: Definition: *A point x of a metric space X is said to be an* ***isolated point*** *of a subset A of X if $x \in A$ but x is not a limit point of A.*

Perfect set : Definition: *A closed set which has no isolated points is said to be* **perfect.**

If follows from the above definitions that an adherent point is either an isolated point or a limit point.

Theorem 1: *Let (X, d) be a metric space and let A be a subset of X. A point $x \in X$ is a limit point of A if and only if every open sphere centred at x contains infinitely many points of A.*

Proof: Let every open sphere centred at x contain infinitely many points of A. Then surely every open sphere centred at x must contain a point of A other than x. Hence x is a limit point of A.

Conversely, let x be a limit point of A. To show that every open sphere centred at x contains infinitely many points of A. Suppose if possible, there exists an open sphere $S(x, r)$ which contains only a finite number of points of A. Let $x_1, x_2, \ldots, x_n$ be those points of $S(x, r) \cap A$ which are distinct from x and put

$$r_0 = \min \{d(x, x_m) : 1 \le m \le n\}.$$

Since the minimum of a finite set of positive numbers is positive, we have $r_0 > 0$. The open sphere $S(x, r_0)$ then contains no point of A distinct from x so that x is not a limit point of A. But this is a contradiction. Hence every open sphere centred at x must contain infinitely many points of A.

Cor: *Let (X, d) be a metric space and let A be a subset of X. A point $x \in X$ is a limit point of A if and only if every neighbourhood of x contains infinitely many points of A.*

Theorem 2: *Let (X, d) be a metric space and let A be a subset of X. A point x of X is an adherent point of A if and only if $d(x, A) = 0$.*

Proof: Let $d(x, A) = 0$. Then by definition,

$$d(x, A) = \inf \{d(x, y) : y \in A\} = 0.$$

To show that x is an adherent point of A. Let $S(x, r)$ be any open sphere centred at x. Now by definition of infimum, there exists a $y_0 \in A$ such that $0 \le d(x, y_0) < r$. This means that

$$y_0 \in S(x, r).$$

It follows that x is an adherent point of A.

Conversely, let x be an adherent point of A. Then every open sphere centred at x must contain a point of A. This means that to each $r > 0$, there exists a $y \in A$ such that

$$0 \le d(x, y) < r. \qquad \ldots(1)$$

Now r is arbitrary and so can be taken as small as we please. It follows from (1) that

$$\inf \{d(x, y) : y \in A\} = 0 \quad \text{i.e.,} \quad d(x, A) = 0.$$

This completes the proof.

Theorem 3: *Let (X, d) be a metric space and let A be a subset of X. A point x of X is a limit point of A if and only if $d(x, [A \sim \{x\}]) = 0$.*

Proof: The proof of this theorem is similar to that of the previous theorem and is left as an exercise for the readers.

Theorem 4: A characterization of closed sets: *A subset A of a metric space X is closed if and only if $D(A) \subseteq A$ i.e., A is closed if and only if A contains all its limit points.*

Proof: Let A be closed so that A' is open. If $D(A) = \varnothing$, then obviously $D(A) \subseteq A$. So let $D(A) \neq \varnothing$. Let $p \in D(A)$ i.e., let p be a limit point of A. We wish to show that $p \in A$.

Suppose, if possible, $p \notin A$. Then $p \in A'$ and since A' is open, there exists $r > 0$ such that $S(p, r) \subseteq A'$. Consequently $S(p, r) \cap A = \varnothing$ i.e., the open sphere $S(p, r)$ which is a neighbourhood of p contains no point of A, contradicting the fact that p is a limit point of A. Hence we must have $p \in A$.

Thus if A is closed, then $D(A) \subseteq A$.

Conversely, we assume that $D(A) \subseteq A$ i.e., A contains all its limit points. Then we wish to show that A is closed. By definition A will be closed if A' is open.

Let $p \in A'$. Then $p \notin A$, and so $p \notin D(A)$ because by hypothesis $D(A) \subseteq A$.

Now $p \notin D(A) \Rightarrow$ p is not a limit point of A

$\Rightarrow$ there exists $r > 0$ such that $S(p, r)$ contains no point of A other than p

$\Rightarrow$ there exists $r > 0$ such that $S(p, r)$ contains no point of A because $p \notin A$

$\Rightarrow$ there exists $r > 0$ such that $S(p, r) \subseteq A'$.

Thus to each $p \in A'$ there exists $r > 0$ such that $S(p, r) \subseteq A'$ and so A' is open. Hence, A is closed.

Remark: On account of the characteristic property of a closed set given in theorem 4, sometimes we define a closed set as follows :

Closed set: Definition: *A subset A of a metric space (X, d) is said to be a closed set iff A contains all its limit points i.e., iff $D(A) \subseteq A$.*

In view of this definition of a closed set the theorem 4 given above may be stated as follows :

Theorem 5: *Let (X, d) be a metric space and let $A \subseteq X$. Then A is closed if and only if its complement $X - A$ is open.*

Proof: In order to prove that a set is closed iff its complement is open, we define a set A to be closed if it contains all its limit points *i.e.*, if $D(A) \subseteq A$.

'Only if' part of the theorem: Assume that the set A is closed *i.e.*, $D(A) \subseteq A$ *i.e.*, A contains all its limit points. Then to prove that $X - A$ *i.e.*, A' is open.

Let $p \in A'$. Then $p \notin A$, and so $p \notin D(A)$ because by hypothesis $D(A) \subseteq A$.

Now $p \notin D(A) \Rightarrow p$ is not a limit point of A

$\Rightarrow$ there exists $r > 0$ such that $S(p, r)$ contains no point of A other than p

$\Rightarrow$ there exists $r > 0$ such that $S(p, r)$ contains no point of A because $p \notin A$

$\Rightarrow$ there exists $r > 0$ such that $S(p, r) \subseteq A'$.

Thus to each $p \in A'$ there exists $r > 0$ such that $S(p, r) \subseteq A'$. Hence, A' is open.

'If part' of the theorem: Now assume that A' is open. Then to prove that A is closed *i.e.*, to prove that $D(A) \subseteq A$.

If $D(A) = \varnothing$, then obviously $D(A) \subseteq A$. So let $D(A) \neq \varnothing$. Let $p \in D(A)$ *i.e.*, let p be a limit point of A. We wish to show that $p \in A$.

Suppose, if possible, $p \notin A$. Then $p \in A'$ and since A' is open, there exists $r > 0$ such that $S(p, r) \subseteq A'$. Consequently

$$S(p, r) \cap A = \varnothing$$

i.e., $S(p, r)$ which is a neighbourhood of p contains no point of A, contradicting the fact that p is a limit point of A. Hence we must have $p \in A$.

Thus if A' is open, then $D(A) \subseteq A$ *i.e.*, A contains all its limit points and so A is closed by our definition of a closed set.

Theorem 6: *Let A be any subset of a metric space (X, d). Then the derived set of A i.e., $D(A)$ is a closed set.*

Proof: In order to prove that $D(A)$ is closed we shall show that $D(A)$ contains all its limit points.

Let p be any limit point of $D(A)$. Then for every $r > 0$, the open sphere $S(p, r)$ contains infinitely many points of $D(A)$ and since each point of $D(A)$ is a limit point of A, every open sphere $S(p, r)$ must contain infinitely many points of A.

Thus, p is also a limit point of A and so $p \in D(A)$. Therefore $D(A)$ contains all its limit points and so $D(A)$ is closed.

Theorem 7: *Let A and B be any subsets of a metric space X. Then*

(i) $D(\varnothing) = \varnothing$ *(ii)* $A \subseteq B \Rightarrow D(A) \subseteq D(B)$

(iii) $D(A \cap B) \subseteq D(A) \cap D(B)$ *(iv)* $D(A \cup B) = D(A) \cup D(B)$.

Proof: (i) If p be any point of X, then X is a nhd of p and $X \cap \varnothing = \varnothing$ *i.e.*, X contains no point of $\varnothing$ because the empty set $\varnothing$ has no points at all. Therefore p is not a limit point of $\varnothing$. Thus no point of X is a limit point of $\varnothing$ and hence $D(\varnothing) = \varnothing$.

(ii) It is given that $A \subseteq B$.

Let $x \in D(A)$ *i.e.*, let x be a limit point of A. Then

$x \in D(A) \Rightarrow$ every nhd N of x contains a point of A other than x

$\Rightarrow$ every nhd N of x contains a point of B other than x $[\because B \supseteq A]$

$\Rightarrow$ x is a limit point of $B \Rightarrow x \in D(B)$.

$\therefore$ $A \subseteq B \Rightarrow D(A) \subseteq D(B)$.

(iii) We know that $A \subseteq B \Rightarrow D(A) \subseteq D(B)$.

Now $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

$\therefore$ $D(A \cap B) \subseteq D(A)$ and $D(A \cap B) \subseteq D(B)$.

Hence, $D(A \cap B) \subseteq D(A) \cap D(B)$.

(iv) We know that $A \subseteq B \Rightarrow D(A) \subseteq D(B)$.

Now $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

$\therefore$ $D(A) \subseteq D(A \cup B)$ and $D(B) \subseteq D(A \cup B)$.

$\therefore$ $D(A) \cup D(B) \subseteq D(A \cup B)$. ...(1)

Now we shall show that $D(A \cup B) \subseteq D(A) \cup D(B)$.

Let $x \in D(A \cup B)$. Then to show that $x \in D(A) \cup D(B)$.

For this we shall show that $x \notin D(A) \cup D(B) \Rightarrow x \notin D(A \cup B)$.

Suppose $x \notin D(A) \cup D(B)$. Then $x \notin D(A)$ and $x \notin D(B)$ *i.e.*, x is neither a limit point of A nor a limit point of B

$\therefore$ there exists a neighbourhood N of x which contains no point of A other than x and there exists a neighbourhood M of x which contains no point of B other than x.

Since the intersection of two nhds of x is also a nhd of x, therefore $N \cap M$ is a nhd of x which contains no point of A other than x and no point of B other than x *i.e.*, which contains no point of $A \cup B$ other than x. Consequently x is not a limit point of $A \cup B$

i.e., $x \notin D(A \cup B)$.

Thus $x \notin D(A) \cup D(B) \Rightarrow x \notin D(A \cup B)$.

$\therefore$ $x \in D(A \cup B) \Rightarrow x \cup D(A) \cup D(B)$.

$\therefore$ $D(A \cup B) \subseteq D(A) \cup D(B)$. ...(2)

From (1) and (2), we conclude that

$$D(A \cup B) = D(A) \cup D(B).$$

Illustrative Examples

Example 39: *Show that a finite set in a metric space has no limit points.*

Solution: Let S be any finite subset of a metric space X. We shall show that every $p \in X$ is not a limit point of S.

Since the set S is finite, therefore if we take any $r > 0$, then the open sphere $S(p, r)$ contains only finitely many points of the set S. Thus $S(p, r)$ is a nhd of p which does not contain infinitely many points of S and so p is not a limit point of S.

Thus every $p \in X$ is not a limit point of S and so a finite set S has no limit points *i.e.*, the derived set of a finite set is empty.

Example 40: *In the usual metric space* $(\mathbf{R}, d)$ *find the derived set of the set* $\mathbf{Q}$ *of all rational numbers.*

Solution: We shall show that every real number is a limit point of the set $\mathbf{Q}$.

Let p be any real number and let $\varepsilon > 0$ be given. Then $p - \varepsilon$ and $p + \varepsilon$ are two distinct real numbers and we know that between two distinct real numbers there lie infinitely many rational numbers. Therefore for every $\varepsilon > 0$, the open interval $]p - \varepsilon, p + \varepsilon[$ contains at least one point of $\mathbf{Q}$ other than p. Hence, p is a limit point of $\mathbf{Q}$.

Since p is an arbitrary real number, therefore every real number is a limit point of $\mathbf{Q}$.

Hence the set of the limit points of $\mathbf{Q}$ is the set of all real numbers $\mathbf{R}$.

$\therefore$ $D(\mathbf{Q}) = \mathbf{R}$.

Example 41: *On the real line, show that the set* $\mathbf{Z}$ *of all integers has no limit points where the metric on* $\mathbf{R}$ *is the usual metric* $d(x, y) = |x - y|, \forall\, x, y \in \mathbf{R}$.

Solution: Let p be any real number. The open interval $\left]p - \frac{1}{4}, p + \frac{1}{4}\right[$ whose length is $\frac{1}{2}$ contains at the most one integer. Thus $\left]p - \frac{1}{4}, p + \frac{1}{4}\right[$ is a nhd of p which does not contain infinitely many points of the set $\mathbf{Z}$. Hence, p is not a limit point of $\mathbf{Z}$.

Thus no real number p is a limit point of $\mathbf{Z}$ *i.e.*, $D(\mathbf{Z}) = \varnothing$.

10.18 Closure

Definition: *Let A be a subset of a metric space X. The closure of A, denoted by $\overline{A}$ or by* **cl** *A, is the intersection of all closed sets containing A, i.e.,*

$$\overline{A} = \cap \{F : F \text{ is closed and } F \supseteq A\}.$$

A point $x \in X$ is called a point of closure of A if $x \in \overline{A}$.

Theorem 1: *If A is any subset of a metric space X, then $\overline{A}$ is the smallest closed set containing A.*

Proof: By the definition of the closure of a set A, we have

$$\overline{A} = \cap \{F : F \text{ is closed and } F \supseteq A\}.$$

Since in a metric space the intersection of any arbitrary collection of closed sets is a closed set, therefore $\overline{A}$ is a closed set.

Also, since $A \subseteq F, \forall F$, therefore $A \subseteq \cap \{F : F \text{ is closed and } F \supseteq A\}$

i.e., $A \subseteq \overline{A}$.

Thus, $\overline{A}$ is a closed set containing A.

Now if B is any closed set containing A, then B is a member of the family of closed sets containing A and so by the definition of $\overline{A}$, we have $\overline{A} \subseteq B$.

Thus $\overline{A}$ is a closed set containing A and if B is any closed set containing A, then $\overline{A} \subseteq B$.

Hence, $\overline{A}$ is the smallest closed set containing A.

Theorem 2: *A subset A of a metric space is closed if and only if $\overline{A} = A$.*

Proof: If A is closed, then A is the smallest closed set containing A and hence $\overline{A} = A$. Conversely, if $\overline{A} = A$, then since $\overline{A}$ is closed, we see that A is also closed.

Theorem 3: *Let A be a subset of a metric space. Then*

$$\overline{A} = A \cup D(A).$$

Proof: We shall first prove that $A \cup D(A)$ is a closed set.

Let p be any limit point of $A \cup D(A)$.

Then either p is a limit point of A or a limit point of $D(A)$. If p is a limit point of A, then $p \in D(A)$. If p is a limit point of $D(A)$, then since $D(A)$ is closed, $p \in D(A)$. So in either case $p \in D(A)$ and surely then $p \in A \cup D(A)$. Thus $A \cup D(A)$ contains all its limit points and hence $A \cup D(A)$ is closed.

Now $A \cup D(A)$ is a closed set containing A, and since $\overline{A}$ is the smallest closed set containing A, we have

$$\overline{A} \subseteq A \cup D(A). \qquad \ldots(1)$$

Also $$A \subseteq \overline{A} \Rightarrow D(A) \subseteq D(\overline{A}). \qquad \ldots(2)$$

Since $\overline{A}$ is closed, we have

$$D(\overline{A}) \subseteq \overline{A}. \qquad \ldots(3)$$

$\therefore$ from (2) and (3), we have $D(A) \subseteq \overline{A}$.

Moreover, $A \subseteq \overline{A}$, and $D(A) \subseteq \overline{A}$

$\Rightarrow$ $$A \cup D(A) \subseteq \overline{A}. \qquad \ldots(4)$$

$\therefore$ from (1) and (4), we have

$$\overline{A} = A \cup D(A).$$

Cor: *Since Adh $A = A \cup D(A)$, it follows from the above theorem that $\overline{A} = Adh\ A$.*

Theorem 4: *Let (X, d) be a metric space and let A, B be subsets of X. Then*

(i) $\overline{\varnothing} = \varnothing$; *(ii)* $A \subseteq \overline{A}$;

(iii) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$; *(iv)* $\overline{A \subseteq B} = \overline{A} \cup \overline{B}$;

(v) $\overline{(A \cap B)} \subseteq \overline{A} \cap \overline{B}$; *(vi)* $\overline{\overline{A}} = \overline{A}$.

Proof: (i) Since $\varnothing$ is a closed set, we have

$$\overline{\varnothing} = \varnothing. \qquad \text{[By Theorem 2]}$$

(ii) Since $\overline{A}$ is the smallest closed set containing A, we surely have $A \subseteq \overline{A}$.

(iii) By (ii), $B \subseteq \overline{B}$. Since $A \subseteq B$, we have, $A \subseteq \overline{B}$. But $\overline{B}$ is closed. Thus $\overline{B}$ is a closed set containing A. Since $\overline{A}$ is the smallest closed set containing A, we have

$$\overline{A} \subseteq \overline{B}.$$

(iv) Since $A \subseteq A \cup B$, we have by (iii)

$$\overline{A} \subseteq \overline{A \cup B}.$$

Similarly, $\overline{B} \subseteq \overline{A \cup B}$.

Hence $$\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}. \qquad \ldots(1)$$

Since $\overline{A}$ and $\overline{B}$ are closed sets, therefore $\overline{A} \cup \overline{B}$ is also a closed set. Also $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$ so that $A \cup B \subseteq \overline{A} \cup \overline{B}$. Thus $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$.

Since $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$, we have

$$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}. \qquad \ldots(2)$$

From (1) and (2), we have

$$\overline{(A \cup B)} = \overline{A} \cup \overline{B}.$$

(v) We have $A \cap B \subseteq A \Rightarrow \overline{(A \cap B)} \subseteq \overline{A}$

and $A \cap B \subseteq B \Rightarrow \overline{(A \cap B)} \subseteq \overline{B}$.

Hence, $$\overline{(A \cap B)} \subseteq \overline{A} \cap \overline{B}.$$

(vi) Since $\overline{A}$ is a closed set, we have by theorem 2,

$$\overline{(\overline{A})} = \overline{A} \quad i.e., \quad \overline{\overline{A}} = \overline{A}.$$

Theorem 5: *Let (X, d) be a metric space and let A be a subset of X. Then the following statements are equivalent :*

(i) *A is closed i.e., the complement of A is open.*

(ii) *A contains all its limiting points.*

(iii) $\overline{A} = A$.

Proof: By theorem 4 of article 10.17, we have

(i) $\Leftrightarrow$ (ii)

And by theorem 2 of article 10.18, we have

(i) $\Leftrightarrow$ (iii)

Remark: In view of the above theorem, any of the above statements can be taken as the definition of closed sets and the remaining statements can be deduced from it.

Illustrative Examples

Example 42: *Consider the usual metric space* $(\mathbf{R}, d)$ *and find the closures of the following subsets of* $\mathbf{R}$.

(i) $A = \left\{\frac{1}{n} : n \in \mathbf{N}\right\}$, *(ii) the set of integers* $\mathbf{Z}$,

(iii) $\mathbf{Q}$, *(iv)* $]0, 1[$,

(v) $[0, 1[$, *(vi)* $[0, 1]$.

Solution: (i) Here $D(A) = \{0\}$ and so

$$\overline{A} = \left\{\frac{1}{n} : n \in \mathbf{N}\right\} \cup \{0\}.$$

(ii) Since $D(\mathbf{Z}) = \varnothing$, therefore $\overline{\mathbf{Z}} = \mathbf{Z} \cup D(\mathbf{Z}) = \mathbf{Z} \cup \varnothing = \mathbf{Z}$.

(iii) Since $D(\mathbf{Q}) = \mathbf{R}$, therefore $\overline{\mathbf{Q}} = \mathbf{Q} \cup D(\mathbf{Q}) = \mathbf{Q} \cup \mathbf{R} = \mathbf{R}$.

(iv) We have $D(\,]0, 1[\,) = [0, 1]$.

$\therefore$ $\overline{]0, 1[} =]0, 1[\cup [0, 1] = [0, 1]$.

(v) We have $D([0, 1[) = [0, 1]$.

$\therefore$ $\overline{[0, 1[} = [0, 1[\cup [0, 1] = [0, 1]$.

(vi) We have $D([0, 1]) = [0, 1]$.

$\therefore$ $\overline{[0, 1]} = [0, 1] \cup [0, 1] = [0, 1]$.

Example 43: *Give an example to show that in a metric space it is not necessary that* $\overline{(A \cap B)} = \overline{A} \cap \overline{B}$.

Solution: Consider the usual metric space $(\mathbf{R}, d)$.

Let $A = [0, 1[$, $B =]1, 2]$.

We have $A \cap B = \varnothing$ and so $\overline{(A \cap B)} = \overline{\varnothing} = \varnothing$.

But $\overline{A} = [0, 1]$ and $\overline{B} = [1, 2]$ so that $\overline{A} \cap \overline{B} = \{1\}$.

Hence, $\overline{(A \cap B)} \neq \overline{A} \cap \overline{B}$.

10.19 Interior, Exterior, Frontier and Boundary of a Set

Interior points and the interior of a set :

Definition: *Let (X, d) be a metric space and let $A \subseteq X$. A point $x \in X$ is said to be an* **interior point** *of A if A is a neighbourhood of x.*

The set of all interior points of A is called the **interior** *of A and is denoted by A° or A^i or* **i** (A) *or* **int** (A).

Evidently A contains all its interior points *i.e.*, $A^\circ \subseteq A$.

Theorem 1: *Let (X, d) be a metric space and let A be a subset of X. Then*

(i) *A° is an open set.*

(ii) *A° is the largest open set contained in A.*

(iii) *A is open if and only if $A^\circ = A$.*

Proof: (i) Let x be any arbitrary point of A°. Then x is an interior point of A. Hence by definition, A is a neighbourhood of x so that there exists an open set G such that

$$x \in G \subseteq A.$$

Since G is open, it is a nhd of each of its points and so A is also a nhd of each point of G because $A \supseteq G$. It follows that every point of G is an interior point of A and so

$$G \subseteq A^\circ.$$

Thus it is shown that to each $x \in A^\circ$, there exists an open set G such that

$$x \in G \subseteq A^\circ.$$

Hence A° is a nhd of each of its points and consequently A° is open.

(ii) We have shown in (i) that A° is an open set. Also, $A^\circ \subseteq A$ *i.e.*, A° is contained in A. Now let G be any open subset of A and let $x \in G$. Then

$$x \in G \subseteq A.$$

Since G is open, A is a nhd of x and consequently x is an interior point of A. Hence $x \in A^\circ$. Thus we have shown that

$$x \in G \Rightarrow x \in A^\circ \quad \therefore \quad G \subseteq A^\circ.$$

Thus A° contains every open subset of A and it is therefore the largest open subset of A.

Hence, A° is the largest open set contained in A.

(iii) Let $A = A^\circ$. By (i) A° is an open set and therefore A is also open.

Conversely let A be open. Then A is surely identical with the largest open subset of A. But by (ii), A° is the largest open subset of A.

Hence $A = A^\circ$.

Thus it is shown that A is open if and only if

$$A^\circ = A.$$

Example: Let X be a discrete mertic space and let A be any subset of X. Since A is open, we have

$$A^\circ = A.$$

Thus in a discrete space every subset of X coincides with its interior.

Theorem 2: *Let X be a metric space and let A be a subset of X. Then A° equals the set of all those points of A which are not limit points of A'.*

Proof: Let x be a point of A which is not a limit point of A'. Then there exists a nhd N of x which contains no point of A' and so $N \subseteq A$. This implies that A is also a nbd of x and so $x \in A^\circ$.

Conversely let $x \in A^\circ$. Since A° is open, it is a nhd of x. Also A° contains no point of A'. [$\because$ $A^\circ \subseteq A$ and A contains no point of A']

It follows that x is not a limit point of A'. Thus no point of A° is a limit point of A'.

Hence A° consists of precisely those points of A which are not limit points of A'.

Exterior points and the exterior of a set:

Definition: *Let A be a subset of a metric space X. A point $x \in X$ is said to be an* **exterior point** *of A if it is an interior point of the complement A' of A.*

The set of all exterior points of A is called the **exterior** *of A and is denoted by* **ext** (A) *or* $e(A)$ *or* A^e.

Thus **ext** $(A) = (A')^\circ$.

It follows that **ext** $(A') = (A'')^\circ = A^\circ$. [Here A'' stands for $(A')'$]

We also have

$$A \cap \textbf{ext}\ (A) = \varnothing, \quad [\because\ A \cap A' = \varnothing \text{ and } \textbf{ext}\ (A) = (A')^\circ \subseteq A']$$

that is, no point of A can be an exterior point of A.

Remark: Since **ext** (A) is the interior of A', it follows from theorem 1 that **ext** A is open and is the largest open set contained in A'.

Theorem 3: *Let X be a metric space and let $A \subseteq X$. Then*

(i) $A^\circ = \cup \{G : G \text{ is open}, G \subseteq A\}$.

(ii) **ext** $(A) = \cup \{G : G \text{ is open}, G \subseteq A'\}$.

Proof: (i) Let $x \in A^\circ$. Then A is a neighbourhood of x so that there exists an open set G such that

$$x \in G \subseteq A.$$

It follows that

$$x \in \cup \{G : G \text{ is open}, G \subseteq A\}.$$

Conversely, let $x \in \cup \{G : G \text{ is open}, G \subseteq A\}$. Then $x \in G$ for some open set G such that $G \subseteq A$. Hence A is a neighbourhood of x and consequently $x \in A^\circ$.

Thus we have shown that

$$x \in A^\circ \Leftrightarrow x \in \cup \{G : G \text{ is open}, G \subseteq A\}.$$

Therefore $A^\circ = \cup \{G : G \text{ is open}, G \subseteq A\}$.

(ii) We have by definition, **ext** $(A) = (A')^\circ$.

But by (i), $(A')^\circ = \cup \{G : G \text{ is open}, G \subseteq A'\}$.

Hence, **ext** $(A) = \cup \{G : G \text{ is open}, G \subseteq A'\}$.

Theorem 4: *Let A be a subset of a metric space X. Then a point $x \in X$ is an exterior point of A if and only if x is not an adherent point of A, that is, iff $x \in (\overline{A})'$.*

Proof: Let x be an exterior point of A. Then x is an interior point of A' so that A' is a neighbourhood of x containing no point of A. It follows that x is not an adherent point of A

i.e., $x \in (\overline{A})'$. $[\because \overline{A} = \text{Adh } A]$

Conversely, suppose that x is not an adherent point of A. Then there exists a neighbourhood N of x which contains no point of A. This implies that $x \in N \subseteq A'$. It follows that A' is a neighbourhood of x and consequently x is an interior point of A' *i.e.*, x is an exterior point of A.

Cor: It follows from the above theorem that

$$\textbf{ext } (A) = (\overline{A})'. \qquad \ldots(1)$$

Hence, $(A')^\circ = (\overline{A})'$.

Also, $[\textbf{ext } (A)]' = [(\overline{A})']' = \overline{A}$.

Replacing A by A' in (1), we conclude that

$$A^\circ = \textbf{ext } (A') = A'^{-\prime}.$$

[Note that we have used the abbreviated notation $A'^{-\prime}$ for $[\overline{(A')}]'$.

Frontier points and the frontier of a set:

Definition: *A point x of a metric space X is said to be a* **frontier point** *of a subset A of X iff it is neither an interior nor an exterior point of A. The set of all frontier points of A is called the frontier of A and shall be denoted by* **Fr** (A).

Theorem 5: *Let X be a metric space and let $A \subseteq X$. Then a point x in X is a frontier point of A if and only if every neighbourhood of x intersects both A and A'.*

Proof: We have

$x \in \textbf{Fr } (A) \Leftrightarrow x \notin A^\circ$ and $x \notin \textbf{ext } A = (A')^\circ$

$\Leftrightarrow$ neither A nor A' is a nhd of x

$\Leftrightarrow$ no nhd of x can be contained in A or in A'

$[\because$ if a nhd N of x is contained in A, then A is also a nhd of $x]$

every nhd of x intersects both A and A'.

Cor: **Fr** $(A) =$ **Fr** (A').

For we have $x \in \textbf{F } (A) \Leftrightarrow$ every nhd of x intersects both A and A'

$\Leftrightarrow$ every nhd of x intersects both $(A')'$ and A' $[\because (A')' = A]$

$\Leftrightarrow$ $x \in \mathbf{Fr}(A')$.

Theorem 6: *Let A be any subset of a metric space X. Then A°, **ext** (A) and **Fr** (A) are disjoint and $X = A^\circ \cup \mathbf{ext}(A) \cup \mathbf{Fr}(A)$. Further **Fr** (A) is a closed set.*

Proof: We have $\mathbf{ext}(A) = (A')^\circ$. Also $A^\circ \subseteq A$ and $(A')^\circ \subseteq A'$. Since $A \cap A' = \varnothing$, it follows that $A^\circ \cap \mathbf{ext}(A) = A^\circ \cap (A')^\circ = \varnothing$. Again by the definition of frontier, we have

$x \in \mathbf{Fr}(A) \Leftrightarrow x \notin A^\circ$ and $x \approx \mathbf{ext}(A)$

$\Leftrightarrow$ $x \notin A^\circ \cup \mathbf{ext}(A) \Leftrightarrow x \in [A^\circ \cup \mathbf{ext}(A)]'$.

Thus $\mathbf{Fr}(A) = [A^\circ \cup \mathbf{ext}(A)]'$. ...(1)

It follows that $\mathbf{Fr}(A) \cap A^\circ = \varnothing$ and $\mathbf{Fr}(A) \cap \mathbf{ext}(A) = \varnothing$

and $X = A^\circ \cup \mathbf{ext}(A) \cup \mathbf{Fr}(A)$. ...(2)

Since A° and **ext** (A) are open sets, we see from (1) that **Fr** (A) is a closed set.

From (2), we have $[A^\circ \cup \mathbf{Fr}(A)] \cup \mathbf{ext}(A) = X$.

But $[A^\circ \cup \mathbf{Fr}(A)] \cap \mathbf{ext}(A) = \varnothing$.

$[\because A^\circ, \mathbf{ext}(A)$ and $\mathbf{Fr}(A)$ are mutually disjoint$]$

$\therefore$ $A^\circ \cup \mathbf{Fr}(A) = [\mathbf{ext}(A)]'$.

Boundary points and the boundary of a set :

Definition: *A point x of a metric space X is said to be a* **boundary point** *of a subset A of X if it is a frontier point of A and belongs to A.*

The set of all boundary points of A is called the **boundary** *of A and shall be denoted by* $\mathbf{b}(A)$.

Note that a boundary point of A is essentially a point of A whereas a frontier point may not belong to A. Thus every boundary point of A is also a frontier point of A but not conversely. Thus

$\mathbf{b}(A) \subseteq \mathbf{Fr}(A)$.

Note: Some authors do not make any distinction between the boundary and the frontier of a set. They use the term 'boundary' for the term 'frontier' as defined here.

Dense and non-dense sets :

Definition: *Let X be a metric space and let A, B be subsets of X. Then*

(i) A is said to be dense in B if $B \subseteq \overline{A}$.

(ii) A is said to be **dense in** X *or* **everywhere dense** *if $\overline{A} = X$.*

It follows that A is everywhere dense iff every point of X is an adherent point of A.

Example of a dense set: In the usual metric space $(\mathbf{R}, d)$, the set $\mathbf{Q}$ of rational numbers is dense because $\overline{\mathbf{Q}} = \mathbf{R}$.

(iii) A is said to be **nowhere dense** *or* **non-dense** *in X if* $(\overline{A})^\circ = \varnothing$ *i.e., if the interior of the closure of A is empty.*

(iv) A is said to be **dense-in itself** *if* $A \subseteq D(A)$.

It follows from the definition of a perfect set given in 10.17, and (iv) of the above definition that a set A is **perfect** iff A is dense in itself and closed. This implies that A is perfect if and only if

$$A = D(A).$$

For A is perfect

A is closed and A has no isolated points

A is closed and every point of A is a limit point of A

$D(A) \subseteq A$ and $A \subseteq D(A) \Leftrightarrow A = D(A)$.

Separable spaces: Definition: *A metric space X is said to be* **separable** *if X contains a countable dense subset, that is, there exists a countable subset A of X such that* $\overline{A} = X$.

For example, the usual metric space $(\mathbf{R}, d)$ is separable since the set $\mathbf{Q}$ of rational numbers is a countable dense subset of $\mathbf{R}$.

Illustrative Examples

Example 44: *Consider the usual metric* $d(x, y) = |x - y|$ *on* $\mathbf{R}$ *and find*

(i) interior *(ii) exterior*

(iii) frontier and

(iv) boundary of each of the following subsets of $\mathbf{R}$:

(a) $A =]0, 1[$; *(b)* $B = [0, 1[$;

(c) $C = [0, 1]$; *(d)* $D = \left\{\frac{1}{n} : n \in \mathbf{N}\right\}$;

(e) $\mathbf{N}$; *(f)* $\mathbf{Q}$.

Solution: **(a)** (i) Since A is an open set, it is a nhd of each its points and so every point of A is its interior point.

Hence, $A^\circ = A =]0, 1[$.

(ii) We have $A' =]-\infty, 0\,] \cup [\,1, \infty[$.

Here A' is a nhd of each of its points except 0 and 1.

Hence, $\mathbf{ext}(A) = (A')^\circ$

$=]-\infty, 0[\cup]1, \infty[$.

(iii) $\mathbf{Fr}(A) = [A^\circ \cup \mathbf{ext}(A)]' = \{0, 1\}$.

Thus the frontier of A consists of two points 0 and 1 only.

(iv) $\mathbf{b}(A) = \varnothing$, since no frontier point of A is a point of A.

(b). Proceeding as in (a), we have

(i) $B^\circ =]0, 1[$.

(ii) Here $B' =]-\infty, 0[\cup [1, \infty[$.

Then **ext** $(B) = (B')^\circ =]-\infty, 0[\cup]1, \infty[$.

(iii) **Fr** $(B) = \{0, 1\}$.

(iv) **b** $(B) = \{0\}$, since 0 is the only frontier point which belongs to B.

(c). (i) $C^\circ =]0, 1[$.

(ii) Here $C' =]-\infty, 0[\cup]1, \infty[$.

Hence **ext** $(C) = (C')^\circ =]-\infty, 0[\cup]1, \infty[$.

(iii) **Fr** $(C) = \{0, 1\}$.

(iv) **b** $(C) = \{0, 1\}$, since the frontier points 0 and 1 are also members of C.

(d). (i) Here D cannot be a nhd of any of its points $\frac{1}{n}$, $n = 1, 2, 3, \ldots$, since there exists no $\varepsilon > 0$ such that

$$\left]\frac{1}{n} - \varepsilon, \frac{1}{n} + \varepsilon\right[\subseteq D.$$

Hence no point of D can be its interior point so that $D^\circ = \varnothing$.

(ii) It is easy to see that $D' = \mathbf{R} - D$ is a nhd of each of its points except 0. Hence

ext $(D) = (D')^\circ = \mathbf{R} - (D \cup \{0\})$.

(iii) **Fr** $(D) = [D^\circ \cup \textbf{ext}\ (D)]' = D \cup \{0\}$.

(iv) **b** $(D) = D$, since all the points of D are frontier points of D and the frontier point 0 does not belong to D.

(e). (i) It is easy to see that $\mathbf{N}$ is not a nhd of any of its points and so $\mathbf{N}^\circ = \varnothing$.

(ii) Since $\mathbf{N}$ has no limit points we have $\overline{\mathbf{N}} = \mathbf{N}$ so that $\mathbf{N}$ is a closed set. Its complement $\mathbf{N}'$ is therefore an open set.

Hence **ext** $(\mathbf{N}) = (\mathbf{N}')^\circ = \mathbf{N}' = \mathbf{R} - \mathbf{N}$.

(iii) **Fr** $(\mathbf{N}) = [\mathbf{N}^\circ \cup \textbf{ext}\ (\mathbf{N})]' = [\varnothing \cup \textbf{ext}\ (\mathbf{N})]' = [\textbf{ext}\ (\mathbf{N})]' = (\mathbf{N}')' = \mathbf{N}$.

(iv) **b** $(\mathbf{N}) = \mathbf{N}$, since every point of $\mathbf{N}$ is a frontier point of $\mathbf{N}$.

Note that $\mathbf{N}$ is non-dense since $(\overline{\mathbf{N}})^\circ = \varnothing$.

(f). (i) Since $\mathbf{Q}$ is not a nhd of any of its points, we have

$\mathbf{Q}^\circ = \varnothing$.

(ii) Also $\mathbf{Q}' = \mathbf{R} - \mathbf{Q}$ is the set of all irrational points which is also not a nhd of any of its points so that

$(\mathbf{Q}')^\circ = \varnothing$.

Hence, **ext** $(\mathbf{Q}) = (\mathbf{Q}')^\circ = \varnothing$.

(iii) **Fr** $(\mathbf{Q}) = [\mathbf{Q}^\circ \cup \textbf{ext}\ (\mathbf{Q})]' = [\varnothing \cup \varnothing]' = \varnothing' = \mathbf{R}$.

(iv) **b** $(\mathbf{Q}) = \mathbf{Q}$, since all the rational points are frontier points of $\mathbf{Q}$.

Note that $\mathbf{Q}$ is everywhere dense since $\overline{\mathbf{Q}} = \mathbf{R}$.

Example 45: *Let (X, d) be a discrete metric space and let A be a subset of X. Find*
(i) A°, *(ii)* **ext** (A), *(iii)* **Fr** (A) *and (iv)* **b** (A).

Solution: (i) Since in a discrete metric space, every set is open, it follows that A is a nhd of each of its points. Hence

$$A^\circ = A.$$

(ii) **ext** $(A) = (A')^\circ = A'$ [$\because$ A' is open because every subset of X is open]

(iii) **Fr** $(A) = [A^\circ \cup \textbf{ext}\ (A)]' = [A \cup A']' = X' = \varnothing$.

(iv) **b** $(A) = \varnothing$.

Properties of interior and exterior

Theorem 7: *Let X be a metric space and let A, B be any subsets of X. Then*

(i) $X^\circ = X, \varnothing^\circ = \varnothing$ *(ii)* $A^\circ \subseteq A$

(iii) $A \subseteq B \Rightarrow A^\circ \subseteq B^\circ$ *(iv)* $(A \cap B)^\circ = A^\circ \cap B^\circ$

(v) $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$ *(vi)* $A^{\circ\circ} = A^\circ$.

Proof: (i) We know that if A is an open set, then $A^\circ = A$. Since X and $\varnothing$ are open sets, we have

$$X^\circ = X, \varnothing^\circ = \varnothing.$$

(ii) Let $x \in A^\circ$. Then x is an interior point of A and consequently A is a nhd of x. Hence $x \in A$. Thus

$$x \in A^\circ \Rightarrow x \in A.$$

Hence, $A^\circ \subseteq A$.

(iii) It is given that $A \subseteq B$ and we have to show that $A^\circ \subseteq B^\circ$. Let $x \in A^\circ$. Then x is an interior point of A and so A is a nhd of x. Since $B \supseteq A$, B is also a nhd of x. This implies that $x \in B^\circ$. Thus we have shown that

$$x \in A^\circ \Rightarrow x \in B^\circ.$$

$\therefore$ $A^\circ \subseteq B^\circ$.

(iv) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, we have by (iii),

$$(A \cap B)^\circ \subseteq A^\circ \text{ and } (A \cap B)^\circ \subseteq B^\circ.$$

This implies that

$$(A \cap B)^\circ \subseteq A^\circ \cap B^\circ. \qquad \ldots(1)$$

Again let $x \in A^\circ \cap B^\circ$. Then $x \in A^\circ$ and $x \in B^\circ$. Hence x is an interior point of each of the sets A and B. It follows that A and B are nhds of x so that their intersection $A \cap B$ is also a nhd of x. Hence $x \in (A \cap B)^\circ$. Thus

$$x \in A^\circ \cap B^\circ \Rightarrow x \in (A \cap B)^\circ.$$

$\therefore$ $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$. $\ldots(2)$

From (1) and (2), we get $(A \cap B)^\circ = A^\circ \cap B^\circ$.

(v) We have $A \subseteq A \cup B \Rightarrow A^\circ \subseteq (A \cup B)^\circ$ [By (iii)]

and $B \subseteq A \cup B \Rightarrow B^\circ \subseteq (A \cup B)^\circ$. [By (iii)]

Hence, $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$.

Note that in general,

$$A^\circ \cup B^\circ \neq (A \cup B)^\circ.$$

For example consider the usual metric d for $\mathbf{R}$ and let

$$A = [0, 1] \text{ and } B = [1, 2].$$

Then $A^\circ =]0, 1[$ and $B^\circ =]1, 2[$.

$\therefore$ $A^\circ \cup B^\circ =]0, 1[\cup]1, 2[=]0, 2[- \{1\}$.

Also, $A \cup B = [0, 2]$.

Hence, $(A \cup B)^\circ =]0, 2[$.

Thus, in this case, $A^\circ \cup B^\circ$ is a proper subset of $(A \cup B)^\circ$.

$\therefore$ $A^\circ \cup B^\circ \neq (A \cup B)^\circ$.

(vi) We know that $A^\circ = A$ iff A is an open set. Since A° is an open set, therefore

$$A^{\circ\circ} = (A^\circ)^\circ = A^\circ.$$

Theorem 8: *Let X be a metric space and let A, B be subsets of X. Then*

(i) $\mathbf{ext}\ (X) = \varnothing, \mathbf{ext}\ (\varnothing) = X$ *(ii)* $\mathbf{ext}\ (A) \subseteq A'$

(iii) $\mathbf{ext}\ (A) = \mathbf{ext}\ [(\mathbf{ext}\ (A))']$ *(iv)* $A \subseteq B \Rightarrow \mathbf{ext}\ (B) \subseteq \mathbf{ext}\ (A)$

(v) $A^\circ \subseteq \mathbf{ext}\ (\mathbf{ext}\ (A))$ *(vi)* $\mathbf{ext}\ (A \cup B) = \mathbf{ext}\ (A) \cap \mathbf{ext}\ (B)$.

Proof: (i) We have $\mathbf{ext}\ (X) = (X')^\circ = \varnothing^\circ = \varnothing$

and $\mathbf{ext}\ (\varnothing) = (\varnothing')^\circ = X^\circ = X$.

(ii) We have $\mathbf{ext}\ (A) = (A')^\circ \subseteq A'$. $[\because\ A^\circ \subseteq A]$

(iii) We have, $\mathbf{ext}\ [(\mathbf{ext}\ (A))'] = \mathbf{ext}\ [(A'^\circ)'] = \mathbf{ext}\ (A'^{\circ\prime})$

$= [(A'^{\circ\prime})']^\circ = [A'^{\circ\prime\prime}]^\circ = (A'^\circ)^\circ$ $[\because\ S'' = S$ for any set $S]$

$= A'^{\circ\circ} = A'^\circ$ $[\because\ A^{\circ\circ} = A^\circ]$

$= \mathbf{ext}\ (A)$.

(iv) We have $A \subseteq B \Rightarrow B' \subseteq A' \Rightarrow (B')^\circ \subseteq (A')^\circ \Rightarrow \mathbf{ext}\ (B) \subseteq \mathbf{ext}\ (A)$.

(v) By (ii), we have $\mathbf{ext}\ (A) \subseteq A'$.

Then (iv) gives $\mathbf{ext}\ (A') \subseteq \mathbf{ext}\ (\mathbf{ext}\ (A))$.

Also $A^\circ = \mathbf{ext}\ (A')$.

Hence $A^\circ \subseteq \mathbf{ext}\ (\mathbf{ext}\ (A))$.

(vi) We have $\mathbf{ext}\ (A \cup B) = [(A \cup B)']^\circ = (A' \cap B')^\circ$ [By De-Morgan's law]

$= A'^\circ \cap B'^\circ$ $[\because\ (A \cap B)^\circ = A^\circ \cap B^\circ]$

$= \mathbf{ext}\ (A) \cap \mathbf{ext}\ (B)$.

Relation between closure, interior and frontier:

Theorem 9: *Let X be a metric space and let $A \subseteq X$. Then*

(i) $A'^- = A^{\circ\prime}$,

i.e., the closure of the complement of A is the complement of the interior of A.

(ii) $A^\circ = A'^{-\prime}$,

i.e., the interior of A is the complement of the closure of the complement of A.

(iii) $\overline{A} = A'^{\circ\prime}$,

i.e., the closure of A is the complement of the interior of the complement of A.

Proof: (i) By the corollary of theorem 4, $A^\circ = A'^{-\prime}$.

Hence $(A^\circ)' = (A'^{-\prime})' = A'^{-\prime\prime}$ or $A^{\circ\prime} = A'^{-}$. [$\because S'' = S$ for any set S]

(ii) As proved in the corollary of theorem 4, we have $A^\circ = A'^{-\prime}$.

(iii) By (i), $A'^{-} = A^{\circ\prime}$.

Replacing A by A' in this, we get

$$(A')'^{-} = (A')^{\circ\prime} \quad \text{or} \quad A''^{-} = A'^{\circ\prime} \quad \text{or} \quad \overline{A} = A'^{\circ\prime} \qquad [\because A'' = A]$$

Theorem 10: *Let X be a metric space and let $A \subseteq X$. Then*

$$\overline{A} = A^\circ \cup \mathbf{Fr}\,(A).$$

Proof: By definition of $\overline{A}$, we have $\overline{A} = \cap \{F : F \text{ is closed and } F \supseteq A\}$.

Then by De-Morgan's law, we have

$$(\overline{A})' = \cup \{F' : F' \text{ is open and } F' \cup A'\}$$

$$= (A')^\circ \qquad \text{[By theorem 3, part (i) of article 10.19]}$$

$$= \mathbf{ext}\,(A).$$

Taking complements, we get

$$(\overline{A})'' = [\mathbf{ext}\,(A)]' = A^\circ \cup \mathbf{Fr}\,(A).$$

[By theorem 6 of article 10.19]

Hence $\overline{A} = A^\circ \cup \mathbf{Fr}\,(A)$. [$\because (\overline{A})'' = \overline{A}$]

Cor. 1: $\mathbf{Fr}\,(A) \subseteq \overline{A}$.

Cor. 2: $\overline{A} = A \cup \mathbf{Fr}\,(A)$.

Proof: Since $A \subseteq \overline{A}$ and $\mathbf{Fr}\,(A) \subseteq \overline{A}$, we have

$$A \cup \mathbf{Fr}\,(A) \subseteq \overline{A}. \qquad \ldots(1)$$

Also $\mathbf{Fr}\,(A) = [A^\circ \cup \mathbf{ext}\,(A)]'$ [By theorem 6 of article 10.19]

$= A^{\circ\prime} \cap [\mathbf{ext}\,(A)]'$ [By De-Morgan's law]

Again since $A^\circ \subseteq A$ and $\overline{A} = A^\circ \cup \mathbf{Fr}\,(A)$, it follows that

$$\overline{A} \subseteq A \cup \mathbf{Fr}\,(A). \qquad \ldots(2)$$

From (1) and (2), we get $\overline{A} = A \cup \mathbf{Fr}\,(A)$.

Theorem 11: *Let X be a metric space and let A, B be subsets of X. Then*

(i) $\mathbf{Fr}\,(A) = \overline{A} \cap A'^{-} = \overline{A} - A^\circ$ *(ii)* $A^\circ = A - \mathbf{Fr}\,(A)$

(iii) $[\mathbf{Fr}\,(A)]' = A^\circ \cup A'^\circ$ *(iv)* $\mathbf{Fr}\,(A^\circ) \subseteq \mathbf{Fr}\,(A)$

(v) $\mathbf{Fr}\,(\overline{A}) \subseteq \mathbf{Fr}\,(A)$ *(vi)* $\mathbf{Fr}\,(A \cup B) \subseteq \mathbf{Fr}\,(A) \cup \mathbf{Fr}\,(B)$

(vii) $\mathbf{Fr}\,(A \cap B) \subseteq \mathbf{Fr}\,(A) \cup \mathbf{Fr}\,(B)$

Proof: (i) We have

$$\mathbf{Fr}\,(A) = [A^\circ \cup \mathbf{ext}\,(A)]' = A^{\circ\prime} \cap [\mathbf{ext}\,(A)]' \quad \text{[By De-Morgan's law]}$$
$$= A'^{-\prime\prime} \cap A^{-\prime\prime} \quad \text{[By cor. of theorem 4]}$$
$$= A'^{-} \cap \overline{A} \quad [\because\ S'' = S \text{ for any set } S]$$

Now $\overline{A} \cap A'^{-} = \overline{A} - A'^{-\prime}$ [$\because\ S \cap T = S - T'$ for any sets S and T]

$$= \overline{A} - A^\circ \quad \text{[By theorem 9, part (ii)]}$$

Hence, $\mathbf{Fr}\,(A) = \overline{A} \cap A'^{-} = \overline{A} - A^\circ$.

(ii) We have

$$A - \mathbf{Fr}\,(A) = A - (\overline{A} - A^\circ) \quad \text{[By (i)]}$$
$$= A^\circ \quad [\because\ A^\circ \subseteq A \subseteq \overline{A}]$$

(iii) We have

$$[\mathbf{Fr}\,(A)]' = [\overline{A} \cap A'^{-}]' \quad \text{[By (i)]}$$
$$= A^{-\prime} \cap A'^{-\prime}. \quad \text{[By De-Morgan's law]}$$

By theorem 9 part (ii), $A'^{-\prime} = A^\circ$ and so

$$A'^\circ = (A')^\circ = (A')'^{-\prime} = A''^{-\prime} = A^{-\prime} \quad [\because\ A'' = A]$$

$\therefore$ $[\mathbf{Fr}\,(A)]' = A'^\circ \cup A^\circ = A^\circ \cup A'^\circ$.

(iv) We have

$$\mathbf{Fr}\,(A^\circ) = A^{\circ-} \cap A^{\circ\prime-} \quad \text{[By (i)]}$$
$$= A^{\circ-} \cap A'^{-\prime\prime-} \quad [\because\ A'^{-\prime} = A^\circ]$$
$$= A^{\circ-} \cap A'^{-} \quad [\because\ S'' = S \text{ and } \overline{\overline{S}} = \overline{S} \text{ for any set } S]$$
$$\subseteq \overline{A} \cap A'^{-} \quad [\because\ A^\circ \subseteq A \Rightarrow A^{\circ-} \subseteq \overline{A}]$$
$$= \mathbf{Fr}\,(A) \quad \text{[By (i)]}$$

Hence $\mathbf{Fr}\,(A^\circ) \subseteq \mathbf{Fr}\,(A)$.

(v) We have $\mathbf{Fr}\,(\overline{A}) = \overline{\overline{A}} \cap A^{-\prime-}$ [By (i)]

$$= \overline{A} \cap A^{-\prime-}.$$

Now $A \subseteq \overline{A} \Rightarrow (\overline{A})' \subseteq A'$ [$\because\ A \subseteq B \Rightarrow B' \subseteq A'$]

$\Rightarrow A^{-\prime-} \subseteq A'^{-}$ [$\because\ A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$]

Hence $\mathbf{Fr}\,(\overline{A}) \subseteq \overline{A} \cap A'^{-} = \mathbf{Fr}\,(A)$.

Thus $\mathbf{Fr}\,(\overline{A}) \subseteq \mathbf{Fr}\,(A)$.

(vi) We have $\mathbf{Fr}\,(A \cup B) = \overline{(A \cup B)} \cap (A \cup B)'^{-}$ [By (i)]

$$= (\overline{A} \cup \overline{B}) \cap (A' \cap B')^{-} \quad \text{[By De-Morgan's law]}$$
$$\subseteq (\overline{A} \cup \overline{B}) \cap (A'^{-} \cap B'^{-}) \quad [\because\ \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}]$$
$$= [\overline{A} \cap (A'^{-} \cap B'^{-})] \cup [(\overline{B} \cap (A'^{-} \cap B'^{-})]$$

[By distributive law]

$$= [(\overline{A} \cap A'^{-}) \cap B'^{-}] \cup [(\overline{B} \cap B'^{-} \cap A'^{-}]$$

$$= [\mathbf{Fr}(A) \cap B'^{-}] \cup [\mathbf{Fr}(B) \cap A'^{-}]$$

[By (i)]

$$\subseteq \mathbf{Fr}(A) \cup \mathbf{Fr}(B) \quad [\because \mathbf{Fr}(A) \cap B'^{-} \subseteq \mathbf{Fr}(A) \text{ etc.}]$$

Hence, $\mathbf{Fr}(A \cup B) \subseteq \mathbf{Fr}(A) \cup \mathbf{Fr}(B)$.

(vii) We have

$$\mathbf{Fr}(A \cap B) = \overline{(A \cap B)} \cap (A \cap B)'^{-} \quad \text{[By (i)]}$$

$$\subseteq (\overline{A} \cap \overline{B}) \cap (A' \cup B')^{-} \quad \text{(why ?)}$$

$$= (\overline{A} \cap \overline{B}) \cap (A'^{-} \cup B'^{-}) \quad [\because \overline{A \cup B} = \overline{A} \cup \overline{B}]$$

$$= [(\overline{A} \cap \overline{B}) \cap A'^{-}] \cup [(\overline{A} \cap \overline{B}) \cap B'^{-}]$$

[By distributive law]

$$= [(\overline{A} \cap A'^{-}) \cap \overline{B}] \cup [\overline{A} \cap (\overline{B} \cap B'^{-})]$$

$$= [\mathbf{Fr}(A) \cap \overline{B}] \cup [\overline{A} \cap \mathbf{Fr}(B)] \quad \text{[By (i)]}$$

$$\subseteq \mathbf{Fr}(A) \cup \mathbf{Fr}(B).$$

Hence, $\mathbf{Fr}(A \cap B) \subseteq \mathbf{Fr}(A) \cup \mathbf{Fr}(B)$.

Theorem 12: *Let X be a metric space and let $A \subseteq X$.*

(i) If A is open, then $\mathbf{Fr}(A) = \overline{A} - A$.

(ii) $\mathbf{Fr}(A) = \varnothing$ *if and only if A is open as well as closed.*

(iii) A is open if and only if

$A \cap \mathbf{Fr}(A) = \varnothing$ *i.e., iff* $\mathbf{Fr}(A) \subseteq A'$.

(iv) A is closed if and only if $\mathbf{Fr}(A) \subseteq A$.

Proof: (i) By (i) of theorem 11, we have

$$\mathbf{Fr}(A) = \overline{A} - A^{\circ}.$$

Since A is open, therefore $A^{\circ} = A$.

Hence, $\mathbf{Fr}(A) = \overline{A} - A$.

(ii) Let $\mathbf{Fr}(A) = \varnothing$. We have

$\mathbf{Fr}(A) = \varnothing \Rightarrow \overline{A} - A^{\circ} = \varnothing$ [By part (i) of theorem 11]

$\Rightarrow \overline{A} \subseteq A^{\circ} \Rightarrow \overline{A} \subseteq A$ $[\because A^{\circ} \subseteq A]$

$\Rightarrow A \cup D(A) \subseteq A$ $[\because \overline{A} = A \cup D(A)]$

$\Rightarrow D(A) \subseteq A$

$\Rightarrow A$ is closed.

Again, $\mathbf{Fr}(A) = \varnothing \Rightarrow \overline{A} - A^{\circ} = \varnothing \Rightarrow \overline{A} \subseteq A^{\circ}$

$\Rightarrow A \cup D(A) \subseteq A^{\circ} \Rightarrow A \subseteq A^{\circ}$.

But $A^{\circ} \subseteq A$. Hence $A^{\circ} = A$.

It follows that A is an open set. $[\because A^{\circ}$ is open$]$

Thus we have shown that if $\mathbf{Fr}(A) = \varnothing$, then A is open as well as closed.

Conversely, let A be open as well as closed. To show that

$\mathbf{Fr}(A) = \varnothing$. Now $\mathbf{Fr}(A) = \overline{A} - A^\circ$.

Since A is closed, we have

$$A = \overline{A}.$$

Again since A is open, we have

$$A^\circ = A.$$

Hence, $\mathbf{Fr}(A) = A - A = \varnothing$.

(iii) By (i) of theorem 11, we have

$$\mathbf{Fr}(A) = \overline{A} \cap A'^{-} \qquad \ldots(1)$$

Let A be open. Then A' is closed.

Hence, $A'^{-} = A'$. $\qquad \ldots(2)$

Now $A \cap \mathbf{Fr}(A) = A \cap [\overline{A} \cap A'^{-}]$ [By (1)]

$= A \cap [\overline{A} \cap A']$ [By (2)]

$= (A \cap \overline{A}) \cap A' = A \cap A'$ $[\because A \subseteq \overline{A}]$

$= \varnothing$.

Conversely, let $A \cap \mathbf{Fr}(A) = \varnothing$.

Then by (1),

$A \cap \mathbf{Fr}(A) = \varnothing \Rightarrow A \cap (\overline{A} \cap A'^{-}) = \varnothing$

$\Rightarrow (A \cap \overline{A}) \cap A'^{-} = \varnothing$

$\Rightarrow A \cap A'^{-} = \varnothing$ $[\because A \subseteq \overline{A}]$

$\Rightarrow A \subseteq A'^{-\prime}$ $[\because A \cap B = \varnothing \Rightarrow A \subseteq B']$

$\Rightarrow A \subseteq A^\circ$. $[\because A'^{-\prime} = A^\circ]$

But $A^\circ \subseteq A$. Hence $A^\circ = A$.

It follows that A is open.

(iv) Let A be closed. Then $\overline{A} = A$. $\qquad \ldots(3)$

We have $\mathbf{Fr}(A) = \overline{A} \cap A'^{-}$ [By (1)]

$= A \cap A'^{-}$ [By (3)]

$\subseteq A$.

Thus if A is closed, then $\mathbf{Fr}(A) \subseteq A$.

Conversely, let $\mathbf{Fr}(A) \subseteq A$. Then $A \cup \mathbf{Fr}(A) = A$.

But $A \cup \mathbf{Fr}(A) = \overline{A}$ [By corollary 2 of theorem 10]

It follows that $A = \overline{A}$. Hence A is closed.

10.20 Bolzano-Weierstrass Theorem

Every infinite subset of a compact metric space X has a limit point in X.

A metric space X is said to have the Bolzano-Weierstrass property if every infinite subset of X has a limit point. Thus, every compact metric space has the Bolzano-Weierstrass property.

Here, we shall prove the Bolzano-Weierstrass theorem only for the metric space $(\mathbf{R}, d)$ where $d = |x - y|, \forall\ x, y \in \mathbf{R}$ is the usual metric on $\mathbf{R}$.

Bolzano-Weierstrass theorem: *Every infinite bounded set of real numbers has a limit point.*

Proof: Let S be an infinite bounded set of real numbers and let h and k be its infimum and supremum respectively. Note that both h and k exist by order-completeness property of real numbers.

Let us define a set H as follows.

A real number x belongs to H iff it exceeds at most a finite number of members of S *i.e.*,

$H = \{x : x \in \mathbf{R}$ and the number of elements of S which are less than x is finite$\}$.

Since $h = \inf S$, therefore $x \geq h,\ \forall\ x \in S$.

$\therefore$ h exceeds no member of S and so by our definition of H, $h \in H$. Thus $H \neq \varnothing$.

Also no number greater than k is in H. For let $y > k = \sup S$. Then $x < y,\ \forall\ x \in S$. Therefore y exceeds all the members of S which are infinite because S is an infinite set. Therefore if $y > k$, then by our definition of H, $y \notin H$.

$\therefore$ the set H is bounded above with k as an upper bound.

Thus H is a non-empty subset of $\mathbf{R}$ bounded above by k. Hence by order-completeness property, H must have a supremum in $\mathbf{R}$.

Let $\sup H = p$. We shall show that p is a limit point of S.

Let $\varepsilon > 0$ be given. Since $\sup H = p$, therefore $p - \varepsilon$ cannot be an upper bound for H. Therefore there exists $x \in H$ such that

$$x > p - \varepsilon \quad i.e.,\quad p - \varepsilon < x.$$

Since $x \in H$, therefore, it exceeds at most a finite number of members of S and consequently $p - \varepsilon$ also exceeds at most a finite number of members of S.

Again since $p = \sup H$, $p + \varepsilon$ cannot belong to H and so $p + \varepsilon$ exceeds an infinite number of members of S.

Now $p - \varepsilon$ exceeds only a finite number of members of S and $p + \varepsilon$ exceeds an infinite number of members of S. It means that the nbd $]\, p - \varepsilon, p + \varepsilon\, [$ of p contains infinitely many points of S.

Since ε is arbitrary, therefore every ε-nbd of p contains infinitely many points of S. Hence p is a limit point of S. This proves the theorem.

10.21 Bases

Base of the neighbourhood system of a point :

Definition: *Let $N(x)$ denote the family of all neighbourhoods of a point x in a metric space X. A subfamily $\mathbf{B}(x)$ of $N(x)$ is said to be a base for $N(x)$ (i.e.,* **a base for the**

neighbourhood system of x) *if to each member* N *of* $\boldsymbol{N}(x)$ *there exits* $B \in \boldsymbol{B}(x)$ *such that* $B \subseteq N$.

We then also say that $\boldsymbol{B}(x)$ is a **local base** at x or a **fundamental system of neighbourhoods** of x.

Example: *Consider the usual metric* $d(x, y) = |x - y|$ *for* $\mathbf{R}$ *and any* $x \in \mathbf{R}$. *Then the collection*

$$\boldsymbol{B}(x) = \{]x - \varepsilon, x + \varepsilon[: 0 < \varepsilon \in \mathbf{R}\}$$

of all open intervals with x *as mid-point constitutes a base for the neighbourhood system of* x ; *for any nhd* N *of* x *we can choose* ε *so small that*

$$]x - \varepsilon, x + \varepsilon[\subseteq N.$$

Another local base at x is the collection

$$\boldsymbol{B}'(x) = \left\{\left]x - \frac{1}{n}, x + \frac{1}{n}\right[: n \in \mathbf{N}\right\}.$$

Base for the open sets of a metric space:

Definition: *Let* $\boldsymbol{G}$ *denote the family of all open subsets of a metric space* (X, d). *A sub-family* $\boldsymbol{B}$ *of* $\boldsymbol{G}$ *is said to be a* **base** *(or* **basis***) for* G *if for each point* $x \in X$ *and each neighbourhood* N *of* x, *there exists some* $B \in \boldsymbol{B}$ *such that*

$$x \in B \subseteq N.$$

Example: *Consider the usual metric* $d(x, y) = |(x - y)|$ *for* $\mathbf{R}$. *Let* $\boldsymbol{B}$ *be the collection of all open intervals on* $\mathbf{R}$. *Then* $\boldsymbol{B}$ *is a base for the family* $\boldsymbol{G}$ *of all open subsets of* $\mathbf{R}$.

For we have

(i) each open interval is an open set so that $\boldsymbol{B} \subseteq \boldsymbol{G}$.

(ii) for each $x \subseteq \mathbf{R}$ *and each nhd* N *of* x, *there exists an open set* G *such that*

$$x \in G \subseteq N.$$

Then by definition of open sets, there exists $\varepsilon > 0$ *such that*

$$]x - \varepsilon, x + \varepsilon[\subseteq G \subseteq N.$$

A characterization for a base:

Theorem: *Let* (X, d) *be a metric space and let* $\boldsymbol{G}$ *be the family of all open subsets of* X. *A sub-family* $\boldsymbol{B}$ *of* $\boldsymbol{G}$ *is a base for* $\boldsymbol{G}$ *iff every member of* $\boldsymbol{G}$ *can be expressed as a union of members of* $\boldsymbol{B}$.

Proof: Let $\boldsymbol{B}$ be a base for $\boldsymbol{G}$ and let $G \in \boldsymbol{G}$.

Let $H = \cup \{B : B \in \boldsymbol{B} \text{ and } B \subseteq G\}$,

that is, let H be the union of members of $\boldsymbol{B}$ which are subsets of G and let $x \in G$.

Since G is open, it is a nhd of x and consequently by definition of base $\boldsymbol{B}$, there exists $B \in \boldsymbol{B}$ such that $x \in B \subseteq G$. Since $B \subseteq H$, it follows that $x \in H$. Hence

$$G \subseteq H. \qquad \ldots(1)$$

Also if $x \in H$, then $x \in B$ for some $B \in \boldsymbol{B}$. Since $B \subseteq G$, it follows that $x \in G$. Hence

$$H \subseteq G. \qquad \ldots(2)$$

From (1) and (2), we get

$$G = H.$$

This shows that every open set can be expressed as a union of members of $\boldsymbol{B}$.

Conversely, let $\boldsymbol{B} \subseteq \boldsymbol{G}$ and let every open set G be the union of members of $\boldsymbol{B}$. We have to show that $\boldsymbol{B}$ is a base for $\boldsymbol{G}$. We have

(i) $\boldsymbol{B} \subseteq \boldsymbol{G}$ (given).

(ii) For each $x \in X$ and each nhd N of x, there exists an open set G such that $x \in G \subseteq N$. But G is a union of a sub-family of $\boldsymbol{B}$. Hence there exists $B \in \boldsymbol{B}$ such that $x \in B \subseteq G \subseteq N$. Accordingly, $\boldsymbol{B}$ is a base for $\boldsymbol{G}$.

First and Second Countable Spaces:

Definition 1: *A metric space (X, d) is said to satisfy the* **first axiom of countability** *if each point of X posseses a* **countable** *local base. Such a space is said to be a* **first countable** *(or a* **first axiom***) space.*

Definition 2: *A metric space (X, d) is said to satisfy the* **second axiom of countability** *if there exists a* **countable base** *for $\boldsymbol{G}$ where $\boldsymbol{G}$ is the family of all open subsets of X. Such a space is said to be a* **second countable** *(or a* **second axiom***) space.*

Example 46: *The usual metric space $(\mathbf{R}, d)$ is a first as well as a second countable space.*

Solution: (i) To each $x \in \mathbf{R}$, the family

$$\left\{ \left] x - \frac{1}{n}, x + \frac{1}{n} \right[: n \in \mathbf{N} \right\}$$

is a countable local base at x.

(ii) The collection of all open intervals $]a, b[$ where a, b are rational numbers forms a countable base for $\boldsymbol{G}$.

Theorem 1: *Every metric space is first countable.*

Proof: Let (X, d) be a metric space and let $x \in X$. We have to show that there exists a countable local base at x. Consider the collection of open spheres

$$\boldsymbol{B}(x) = \left\{ S\left(x, \frac{1}{n}\right) : n \in \mathbf{N} \right\}.$$

We claim that this collection forms a countable local base at x. Let N be a nhd of x. Then there exists an open set G such that

$$x \in G \subseteq N.$$

By definition of open sets, there exists $\varepsilon > 0$ such that

$$S(x, \varepsilon) \subseteq G \subseteq N.$$

Choose n so large that $\frac{1}{n} < \varepsilon$.

We then have $S\left(x, \frac{1}{n}\right) \subseteq S(x, \varepsilon) \subseteq G \subseteq N$.

It follows that every nhd of x contains a member of $\boldsymbol{B}(x)$ and hence $\boldsymbol{B}(x)$ forms a countable local base at x. Accordingly (X, d) is first countable.

Theorem 2: *A metric space is separable if and only if it is second countable.*

Proof: Recall that a metric space X is said to be **separable** if there exists a countable dense subset A of X such that $\overline{A} = X$.

The 'only if' part: Let (X, d) be a separable metric space and let

$$A = \{x_n : n \in \mathbf{N}\}$$

be a countable dense subset of X so that $\overline{A} = X$. We claim that the collection

$$\boldsymbol{B} = \left\{S\left(x_n, \frac{1}{m}\right) : n \in \mathbf{N}, m \in \mathbf{N}\right\}$$

is a countable base for $\boldsymbol{G}$ (the family of all open subsets of X). Since $\boldsymbol{B}$ is a countable union of countable families, it follows, that $\boldsymbol{B}$ is a countable set. Also each $S\left(x_n, \frac{1}{m}\right)$ is an open set being an open sphere. It remains to be shown that $\boldsymbol{B}$ is a base for $\boldsymbol{G}$. Let G be any open set and let $x \in G$. By definition of open sets, there exists $\varepsilon > 0$ such that

$$S(x, \varepsilon) \subseteq G.$$

Now choose a positive integer $m_0 > \frac{2}{\varepsilon}$.

Since $\overline{A} = X$, every point of X is an adherent point of A. It follows that every open sphere centred at x must contain a point of A.

In particular, $S\left(x, \frac{1}{m_0}\right)$ must contain a point of A, say x_i. But

$$x_i \in S\left(x, \frac{1}{m_0}\right) \Rightarrow d(x, x_i) < \frac{1}{m_0} \Rightarrow x \in S\left(x_i, \frac{1}{m_0}\right).$$

We now show that $S\left(x_i, \frac{1}{m_0}\right) \subseteq S(x, \varepsilon)$.

Let $y \in S\left(x_i, \frac{1}{m_0}\right)$. Then $d(y, x_i) < \frac{1}{m_0}$.

But then

$$d(x, y) \leq d(x, x_i) + d(x_i, y)$$
$$< \frac{1}{m_0} + \frac{1}{m_0} = \frac{2}{m_0} < \varepsilon \qquad \text{[See the choice of } m_0\text{]}$$

This implies that $y \in S(x, \varepsilon)$.

Thus we have shown that

$$y \in S\left(x_i, \frac{1}{m_0}\right) \Rightarrow y \in S(x, \varepsilon). \quad \therefore S\left(x_i, \frac{1}{m_0}\right) \subseteq S(x, \varepsilon).$$

It follows that $x \in S\left(x_i, \frac{1}{m_0}\right) \subseteq S(x, \varepsilon) \subseteq G$.

Thus for every point x of G, there exists a member $S\left(x_i, \frac{1}{m_0}\right)$ of $\boldsymbol{B}$ containing x and contained in G.

Hence $\boldsymbol{B}$ is a base for $\boldsymbol{G}$.

It follows that (X, d) is second countable.

The 'if' part. Let (X, d) be a second countable metric space. To show that it is separable. Since the space is second countable, there exists a countable base

$$\boldsymbol{B} = \{B_n : n \in \mathbf{N}\}$$

for the family G of all open subsets of X. For each $n \in \mathbf{N}$, we choose a point $b_n \in B_n$. We thus obtain the set $B = \{b_n : n \in \mathbf{N}\}$. The set B is evidently countable. We now show that $\boldsymbol{B}$ is dense in X. Let x be any point of X and let G be any open nhd of x. Since B is a base, there exists at least one $B_{n_0} \in \boldsymbol{B}$ such that

$$x \in B_{n_0} \subseteq G.$$

By definition of B, $b_{n_0} \in B$ is such that $b_{n_0} \in B_{n_0} \subseteq G$.

Thus G contains a point of B. Since G was any open nhd of x, we have shown that every nhd of x contains a point of B. Hence x is an adherent point of B so that $x \in \overline{B}$.

Thus it is shown that

$$x \in X \Rightarrow x \in \overline{B}.$$

Hence $X \subseteq \overline{B}$. But $\overline{B} \subseteq X$.

$\therefore$ $\overline{B} = X$.

Thus B is a countable dense subset of X. Accordingly (X, d) is separable.

10.22 Sub-Spaces of a Metric Space

Definition: *Let (X, d) be a metric space and let Y be a proper subset of X. Let d^* denote the restriction of d to $Y \times Y$, i.e., $d^*(x, y) = d(x, y)$ whenever x and y are points of Y. Then d^* is a metric for Y called the* **induced metric.** *The set Y with metric d^* is called a* **sub-space** *of X.*

Hereditary Property: Definition: *A property of a metric space is said to be hereditary iff every sub-space of that space has that property.*

If (Y, d^*) is a sub-space of the metric space (X, d), a set which is open in Y is not necessarily open in X. Similarly a closed set in Y need not be closed in X. For example, in the sub-space $[0, 1]$ of $\mathbf{R}$ with usual metric $d(x, y) = |x - y|$, the subset

$\left[0, \frac{1}{2}\right[$ is open but it is not open in $\mathbf{R}$. The following theorem gives a characterization of open sets in a sub-space.

Theorem 1: *Let (X, d) be a metric space and let (Y, d^*) be a sub-space thereof. A subset A of Y is d^*-open if and only if there exists a d-open subset G of X such that $A = G \cap Y$.*

Proof: We shall denote a d^*- open sphere in Y with centre y_0 and radius r by the symbol $S^*(y_0, r)$ so that

$$S^*(y_0, r) = \{y \in Y : d^*(y, y_0) < r\}.$$

It is obvious that

$$S^*(y_0, r) = Y \cap S(y_0, r)$$

where $S(y_0, r)$ is a d-open sphere in X.

Let B be any d^*-open subset of Y. Then to each $y \in B$, there exists a d^*-open sphere $S^*(y, r(y))$ with centre y and radius $r(y) > 0$ such that

$$S^*(y, r(y)) \subseteq B.$$

[Note that the notation for the radius used here is $r(y)$ since $r(y)$ depends upon y].

We then have

$$\begin{aligned} B &= \cup \{S^*(y, r(y)) : y \in B\} \\ &= \cup \{Y \cap S(y, r(y)) : y \in B\} \\ &= Y \cap [\cup \{S(y, r(y)) : y \in B\}] \qquad \text{[By Distributive Law]} \\ &= Y \cap G \end{aligned}$$

where $G = \cup \{S(y, r(y)) : y \in B\}$.

Since G is a d-open set, being a union of d-open spheres, the 'only if' part is proved.

Conversely, let

$$B = G \cap Y,$$

where G is a d-open subset of X. Let y be an arbitrary point of B.

Then y is also a point of G since $B = G \cap Y$. Since G is d-open there exists an open sphere $S(y, r(y))$ such that

$$S(y, r(y)) \subseteq G.$$

Hence $$\begin{aligned} S^*(y, r(y)) &= Y \cap S(y, r(y)) \\ &\subseteq Y \cap G = B. \end{aligned}$$

Thus to each $y \in B$, there exists a d^*- open sphere centred at y and contained in B.

Hence B is a d^*- open subset of Y. This proves the 'if' part.

The theorem is thus completely established.

Theorem 2: *Let (Y, d^*) be a sub-space of a metric space (X, d). In order that every subset B of Y, which is open in Y, be open in X it is necessary and sufficient that Y be open in X.*

Proof: Let every subset B of Y, which is open in Y, be also open in X. Since Y is open in Y, it follows that Y is open in X. Thus the condition is necessary.

Conversely, let B be a subset of Y which is open in Y and let Y be open in X. Since B is open in Y, there exists a subset G open in X such that

$$B = G \cap Y.$$

Since Y is open in X, therefore B is open in X, being the intersection of two open subsets of X. Thus the condition is sufficient.

Theorem 3: *Let (Y, d^*) be a subspace of a metric space (X, d) and let A be a subset of Y. Then*

(i) A subset A of Y is closed in Y if and only if there exists a set F closed in X such that

$$A = F \cap Y.$$

(ii) $\overline{A}^* = \overline{A} \cap Y$

where $\overline{A}^*$ and $\overline{A}$ denote respectively the d^*- closure and the d-closure of $A \subseteq Y$.

(iii) A subset N^* of Y is a d^*- nhd of a point $y \in Y$ if and only if $N^* = N \cap Y$ for some d-nhd N of y.

(iv) A point $y \in Y$ is a d^*- limit point of a subset A of Y if and only if it is a d-limit point of A.

Further $D^*(A) = D(A) \cap Y$

where $D^*(A)$ and $D(A)$ denote respectively the d^*-derived set and the d-derived set of A.

(v) If A is a subset of Y, then

$$A^{\circ *} \supseteq A^{\circ}$$

where $A^{\circ *}$ and A° denote respectively the d^*- interior and the d-interior of A.

(vi) For any subset A of Y, we have

$$\mathbf{Fr}^*(A) \subseteq \mathbf{Fr}(A)$$

where $\mathbf{Fr}^*(A)$ and $\mathbf{Fr}(A)$ denote respectively the d^*- frontier and the d-frontier of A.

Proof: (i) Let A be a subset of Y. Then A is closed in Y

$\Leftrightarrow$ $Y - A$ is open in Y

$\Leftrightarrow$ $Y - A = G \cap Y$, where G is a set open in X [See theorem 1]

$\Leftrightarrow$ $A = Y - (G \cap Y)$

$\Leftrightarrow$ $A = (Y - G) \cup (Y - Y)$ [By De-Morgan's Law]

$\Leftrightarrow$ $A = Y - G$ [$\because Y - Y = \varnothing$]

$\Leftrightarrow$ $A = Y \cap G'$, where G' denotes the complement of G in X

$\Leftrightarrow$ $A = Y \cap F$

where $F = G'$ is closed in X since G is open in X.

(ii) By the definition of closure, $\overline{A}^*$ is the intersection of all d^*- closed supersets of A, *i.e.*,

$$\begin{aligned}\overline{A}^* &= \cap \{K : K \text{ is closed in } Y \text{ and } A \subseteq K\} \\ &= \cap \{F \cap Y : F \text{ is closed in } X \text{ and } A \subseteq F \cap Y\} && \text{[By (i)]} \\ &= \cap \{F \cap Y : F \text{ is closed in } X \text{ and } A \subseteq F\} \\ &= [\cap \{F : F \text{ is closed in } X \text{ and } A \subseteq F\}] \cap Y \\ &= \overline{A} \cap Y. \\ & && [\because \ \overline{A} \text{ is the intersection of all } d\text{-closed supersets of } A]\end{aligned}$$

(iii) Let N^* be a d^*-nhd of y. Then there exists a d^*- open set A^* such that

$$y \in A^* \subseteq N^*$$

$\Rightarrow$ $\exists$ a d-open set A such that $y \in A^* = A \cap Y \subseteq N^*$.

Let $N = N^* \cup A$. Then N is a d-nhd of y since A is a d-open set such that $y \in A \subseteq N$.

Further $N \cap Y = (N^* \cup A) \cap Y = (N^* \cap Y) \cup (A \cap Y)$

$= N^* \cup (A \cap Y)$ $[\because N^* \subseteq Y]$

$= N^*$. $[\because A \cap Y \subseteq N^*]$

Conversely, let $N^* = N \cap Y$ where N is a d-nhd of y. Then there exists a d-open set A such that

$$y \in A \subseteq N$$

$\Rightarrow$ $y \in A \cap Y \subseteq N \cap Y = N^*$ $[\because y \in Y]$

Since $A \cap Y$ is a d^*- open set, it follows that N^* is a d^*- nhd of y.

(iv) y is a d^*-limit point of A

$\Leftrightarrow [N^* - \{y\}] \cap A \neq \varnothing$ for all d^*-nhds N^* of y

$\Leftrightarrow [N \cap Y - \{y\}] \cap A \neq \varnothing$ for all d-nhds N of y [By part (iii)]

$\Leftrightarrow [N - \{y\}] \cap A \neq \varnothing$ for all d-nhds N of y

$[\because y \in Y$ and $A \subseteq Y]$

$\Leftrightarrow$ y is a d-limit point of A.

From this, we at once conclude that

$$D^*(A) = D(A) \cap Y.$$

(v) We have

$x \in A^\circ \Rightarrow x$ is a d-interior point of A

$\Rightarrow$ A is a d - nhd of x

$\Rightarrow$ $A \cap Y$ is a d^*- nhd of x [By part (iii)]

$\Rightarrow$ A is a d^*- nhd of x $[\because A \subseteq Y \Rightarrow A \cap Y = A]$

$\Rightarrow x$ is d^* - interior point of A

$\Rightarrow x \in A^{\circ^*}$.

Hence $A^\circ \subseteq A^{\circ^*}$.

(vi) We have

$y \in \mathbf{Fr}^*(A) \Rightarrow y$ is d^* - frontier point of A

$\Rightarrow$ every d^* - nhd of y intersects both A and $Y - A$

$\Rightarrow N \cap Y$ intersects both A and $Y - A$ for every d - nhd N of y

$\Rightarrow$ every d-nhd N of y intersects both A and $X - A$ (why ?)

$\Rightarrow y$ is d - frontier point of A

$\Rightarrow y \in \mathbf{Fr}(A)$.

Hence $\mathbf{Fr}^* A \subseteq \mathbf{Fr}(A)$.

Theorem 4: *Let Y be a subspace of a metric space X. If $A \subseteq Y$ is open (closed) in X, then A is also open (closed) in Y.*

Proof: Since $A \subseteq Y$, we have $A = A \cap Y$ so that A is the intersection of Y with a set open (closed) in X (namely A). Hence by theorem 1 and theorem 3 part (i), A is open (closed) in Y.

Theorem 5: *Let (Y, d^*) be a subspace of a metric space (X, d). In order that every subset B of Y, which is closed in Y, be closed in X, it is necessary and sufficient that Y be closed in X.*

Proof: Necessity. Let every subset B of Y, which is closed in Y be also closed in X. Since Y is closed in Y, it follows that Y is closed in X.

Sufficiency. Let B be a subset of Y which is closed in Y and let Y be closed in X. Since B is closed in Y, there exists a subset F closed in X such that

$$B = F \cap Y.$$

Since Y is closed in X, therefore B is closed in X, being the intersection of two closed subsets of X.

Theorem 6: *Let (Y, d^*) be a sub-space of a metric space (X, d) and let $\mathbf{B}$ be a base for the family $\mathbf{G}$ of all open subsets of X relative to d. Then*

$$\mathbf{B}^* = \{B \cap Y : B \in \mathbf{B}\}$$

is a base for the family $\mathbf{G}^*$ of all open subsets of Y relative to d^*.

Proof: Let G^* be a d^*-open subset of Y and let $x \in G^*$.

Then there exists a d-open subset G of X such that $G^* = G \cap Y$. Since $\mathbf{B}$ is a base for $\mathbf{G}$, there exists a set $B \in \mathbf{B}$, such that

$$x \in B \subseteq G.$$

Since $G^* \subseteq Y$, it follows that $x \in Y$ and consequently

$$x \in B \cap Y \subseteq G \cap Y = G^*.$$

Hence there exists a set $B \cap Y \in \boldsymbol{B}^*$ such that

$$x \in B \cap Y \subseteq G^*.$$

This shows that $\boldsymbol{B}^*$ is a base for $\boldsymbol{G}^*$.

Theorem 7: *The property of being a second axiom space in a metric space is a hereditary property, that is, every sub-space of a second axiom space is a second axiom space.*

Proof: Let (X, d) be a second axiom space and let

$$\boldsymbol{B} = \{B_n : n \in \mathbf{N}\}$$

be a countable base for the family $\mathbf{G}$ of all d-open subsets of X. Let (Y, d^*) be a subspace of (X, d). By the previous theorem, the family

$$\boldsymbol{B}^* = \{B_n \cap Y : n \in \mathbf{N}\}$$

is a base for the family $\boldsymbol{G}^*$ of all d^*-open subsets of Y. Also $\boldsymbol{B}^*$ is surely countable. Hence there exists a countable base for $\boldsymbol{G}^*$. It follows that (Y, d^*) is a second axiom space.

Hence every sub-space of X is second axiom and so the property of being a second axiom space is hereditary.

Theorem 8: *Every subspace of a separable metric space is separable, that is, separability is a hereditary property in a metric space.*

Proof: Let (X, d) be a separable metric space and let (Y, d^*) be any subspace thereof. Then it follows from theorem 2 of article 10.21 that (X, d) is a second axiom space and so by the previous theorem the sub-space (Y, d^*) is also second axiom space. Hence by theorem 2 of article 10.21 again, the space (Y, d^*) is separable. Thus separability is a hereditary property.

Illustrative Examples

***Example 47**: Show that every sub-space of a discrete metric space is discrete.*

***Solution**:* Let Y be any subspace of a discrete metric space X. To show that Y is also discrete *i.e.*, to show that every subset of Y is open in Y.

Let A be any subset of Y.

We have $A \subseteq Y \Rightarrow A \subseteq X$ $[\because X \supseteq Y]$

$\Rightarrow A$ is open in X $[\because X$ is discrete$]$

$\Rightarrow A \cap Y$ is open in Y [By theorem 1 of article 10.22]

$\Rightarrow A$ is open in Y $[\because A \subseteq Y \Rightarrow A \cap Y = A]$

Thus every subset of Y is open in Y. Therefore Y is a discrete space.

Hence every subspace of a discrete metric space is discrete.

***Example 48**: Show that every subspace of an indiscrete metric space is indiscrete.*

***Solution**:* Let Y be any subspace of an indiscrete metric space X. To show that Y is also indiscrete *i.e.*, to show that the null set $\varnothing$ and Y are the only open sets in Y.

Let A be any subset of Y which is open in Y. Then

$A = G \cap Y$, where G is some open set in X.

[By theorem 1 of article 10.22]

But X is indiscrete. Therefore the null set $\varnothing$ and X are the only open sets in X.

Therefore we must have $G = \varnothing$ or $G = X$.

Now $G = \varnothing \Rightarrow A = \varnothing \cap Y = \varnothing$

and $G = X \Rightarrow A = X \cap Y = Y$ $[\because\ Y \subseteq X]$

Thus $\varnothing$ and Y are the only open sets in Y. Therefore Y is indiscrete.

Hence every sub-space of an indiscrete metric space is indiscrete.

Example 49: *Consider the usual metric $d(x, y) = |x - y|$ for $\mathbf{R}$ and let $Y = [0, 1]$. Find whether or not each of the following subsets of Y are open relative to Y.*

(i) $\left]\frac{1}{2}, 1\right]$ *(ii)* $\left]\frac{1}{3}, \frac{2}{3}\right[$ *(iii)* $\left]0, \frac{1}{2}\right]$.

Solution: We know that in the usual metric space $(\mathbf{R}, d)$ a subset G of $\mathbf{R}$ is open if G is an open interval.

Also a subset A of Y is open in Y if and only if $A = G \cap Y$, where G is some open set in $(\mathbf{R}, d)$.

(i) We have $\left]\frac{1}{2}, 1\right] = \left]\frac{1}{2}, \frac{3}{2}\right[\cap [0, 1]$ where $\left]\frac{1}{2}, \frac{3}{2}\right[$ is open in $\mathbf{R}$.

Thus $\left]\frac{1}{2}, 1\right] = G \cap Y$, where $G = \left]\frac{1}{2}, \frac{3}{2}\right[$ is open in $\mathbf{R}$.

Hence $\left]\frac{1}{2}, 1\right]$ is open relative to Y *i.e.*, relative to $[0, 1]$.

(ii) We have $\left]\frac{1}{3}, \frac{2}{3}\right[= \left]\frac{1}{3}, \frac{2}{3}\right[\cap [0, 1]$ where $\left]\frac{1}{3}, \frac{2}{3}\right[$ is open in $\mathbf{R}$ because it is an open interval.

Thus $\left]\frac{1}{3}, \frac{2}{3}\right[= G \cap Y$, where $G = \left]\frac{1}{3}, \frac{2}{3}\right[$ is open in $\mathbf{R}$.

Hence $\left]\frac{1}{3}, \frac{2}{3}\right[$ is open relative to Y.

(iii) There exists no open interval $]a, b[$ in $\mathbf{R}$ such that

$$\left]0, \frac{1}{2}\right] =]a, b[\cap [0, 1].$$

Thus there exists no open set G in $\mathbf{R}$ such that

$$\left]0, \frac{1}{2}\right] = G \cap Y.$$

Hence, $\left]0, \frac{1}{2}\right]$ is not open relative to Y.

Comprehensive Exercise 2

1. Describe open spheres for the usual metric on $\mathbf{R}$.

   Or

   Show that the open spheres on the real line are precisely the bounded open intervals.

2. Let (X, d) be a metric space and let $p \notin S[x_0, r]$ where $x_0 \in X$ and $r > 0$. Then show that

   $$d(p, S[x_0, r]) \geq d(x_0, p) - r.$$

3. In a metric space (X, d), $S(x_0, r)$ is an open sphere and p is any point of $S(x_0, r)$. Show that there exists an open sphere $S(p, \varepsilon)$ such that $S(p, \varepsilon) \subseteq S(x_0, r)$.

4. Show that in a metric space every open sphere is a neighbourhood of each of its points.

5. Show that in the discrete metric space (X, d), the open sphere $S(x_0, r)$ is given by $S(x_0, r) = \begin{cases} X, & \text{if } r > 1 \\ \{x_0\}, & \text{if } 0 < r \leq 1. \end{cases}$

6. Show that in the discrete metric space (X, d), the closed sphere $S[x_0, r]$ is given by

   $$S[x_0, r] = \begin{cases} X, & \text{if } r \geq 1 \\ \{x_0\}, & \text{if } 0 < r < 1. \end{cases}$$

7. Is the subset $[0, 3]$ open in $X = [0, 3]$ under the metric d given by

   $$d(x, y) = |x - y|, \forall\, x, y \in X\ ?$$

8. If A and B are open sets of a metric space X, show that $A \cap B$ is also an open set of X.

9. If A and B are open sets of a metric space X, show that $A \cup B$ is also an open set of X.

10. If A and B are closed sets of a metric space X, show that $A \cup B$ is also a closed set of X.

11. If A and B are closed sets of a metric space X, show that $A \cap B$ is also a closed set of X.

12. Let (X, d) be a metric space and $N(x)$ be the collection of all neighbourhoods of a point $x \in X$. If $M \in N(x)$ and $N \in N(x)$, then show that $M \cap N \in N(x)$.

Or

Show that in a metric space (X, d) the intersection of two neighbourhoods of a point $x \in X$ is also a neighbourhood of x.

13. Show that on the real line with the usual metric the singleton set $\{x\}$ is not open.

14. Show that every finite subset of $\mathbf{R}$ is closed with respect to the usual metric for $\mathbf{R}$.

15. Let (X, d) be a metric space and $S_r(x_0)$ the open sphere with centre x_0 and radius r. Let A be a subset of X which intersects $S_r(x_0)$ and has diameter less than r. Prove that $A \subseteq S_{2r}(x_0)$.

16. In a metric space (X, d), show that for every pair of distinct points $x, y \in X$, there exist disjoint open sets U, V such that $x \in U$, $y \in V$.

17. Let (X, d) be any metric space and let

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \forall x, y \in X.$$

Show that d_1 is also a metric on X and the two metrics d and d_1 are equivalent.

18. Let $F_n = \left[0, \frac{n}{n+1}\right]$, $n \in \mathbf{N}$. Show that for the usual metric space $(\mathbf{R}, d)$, $\bigcup_{n=1}^{\infty} F_n$ is not a closed set.

19. Give an example of two closed subsets A and B of the real line $\mathbf{R}$ such that

$$d(A, B) = 0, \text{ but } A \cap B = \emptyset.$$

20. Let A be any subset of a metric space X and let G be an open set in X. Prove that G is disjoint from A if and only if G is disjoint from $\overline{A}$.

21. Let (X, d) be a metric space and $A \subseteq X$. Show that

$$\overline{A} = \{x \in X : d(x, A) = 0\}.$$

22. Show that a subset F of a metric space X is closed iff

$$\{x \in X : d(x, F) = 0\} \subseteq F.$$

23. Let (X, d) be a metric space. Show that a subset A of X is closed iff given any $x \in X - A$, $d(x, A) \neq 0$.

24. If A be any subset of a metric space X, prove that $d(A) = d(\overline{A})$, where $d(A)$ denotes the diameter of the set A.

Or

Prove that the diameter of a subset of a metric space is equal to the diameter of its closure.

25. If A_1 is an open set in the metric space (X_1, d_1) and A_2 is an open set in the metric space (X_2, d_2), then show that $A_1 \times A_2$ is open in the product metric space (X, d), where $X = X_1 \times X_2$.

26. Let A be a subset of a metric space (X, d). Prove that A is non-dense in X if and only if $X - \overline{A}$ is dense in X.

27. If A and B are subsets of a metric space (X, d), then show that

(i) $(A \cap B)^\circ = A^\circ \cap B^\circ$ (ii) $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$.

Also show by means of an example that it is not necessary that

$A^\circ \cup B^\circ = (A \cup B)^\circ$.

28. Let (X, d) be any metric space and let A be any subset of X. Prove that

(i) $\overline{X - A} = X - A^\circ$ (ii) $X - \overline{A} = (X - A)^\circ$.

29. In any metric space prove that

$(\overline{A})' = (A')^\circ$.

30. In the metric space $(\mathbf{R}, d)$ where d is the usual metric on $\mathbf{R}$, find the boundary of the closed interval $[a, b]$.

31. In the metric space $(\mathbf{R}, d)$ where d is the usual metric on $\mathbf{R}$, find the boundary of the set of integers $\mathbf{Z}$.

32. Show that the frontier of a subset of a metric space is a closed set.

33. In the metric space $(\mathbf{R}, d)$, where d is the usual metric on $\mathbf{R}$, find the limit points of the open interval $]0, 1[$.

34. Give an example of two subsets A and B of $\mathbf{R}$ such that

$$D(A \cap B) \neq D(A) \cap D(B).$$

The metric on $\mathbf{R}$ is the usual metric

$$d(x, y) = |x - y|, \forall\ x, y \in \mathbf{R}.$$

35. In the metric space $(\mathbf{R}, d)$ where d is the usual metric on $\mathbf{R}$, give an example of a set which has

(i) no limit point (ii) exactly one limit point

(iii) exactly two limit points

(iv) an infinite number of limit points

(v) every point of the set as its limit point.

36. In the usual metric space $\mathbf{R}$ if A' denotes the derived set of A, then find a set A such that

(i) $A \cap A' = \varnothing$ (ii) $A = A'$

(iii) $A' \subseteq A$ (iv) $A \subseteq A'$.

37. Show that in the metric space $(\mathbf{R}, d)$, where d is the usual metric on $\mathbf{R}$, the set of rational numbers $\mathbf{Q}$ is dense.
38. Show that $\mathbf{N}$ and $\mathbf{Z}$ are non-dense sets in $\mathbf{R}$ relative to the usual metric.
39. Define a dense set and give an example.
40. Define a nowhere dense set and give an example.
41. Are the following subsets of $\mathbf{R}$ d-nhds of 3 where d denotes the usual metric $d(x, y) = |x - y|$ for $\mathbf{R}$?
    (a) $]2, 4[$ (b) $[1, 3[$
    (c) $[3, 4[$ (d) $\mathbf{N}$.
42. Consider the following subsets of $\mathbf{R}$ and find their closures relative to the usual metric $d(x, y) = |x - y|$.
    (a) $A = \{1, 2, 3, 4\}$ (b) $B = \{1\}$
    (c) $C =]2, \infty[$ (d) $D =]1, 2[\cup]3, 4[$
    (e) $E = \left\{\frac{n+1}{n} : n \in \mathbf{N}\right\}$.
43. Give an example of an infinite set which is not bounded but has limit points.
44. Give an example to show that the intersection of an infinite collection of open sets is not necessarily an open set.
45. Show that for the usual metric space $\mathbf{R}$ the set $\mathbf{Z}$ of all integers is not a neighbourhood of any of its points.
46. Let x_1 and x_2 be two distinct points in the metric space (X, d). Show that there exist two disjoint open spheres centred at x_1 and x_2 respectively.
47. Show by an example that in a metric space a set which fails to be closed need not be open.
48. Show by an example that a non-empty proper subset A of a metric space (X, d) can be both open and closed.
49. Show that every subset of a discrete metric space is closed.

Answers 2

7. Yes
19. $A = \mathbf{N} = \{1, 2, 3, 4, 5, \ldots\}$, $B = \left\{n + \frac{1}{n} : n \in \mathbf{N}\right\}$
30. $\{a, b\}$
31. $\mathbf{Z}$
33. $D = (]0, 1[) = [0, 1]$
34. $A = [0, 1[$, $B =]1, 2]$

35. (i) The set of positive integers $\mathbf{Z}^+$

(ii) The set $\{1 / n \in n \in \mathbf{Z}^+\}$

(iii) The set $\left[\frac{1}{2}, -\frac{1}{2}, \frac{2}{3}, -\frac{2}{3}, \frac{3}{4}, -\frac{3}{4}, \ldots, \frac{n}{n+1}, \frac{-n}{n+1}\right]$.

(iv) The open interval $]1, 2[$

(v) The closed interval $[2, 5]$

36. (i) The set $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\}$

(ii) The closed interval $[3, 4]$

(iii) The set $A = \left\{0, \frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \left(\frac{1}{2}\right)^4, \ldots\right\}$

(iv) The open interval $]0, 1[$

41. (a) Yes (b) No

(c) No (d) No

42. (a) $\overline{A} = A$ (b) $\overline{B} = B$

(c) $\overline{C} = [2, \infty[$ (d) $\overline{D} = [1, 2] \cup [3, 4]$

(e) $\overline{E} = E \cup \{1\}$

43. In the metric space $(\mathbf{R}, d)$ where d is the usual metric on $\mathbf{R}$, the set $\mathbf{Q}$ of rational numbers is not bounded but has limit points. We have $D(\mathbf{Q}) = \mathbf{R}$.

44. In the metric space $(\mathbf{R}, d)$ where d is the usual metric on $\mathbf{R}$, the intersection of the infinite family of open intervals $\{]-1/n, 1/n[: n \in \mathbf{N}\}$ is the singleton $\{0\}$ which is not an open set.

Complete Metric Spaces

10.23 Sequences and Subsequences in a Metric Space

Sequence in a Metric Space: Definition: *Let (X, d) be a metric space. A sequence in X is a function from $\mathbf{N}$ (the set of positive integers) into X.*

Thus, if s is a sequence in X, then

$$s : \mathbf{N} \to X.$$

If s is a sequence in X, then the image $s(n)$ of $n \in \mathbf{N}$ is usually denoted by s_n. It is customary to denote the sequence s by the symbol $< s_n >$ or by $\{s_n\}$. The image s_n of n is called the nth term of the sequence. Note that the terms of a sequence need not be distinct.

Sometimes we write the sequence $< s_n >$ as

$$< s_1, s_2, s_3, \ldots, s_n, \ldots >.$$

Here s_1 , s_2 , s_3 etc. are elements of X and they need not be distinct.

The range set of a sequence may be finite or infinite. For example, the range of the sequence $< (-1)^n >$ is $\{1, -1\}$ which consists of two elements only whereas the range of the sequence $\left\{\frac{n}{n+1}\right\}$ *i.e.*, the sequence $< \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\ldots >$ is the infinite set

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\ldots\right\}.$$

Subsequences

Definition: *Let s and t be two sequences in a set X. Then t is said to be a subsequence of s if there exists a mapping*

$$\phi : \mathbf{N} \to \mathbf{N}$$

such that

(*i*) $t = s \circ \phi$

(*ii*) *for each n in* **N**, *there exists an m in* **N** *such that*

$$\phi(i) \geq n \text{ for every } i \geq m \text{ in } \mathbf{N}.$$

In other words, if $< s_n >$ is a sequence in X and $< i_n >$ is a strictly increasing sequence in **N** such that $i_1 < i_2 < \ldots\ldots < i_n < \ldots\ldots$, then $< s_{i_n} >$ is called a subsequence of $< s_n >$.

For example, if $s = < \frac{1}{n} >$ is a sequence in **R**, then $t = < \frac{1}{2n-1} >$ is a subsequence of s.

For if we define $\phi : \mathbf{N} \to \mathbf{N}$ such that $\phi(n) = 2n - 1$, then

$$(s \circ \phi)(n) = s(\phi(n)) = s(2n-1) = \frac{1}{2n-1} \text{ and } t(n) = \frac{1}{2n-1}.$$

Hence $s \circ \phi = t$. Thus $< 1, \frac{1}{3}, \frac{1}{5}, \ldots\ldots >$ is a subsequence of the sequence

$$< 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots >.$$

A sequence frequently in a given set

Definition: *A sequence $< s_n >$ is said to be* **eventually** *in a set A if there exists a positive integer m such that $n \geq m \Rightarrow s_n \in A$. A sequence $< s_n >$ is said to be* **frequently** *in a set A if for each positive integer m, there exists a positive integer $n \geq m$ such that $s_n \in A$.*

10.24 Convergence of a Sequence in a Metric Space

Definition: *Let (X, d) be a metric space. A sequence $< s_n >$ in X is said to converge to a point s_0 in X if for any given $\varepsilon > 0$, there exists a positive integer m i.e., there exists $m \in \mathbf{N}$ such that*

$d(s_n, s_0) < \varepsilon$ *for all* $n \geq m$.

Equivalently, a sequence $< s_n >$ in X is said to converge to a point s_0 in X if for any given $\varepsilon > 0$ there exists $m \in \mathbf{N}$ such that

$$s_n \in S_\varepsilon(s_0), \forall \ n \geq m.$$

Convergent sequence in a Metric Space: **Definition:** *A sequence* $< s_n >$ *in a metric space* (X, d) *is called a* **convergent sequence** *if it converges to a point, say* $s_0 \in X$ *i.e., if for any given* $\varepsilon > 0$ *there exists* $m \in \mathbf{N}$ *such that*

$$d(s_n, s_0) < \varepsilon, \forall \ n \geq m.$$

The point $s_0 \in X$ is called the limit of the sequence $< s_n >$ and we write

$$s_n \to s_0 \text{ as } n \to \infty \text{ or } \lim_{n \to \infty} s_n = s_0 \text{ or simply } \lim s_n = s_0 .$$

Remark 1: Every constant sequence $< a, a, a, a \ldots >$ in a metric space (X, d) is always a convergent sequence because it converges to the point a of X.

Remark 2: It is easy to see that if a sequence $< s_n >$ converges to s_0 then any subsequence of $< s_n >$ also converges to s_0.

Cluster points of a sequence:

Definition: *Let* (X, d) *be a metric space. A point* $s_0 \in X$ *is said to be a* **cluster point** *of a sequence* $< s_n >$ *in* X *if* $< s_n >$ *is frequently in every neighbourhood of* s_0 .

More precisely, s_0 *will be a cluster point of the sequence* $< s_n >$ *iff for any* $\varepsilon > 0$ *and any positive integer* m, *there exists an integer* $n \geq m$ *such that*

$$d(s_0, s_n) < \varepsilon .$$

It follows from the above definition that s_0 will be a cluster point of $< s_n >$ iff every open sphere centred at s_0 contains infinitely many terms of the sequence.

Note: The reader will do well to understand the meaning of the phrase "infinitely many terms of the sequence". It is possible that the various terms of a sequence may be the same element of the space in which the sequence is defined but they occur in a definite order so that there is a first term , a second, a third and so on and this order plays an important role in declaring various terms of the sequence as distinct. What the definition requires is that there should be an infinite number of terms of the sequence in the sense that the same element is counted as often as it occurs as a term of the sequence. Consider, for example, the sequence $< s_n > = < (-1)^n >$, that is, the sequence $< -1, 1, -1, 1, \ldots >$ in **R.** This sequence has 1 and -1 as cluster points with respect to the usual metric on **R** since every open sphere centred at 1 contains an infinite number of terms $s_2, s_4, s_6, s_8, \ldots$ and every open sphere centred at -1 contains an infinite number of terms :

$$s_1, s_3, s_5, \ldots$$

Note that here each of the terms $s_2, s_4, s_6, \ldots$ is 1 and that each of the terms $s_1, s_3, s_5, \ldots$ is -1.

Note that this sequence is not convergent.

Theorem 1: *Any convergent sequence in a metric space (X, d) converges to a unique point of X i.e., the limit of a convergent sequence is unique.*

Proof: Let $< s_n >$ be a convergent sequence in a metric space (X, d).

If possible, suppose $< s_n >$ converges to two distinct points s_0 and t_0 of X. Since $s_0 \neq t_0$, therefore $d(s_0, t_0) > 0$.

Let $\varepsilon = \frac{1}{2} d(s_0, t_0)$, then $\varepsilon > 0$.

Now $< s_n >$ converges to $s_0 \Rightarrow$ there exists $m_1 \in \mathbf{N}$ such that

$$d(s_n, s_0) < \varepsilon \text{ for all } n \geq m_1 . \qquad \ldots(1)$$

Similarly $< s_n >$ converges to $t_0 \Rightarrow$ there exists $m_2 \in \mathbf{N}$ such that

$$d(s_n, t_0) < \varepsilon \text{ for all } n \geq m_2 . \qquad \ldots(2)$$

Let $m = \max. \{m_1, m_2\}$.

Then (1) and (2) hold for all $n \geq m$.

For all $n \geq m$, we have

$$d(s_0, t_0) \leq d(s_0, s_n) + d(s_n, t_0) \qquad \text{[By the axiom of triangle inequality in the metric space } (X, d)\text{]}$$

$$= d(s_n, s_0) + d(s_n, t_0) \qquad [\because d(x, y) = d(y, x)]$$

$$< \varepsilon + \varepsilon = 2\varepsilon = d(s_0, t_0).$$

Thus $d(s_0, t_0) < d(s_0, t_0)$ *i.e.*, a positive real number is less than itself, which is absurd.

Hence our initial assumption that $s_0 \neq t_0$ is wrong and we must have $s_0 = t_0$ *i.e.*, the limit of a convergent sequence is unique.

Theorem 2: *Let (X, d) be a metric space. If s_0 is a limit point of a subset S of X, then there exists a sequence $< s_n >$ of points of S, all distinct from s_0, which converges to s_0.*

Proof: Since s_0 is a limit point of S, every sphere with centre s_0 must contain a point of S distinct from s_0. We begin by taking any point s_1 of S and set $r_1 = \min \{1, d(s_0, s_1)\}$. Then the sphere $S(s_0, r_1)$ contains a point s_2 of S, distinct from s_0. Now set $r_2 = \min \left\{\frac{1}{2}, d(s_0, s_2)\right\}$. Then the sphere $S(s_0, r_2)$ contains a point s_3 of S, distinct from s_0. Let

$$r_3 = \min \left(\frac{1}{3}, d(s_0, s_3)\right).$$

Then the sphere $S(s_0, r_3)$ contains a point s_4 of S, distinct from s_0. Continue this process indefinitely. We thus construct a sequence of distinct points $< s_n >$, all

distinct from s_0 such that if $r_n = \min\left(\frac{1}{n}, d(s_0, s_n)\right)$, the sphere $S(s_0, r_n)$ contains the point s_{n+1} of S. Hence $d(s_0, s_n) < r_{n-1} \leq \frac{1}{n-1}$.

It follows that $d(s_0, s_n)$ tends to zero as $n \to \infty$. Hence the sequence $< s_n >$ converges to s_0.

Example 50: *If $x \in \bar{A}$, prove that there exists a sequence $< x_n >$ of elements of A converging to x.*

Solution: Since $x \in \bar{A}$, x is an adherent point of A. Hence for every positive integer n, the open sphere $S\left(x, \frac{1}{n}\right)$ must contain a point x_n of A. We claim that the sequence $< x_n >$ converges to x. Let $\varepsilon > 0$ be given. Choose a positive integer m_0 such that $\frac{1}{m_0} < \varepsilon$. Then $n \geq m_0 \Rightarrow x_n \in S\left(x, \frac{1}{n}\right) \subseteq S(x, \varepsilon)$. Hence the sequence $< x_n >$ converges to x.

Theorem 3: *If the range set of a convergent sequence in a metric space consists of infinitely many distinct points, then the limit of the sequence is a limit point of the range set of the sequence.*

Proof: Let (X, d) be a metric space and let $s = < s_n >$ be a convergent sequence in X with limit s_0. We have to show that s_0 is a limit point of the range set $R(s)$ of s. Suppose, if possible, s_0 is not a limit point of $R(s)$. Then by the definition of a limit point, there exists an open sphere $S(s_0, r)$ centred at s_0 which contains no point of $R(s)$ different from s_0. However, since s_0 is the limit of the sequence, there exists a positive integer m such that

$$s_n \in S(s_0, r), \forall \quad n \geq m$$

and consequently $s_n = s_0$ for all $n \geq m$, since $S(s_0, r)$ contains no point of $R(s)$ different from s_0. It follows that $R(s)$ consists of finitely many distinct points. But this is a contradiction. Hence s_0 must be a limit point of $R(s)$.

Theorem 4: *Let (X, d) be a metric space and let s_0, t_0 be two points of X. If the sequence $< t_n >$ in X converges to t_0, then the sequence $< d(s_0, t_n) >$ of real numbers converges to $d(s_0, t_0)$.*

Proof: Take any given $\varepsilon > 0$.

Since the sequence $< t_n >$ in X converges to t_0, therefore there exists a positive integer m such that

$$d(t_n, t_0) < \varepsilon, \forall \quad n \geq m. \qquad \ldots(1)$$

Now for all $n \geq m$, we have

$$|d(s_0, t_n) - d(s_0, t_0)| = |d(t_n, s_0) - d(s_0, t_0)|$$

$$[\because d(x, y) = d(y, x)]$$

$\leq d(t_n, t_0)$ [$\because$ if x, y, z are any three points of a metric space (X, d), then $|d(x,z) - d(z,y)| \leq d(x,y)$]

$< \varepsilon$. [From (1)]

Thus for any given $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$|d(s_0, t_n) - d(s_0, t_0)| < \varepsilon, \forall\ n \geq m.$$

Hence, the sequence $< d(s_0, t_n) >$ of real numbers converges to $d(s_0, t_0)$.

Theorem 5: *Let (X, d) be a metric space. If the sequences $< s_n >$ and $< t_n >$ in X converge respectively to the points s_0 and t_0 in X, then the sequence $< d(s_n, t_n) >$ of real numbers converges to $d(s_0, t_0)$.*

Proof: Take any given $\varepsilon > 0$.

Since the sequences $< s_n >$ and $< t_n >$ in X converge to s_0 and t_0 respectively, therefore there exist positive integers m_1 and m_2 such that

$$d(s_n, s_0) < \frac{\varepsilon}{2}\ \forall\ n \geq m_1$$

and $$d(t_n, t_0) < \frac{\varepsilon}{2}\ \forall\ n \geq m_2\ .$$

Take $m = \max.\{m_1, m_2\}$.

We then have $d(s_n, s_0) < \frac{\varepsilon}{2}$

and $d(t_n, t_0) < \frac{\varepsilon}{2}$ for all $n \geq m$.

Now for all $n \geq m$, we have

$$\begin{aligned}
&|d(s_n, t_n) - d(s_0, t_0)| \\
&= |d(s_n, t_n) - d(s_n, t_0) + d(s_n, t_0) - d(s_0, t_0)| \\
&\leq |d(s_n, t_n) - d(s_n, t_0)| + |d(s_n, t_0) - d(s_0, t_0)| \\
&\qquad [\because\ |a+b| \leq |a| + |b|,\ \forall\ a, b \in \mathbf{R}] \\
&= |d(t_n, s_n) - d(s_n, t_0)| + |d(s_n, t_0) - d(t_0, s_0)| \\
&\qquad [\because \text{ in a metric space } (X, d),\ d(x,y) = d(y,x)] \\
&\leq d(t_n, t_0) + d(s_n, s_0) \\
&\qquad [\because \text{ in a metric space } (X, d),\ d(x,y) \geq |d(x,z) - d(z,y)| \\
&\qquad\qquad \text{i.e., } |d(x,z) - d(z,y)| \leq d(x,y)] \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\ .
\end{aligned}$$

Thus for any given $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$|d(s_n, t_n) - d(s_0, t_0)| < \varepsilon,\ \forall\ n \geq m.$$

Hence, the sequence $< d(s_n, t_n) >$ of real numbers converges to $d(s_0, t_0)$.

10.25 Cauchy Sequences in a Metric Space

Cauchy sequence in a Metric Space: Definition: *Let (X, d) be a metric space. A sequence $< s_n >$ in X is said to be a* **Cauchy sequence** *in X if for any given $\varepsilon > 0$, there exists a positive integer m such that*

$$d(s_n, s_m) < \varepsilon \text{ for all } n \geq m$$

i.e., $$d(s_p, s_q) < \varepsilon \text{ for all } p, q \geq m.$$

Thus $< s_n >$ is a Cauchy sequence if for every $\varepsilon > 0$, there exists a positive integer $n(\varepsilon)$ such that

$$m, n \geq n(\varepsilon) \Rightarrow d(s_m, s_n) < \varepsilon.$$

Theorem 1: *Every convergent sequence in a metric space is a Cauchy sequence.*

Proof: Let $< s_n >$ be a convergent sequence in a metric space (X, d).

We have to show that $< s_n >$ is a Cauchy sequence in (X, d).

Take any given $\varepsilon > 0$.

Suppose the sequence $< s_n >$ converges to the point s_0 of X.

Since the sequence $< s_n >$ converges to s_0, therefore there exists $m \in \mathbf{N}$ such that

$$d(s_n, s_0) < \frac{\varepsilon}{2}, \forall\ n \geq m.$$

In particular, $d(s_m, s_0) < \frac{\varepsilon}{2}$.

Now for all $n \geq m$, we have

$$d(s_n, s_m) \leq d(s_n, s_0) + d(s_0, s_m)$$

[$\because$ if x, y, z are any three points of a metric space (X, d), then $d(x, y) \leq d(x, z) + d(z, y)$]

$$= d(s_n, s_0) + d(s_m, s_0) \qquad [\because d(x, y) = d(y, x)]$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus for any given $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$d(s_n, s_m) < \varepsilon, \forall\ n \geq m.$$

Hence $< s_n >$ is a Cauchy sequence in X.

Remark: The converse of the above theorem is, however, not true. We shall see later on that there may exist a metric space (X, d) in which a Cauchy sequene $< s_n >$ does not necessarily converge to a point of X.

Theorem 2: *If a Cauchy sequence in a metric space has a convergent subsequence, then the sequence is convergent.*

Proof: Let $s = < s_n >$ be a Cauchy sequence in a metric space (X, d) and let $< s_{k_m} >$ be a subsequence of s converging to $x_0 \in X$ *i.e.*, $s_{k_m} \to x_0$ as $m \to \infty$. We know that every convergent sequence in a metric space is a Cauchy sequence. So, $< s_{k_m} >$ is a

Cauchy sequence and hence, given $\varepsilon > 0$, there exists a positive integer $n(\varepsilon)$ such that

$$m, n \geq n(\varepsilon) \Rightarrow d(s_{k_m}, s_n) < \varepsilon,$$

since $< k_m >$ is a strictly increasing sequence of positive integers. Making $m \to \infty$, we have

$$n \geq n(\varepsilon) \Rightarrow d(x_0, s_n) < \varepsilon.$$

This shows that the sequence $< s_n >$ converges to x_0.

Theorem 3: *Let $< s_n >$ be a sequence in a metric space (X, d) and let the sets E_n be defined by*

$$E_1 = \{s_1, s_2, \ldots\}, \; E_2 = \{s_2, s_3, \ldots\},$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$E_n = \{s_n, s_{n+1}, \ldots\}.$$

Show that $< s_n >$ is a Cauchy sequence if and only if

$$\delta(E_n) \to 0 \text{ as } n \to \infty.$$

Proof: Let $< s_n >$ be a Cauchy sequence. Then for a given $\varepsilon > 0$, there exists a positive integer $n(\varepsilon)$ such that

$$n, m \geq n(\varepsilon) \Rightarrow d(s_n, s_m) < \varepsilon.$$

It follows that $n \geq n(\varepsilon) \Rightarrow \delta(E_n) < \varepsilon$

and consequently $\delta(E_n) \to 0$.

Conversely, let $\delta(E_n) \to 0$. Then for a given $\varepsilon > 0$, there exists a positive integer $m(\varepsilon)$ such that

$$\delta(E_{m(\varepsilon)}) < \varepsilon.$$

Accordingly,

$$n, m \geq m(\varepsilon) \Rightarrow s_n, s_m \in E_{m(\varepsilon)} \qquad \text{[By def. of } E_{m(\varepsilon)}\text{]}$$
$$\Rightarrow \delta(s_n, s_m) < \varepsilon. \qquad [\because \; \delta(E_{m(\varepsilon)}) < \varepsilon]$$

Hence $< s_n >$ is a Cauchy sequence.

Theorem 4: *If $< s_n >$ is a Cauchy sequence in a metric space, then any cluster point of $< s_n >$ is limit of $< s_n >$.*

Proof: Let s_0 be a cluster point of a Cauchy sequence $< s_n >$. Since the sequence is Cauchy, for a given $\varepsilon > 0$, there exists a positive integer $n(\varepsilon)$ such that

$$n \geq n(\varepsilon), m \geq n(\varepsilon) \Rightarrow d(s_m, s_n) < \frac{\varepsilon}{2}. \qquad \ldots(1)$$

Again since s_0 is a cluster point of $< s_n >$, there exists a positive integer $n_0 \geq n(\varepsilon)$ such that

$$d(s_0, s_{n_0}) < \frac{\varepsilon}{2}. \qquad \ldots(2)$$

Now by the triangle inequality, we have

$$d(s_0, s_n) \le d(s_0, s_{n_0}) + d(s_{n_0}, s_n)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \text{[By (2) and (1)]}$$
$$= \varepsilon \text{ for all } n \ge n(\varepsilon).$$

Thus we have shown that to every $\varepsilon > 0$, there exists a positive integer $n(\varepsilon)$ such that

$$n \ge n(\varepsilon) \Rightarrow d(s_0, s_n) < \varepsilon.$$

It follows that s_0 is the limit of the sequence $< s_n >$, that is, the sequence converges to s_0.

Theorem 5: *Let $S = < s_n >$ be a Cauchy sequence in a metric space (X, d) and let $< s_{i_n} >$ be a subsequence of s. Then*

$$\lim_{n \to \infty} d(s_n, s_{i_n}) = 0.$$

Proof: Let $\varepsilon > 0$ be given. Since $< s_n >$ is a Cauchy sequence, there exists a positive integer n_0 such that

$$n \ge n_0, m \ge n_0 \Rightarrow d(s_n, s_m) < \varepsilon.$$

Also since $i_{n_0} \ge n_0$, it follows that

$$d(s_{n_0}, s_{i_{n_0}}) < \varepsilon.$$

Hence $\lim_{n \to \infty} d(s_n, s_{i_n}) = 0.$

Theorem 6: *Let $< s_n >$ be a Cauchy sequence in a metric space (X, d) and let $< s_{i_n} >$ be a subsequence of $< s_n >$ converging to $s_0 \in X$. Then $< s_n >$ also converges to s_0.*

Proof: By the triangle inequality, we have

$$d(s_0, s_n) \le d(s_0, s_{i_n}) + d(s_{i_n}, s_n).$$

$$\therefore \lim_{n \to \infty} d(s_0, s_n) \le \lim_{n \to \infty} d(s_0, s_{i_n}) + \lim_{n \to \infty} d(s_{i_n}, s_n). \qquad \ldots(1)$$

Since $s_{i_n} \to s_0$, we have

$$\lim_{n \to \infty} d(s_0, s_{i_n}) = 0.$$

Also by the previous theorem,

$$\lim_{n \to \infty} d(s_{i_n}, s_n) = 0.$$

Hence (1) gives

$$\lim_{n \to \infty} d(s_0, s_n) \le 0.$$

Since distances cannot be negative, we have

$$\lim_{n \to \infty} d(s_0, s_n) = 0.$$

It follows that $< s_n >$ converges to s_0.

Theorem 7: *Let (X, d) be a metric space and let $< s_n >$ be a Cauchy sequence in X. If $< t_n >$ be a sequence in X such that $d(s_n, t_n) < \frac{1}{n}$ for every positive integer n, then*

(i) $< t_n >$ is also a Cauchy sequence in X.

(ii) $< t_n >$ converges to a point $t_0 \in X$ if and only if $< s_n >$ converges to t_0.

Proof: (i) By the triangle inequality,

$$d(t_m, t_n) \leq d(t_m, s_m) + d(s_m, s_n) + d(s_n, t_n). \quad ...(1)$$

Since $< s_n >$ is a Cauchy sequence, for a given $\varepsilon > 0$, there exists a positive integer m_1, such that

$$n \geq m_1, m \geq m_1 \Rightarrow d(s_n, s_m) < \frac{\varepsilon}{3}. \quad ...(2)$$

Again we can choose a positive integer m_2 such that $\frac{1}{m_2} < \frac{\varepsilon}{3}$. Then by hypothesis

$$n \geq m_2, m \geq m_2 \Rightarrow d(s_m, t_m) < \frac{\varepsilon}{3}$$

and $$d(s_n, t_n) < \frac{\varepsilon}{3} \quad ...(3)$$

Set $m_0 = \max\{m_1, m_2\}$. Then it follows from (1), (2) and (3) that

$$n \geq m_0, m \geq m_0 \Rightarrow d(t_m, t_n) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence $< t_n >$ is also a Cauchy sequence.

(ii) Let $< t_n >$ converge to t_0, so that

$$\lim_{n \to \infty} d(t_n, t_0) = 0. \quad ...(1)$$

By the triangle inequality,

$$d(s_n, t_0) \leq d(s_n, t_n) + d(t_n, t_0).$$

$$\therefore \lim_{n \to \infty} d(s_n, t_0) \leq \lim_{n \to \infty} d(s_n, t_n) + \lim_{n \to \infty} d(t_n, t_0). \quad ...(2)$$

But $$\lim_{n \to \infty} d(s_n, t_n) \leq \lim_{n \to \infty} \frac{1}{n} = 0. \quad ...(3)$$

It follows from (1), (2) and (3) that

$$\lim_{n \to \infty} d(s_n, t_0) \leq 0 + 0 = 0.$$

This implies that

$$\lim_{n \to \infty} d(s_n, t_0) = 0.$$

Hence $< s_n >$ also converges to t_0 .

Similarly, if $s_n \to t_0$, then $t_n \to t_0$.

10.26 Complete Metric Space

Definition: *A metric space (X , d) is said to be* **complete** *if and only if every Cauchy sequence $< s_n >$ in X converges to a point in X.*

A metric space which is not complete is said to be incomplete. Thus a metric space (X , d) is incomplete if there exists a Cauchy sequence $< s_n >$ in X which does not converge to a point in X .

Theorem 1: *Let (X, d) be a complete metric space, and let Y be a sub-space of X. Then Y is complete if and only if Y is closed.*

Proof: Let Y be a complete sub-space of X. To show that Y is closed in X. Let $p \in X$ be a limit point of Y. Then for every positive integer n, the open sphere $S(p, 1/n)$ must contain a point q_n of Y. Evidently the sequence $< q_n >$ converges to p. Then by theorem 1 of article 10.25, $< q_n >$ is a Cauchy sequence in Y. Since Y is complete, $p \in Y$. Thus we have shown that Y contains all its limit points and consequently Y is closed.

Conversely, let Y be closed. We shall show that Y is complete. Let $s = < s_n >$ be any Cauchy sequence in Y. Then s is also a Cauchy sequence in X. Since X is complete, $< s_n >$ must converge to a point $s_0 \in X$. We now show that $s_0 \in Y$. If the range set of s consists of finite number of distinct points, then $< s_n >$ must be of the form

$$< s_1, s_2 , \ldots, s_n , s_0 , s_0 , s_0 , \ldots >$$

where n is finite and hence $s_0 \in Y$. And if the range set of s has infinite many points, then by the theorem 3 of 10.24, s_0 is a limit point of the range set of s and so s_0 is also a limit point of Y. Since Y is closed, $s_0 \in Y$. Thus we have shown that every Cauchy sequence in Y converges to a point in Y. Hence Y is complete.

Nested Sequence :

Definition: *Let (X, d) be a metric space. A sequence $< A_n >$ of subsets of X is said to be* **monotonic decreasing** *iff*

$$A_1 \supset A_2 \supset A_3 \supset \ldots\ldots$$

Such a sequence is also called a **nested sequence.**

Note: Here $A_1 \supset A_2$ means that A_1 is a superset of A_2 or A_2 is a subset of A_1 .

Cantor's intersection Theorem:

Theorem 2: (Cantor's intersection theorem). *Let (X, d) be a metric space and let $< F_n >$ be a nested sequence of non-empty closed subsets of X such that $\delta(F_n) \to 0$ as $n \to \infty$. Then X is complete if and only if $\bigcap_{n=1}^{\infty} F_n$ consists of exactly one point.*

Proof: Let X be complete . For each n, we choose $x_n \in F_n$. Since $\delta(F_n) \to 0$, therefore for every $\varepsilon > 0$, there exists a positive integer m_0 such that $\delta(F_{m_0}) < \varepsilon$. Again since $< F_n >$ is nested , we have $n, m \geq m_0 \Rightarrow F_n , F_m \subset F_{m_0} \Rightarrow x_n, x_m \in F_{m_0}$ $\Rightarrow d(x_n , x_m) < \varepsilon$. Thus $< x_n >$ is a Cauchy sequence. Since X is complete, $x_n \to x_0$ for some $x_0 \in X$. We assert that

$$x_0 \in \bigcap_{n=1}^{\infty} F_n .$$

To prove this, let $m \in \mathbf{N}$ be arbitrary. Then

$$n > m \Rightarrow x_n \in F_m \qquad \ldots(1)$$

$[\because x_n \in F_n$ and $n > m \Rightarrow F_n \subset F_m]$

Since $x_n \to x_0$, the sequence $< x_n >$ is eventually in every nhd of x_0 and so by (1), every nhd of x_0 contains an infinite number of points of F_m. Thus x_0 is a limit point of F_m. Since F_m is closed, $x_0 \in F_m$ and since m is arbitrary, we have

$$x_0 \in \bigcap_{n=1}^{\infty} F_n .$$

Now suppose there is another point $x_0^* \in \bigcap_{n=1}^{\infty} F_n$. Then

$d(x_0 , x_0^*) \leq \delta(F_n)$ for every n. Therefore $d(x_0 , x_0^*) = 0$ since $\delta(F_n) \to 0$ as $n \to \infty$. Hence $x_0 = x_0^*$ and so $\bigcap_{n=1}^{\infty} F_n = \{x_0\}$.

Conversely let $\bigcap_{n=1}^{\infty} F_n$ consist of a single point for every nested sequence $< F_n >$ of non-empty closed subsets F_n of X such that $\delta(F_n) \to 0$. To prove that X is complete. Let $< x_n >$ be any Cauchy sequence in X. Write

$$S_1 = \{x_1 , x_2 , \ldots.\}, S_2 = \{x_2 , x_3 , \ldots.\}, \ldots\ldots, S_n = \{x_n , x_{n+1} , \ldots.\}.$$

Since $< x_n >$ is Cauchy, for a given $\varepsilon > 0$, there exists a positive integer m_0 such that

$$n, m \geq m_0 \Rightarrow d(x_n , x_m) < \varepsilon.$$

It follows that $n \geq m_0 \Rightarrow \delta(S_n) < \varepsilon$ and consequently

$$\delta(S_n) \to 0 \text{ as } n \to \infty.$$

Also $S_1 \supset S_2 \supset S_3 \supset \ldots$ so that

$$\bar{S}_1 \supset \bar{S}_2 \supset \bar{S}_3 \supset \ldots. \qquad [\because A \supset B \Rightarrow \bar{A} \supset \bar{B}\ ]$$

Further it is easy to see that $\delta(\bar{S}) = \delta(S)$.

Hence $< \bar{S}_n >$ is a nested sequence of closed subsets of X whose diameters tend to zero. Then by hypothesis, there exists a unique point x_0 in X such that $x_0 \in \bigcap_{n=1}^{\infty} \bar{S}_n$. We claim that the Cauchy sequence $< x_n >$ converges to x_0 . Since $\delta(\bar{S}_n) \to 0$, therefore for a given $\varepsilon > 0$ there exists $m_0 \in \mathbf{N}$ such that $\delta(\bar{S}_{m_0}) < \varepsilon$ and consequently

$$n \geq m_0 \Rightarrow x_n , x_0 \in \overline{S}_{m_0} \Rightarrow d(x_n , x_0) < \varepsilon.$$

This implies that $< x_n >$ converges to x_0. Thus it is shown that every Cauchy sequence in X converges to a point in X. Hence X is complete.

Some Complete Metric Spaces

Illustrative Examples

Example 51: **Completeness of R.** *The real line is a complete metric space i.e., the metric space* $(\mathbf{R}, d)$ *is complete where* d *is the usual metric for* $\mathbf{R}$.

Solution: Let $x, y \in \mathbf{R}$ be arbitrary. Then d is defined as $d(x, y) = |x - y|$. Let $< s_n >$ be a Cauchy sequence in $\mathbf{R}$. We define the sequence $< n_k >$ of positive integers by induction as follows :

n_{k+1} is the smallest integer $> n_k$ such that

$$n, m \geq n_k \Rightarrow |s_n - s_m| < 1/2^{k+1}. \quad \ldots(1)$$

[This is possible because $< s_n >$ is a Cauchy sequence]

Let I_k be the closed interval

$$[s_{n_k} - 2^{-k}, s_{n_k} + 2^{-k}].$$

Then it is easy to see that $I_{k+1} \subset I_k$. For, we have

$$|s_{n_k} - s_{n_{k+1}}| < \frac{1}{2^{k+1}}$$

i.e., the distance between the centres of I_k and I_{k+1} is less than half the radius of I_k and also the radius of I_{k+1} is half of the radius of I_k.

Also the length of $I_k \to 0$ as $k \to \infty$.

Note that the length of I_k is

$$\frac{2}{2^k} \text{ i.e., } \frac{1}{2^{k-1}}.$$

Hence by the nested interval theorem, $\bigcap_{k=1}^{\infty} I_k$ consists of exactly one point, say $a \in \mathbf{R}$.

Thus $a \in I_k$ for all $k \in \mathbf{N}$ so that

$$|a - s_{n_k}| < \frac{1}{2^k} \quad \ldots(2)$$

for all $k \in \mathbf{N}$.

Again for $n \geq n_k$, we have by (1)

$$|s_{n_k} - s_n| < \frac{1}{2^{k+1}} < \frac{1}{2^k} \quad \ldots(3)$$

Hence for all $n \geq n_k$, we have

$$|a - s_n| = |a - s_{n_k} + s_{n_k} - s_n| \leq |a - s_{n_k}| + |s_{n_k} - s_n|$$
$$< \frac{1}{2^k} + \frac{1}{2^k} = \frac{1}{2^{k-1}}$$

[By (2) and (3)]

Now for any given $\varepsilon > 0$ we can find $k \in \mathbf{N}$ such that $\frac{1}{2^{k-1}} < \varepsilon$.

It follows that $\lim_{n \to \infty} s_n = a$.

Thus every Cauchy sequence in $\mathbf{R}$ converges to a point in $\mathbf{R}$ and consequently $\mathbf{R}$ is complete.

Example 52: *The set* $\mathbf{C}$ *of complex numbers with usual metric is a complete metric space.*

Solution: Let $z_1 = x_1 + i\, y_1$, $z_2 = x_2 + i\, y_2$ be arbitrary elements of $\mathbf{C}$ so that $x_1, y_1, x_2, y_2 \in \mathbf{R}$. The usual metric d on $\mathbf{C}$ is defined as $d(z_1, z_2) = |z_1 - z_2|$.

Let $< z_n >$ be a Cauchy sequence in $\mathbf{C}$.

We shall show that $< z_n >$ converges to a point $z \in \mathbf{C}$.

Take any given $\varepsilon > 0$.

Since $< z_n >$ is a Cauchy sequence in $\mathbf{C}$, therefore there exists $m \in \mathbf{N}$ such that

$$|z_n - z_m| < \varepsilon, \forall \quad n \geq m$$

$$\Rightarrow \quad |(x_n + i\, y_n) - (x_m + i\, y_m)| < \varepsilon, \forall \quad n \geq m$$

$$\Rightarrow \quad |(x_n - x_m) + i\,(y_n - y_m)|^2 < \varepsilon^2, \quad \forall\, n \geq m$$

$$\Rightarrow \quad (x_n - x_m)^2 + (y_n - y_m)^2 < \varepsilon^2, \forall\, n \geq m$$

$$\Rightarrow \quad |x_n - x_m|^2 + |y_n - y_m|^2 < \varepsilon^2, \forall\, n \geq m$$

$$\Rightarrow \quad |x_n - x_m| < \varepsilon, \ |y_n - y_m| < \varepsilon, \forall\, n \geq m.$$

It follows that $< x_n >$ and $< y_n >$ are Cauchy sequences in $\mathbf{R}$. But $\mathbf{R}$ is complete and so $x_n \to x \in \mathbf{R}$ and $y_n \to y \in \mathbf{R}$. Consequently $z_n \to x + i\, y = z \in \mathbf{C}$.

Thus every Cauchy sequence $< z_n >$ in $\mathbf{C}$ converges to a point $z \in \mathbf{C}$. Hence, $(\mathbf{C}, d)$ is a complete metric space.

Example 53: **Completeness of $\mathbf{R}^n$**: *The set* $\mathbf{R}^n$ *of all n-tuples* $x = (x_1, x_2, \ldots, x_n)$ *of real numbers is a complete metric space with respect to the usual metric d defined by*

$$d(x, y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2 \right]^{1/2}.$$

Solution: For convenience, we shall denote a typical element of $\mathbf{R}^n$ by a functional notation. Thus an element of $\mathbf{R}^n$ will be regarded as a real function defined on the set $\{1, 2, \ldots, n\}$. For example, f_m will stand for the element $(f_m(1), f_m(2), \ldots, f_m(n))$ of $\mathbf{R}^n$. We now show that $\mathbf{R}^n$ is complete.

Let $< f_m >$ be a Cauchy sequence in $\mathbf{R}^n$. Then for any given $\varepsilon > 0$, there exists a positive integer m_0 such that

$$p, q \geq m_0 \Rightarrow d(f_p, f_q) < \varepsilon$$

$$\Rightarrow d^2(f_p, f_q) < \varepsilon^2$$

$$\Rightarrow \sum_{i=1}^{n} [f_p(i) - f_q(i)]^2 < \varepsilon^2$$

$$\Rightarrow [f_p(i) - f_q(i)]^2 < \varepsilon^2 \quad (i = 1, 2, \ldots, n)$$

$$\Rightarrow |f_p(i) - f_q(i)| < \varepsilon \; (i = 1, 2, \ldots, n).$$

Hence $< f_m(i) >$ is a Cauchy sequence of real numbers for $i = 1, 2, \ldots, n$. Since $\mathbf{R}$ is complete each of these sequences converges to a point of $\mathbf{R}$. In other words, the sequences $< f_m >$ converge pointwise to a limit function f defined by

$$f(i) = \lim f_m(i).$$

Since the set $\{1, 2, \ldots, n\}$ is finite, this convergence is uniform. Hence there exists a positive integer n_0 such that

$$|f_m(i) - f(i)| < \frac{\varepsilon}{\sqrt{(n)}} \qquad \ldots(1)$$

for all $m \geq n_0$ and for every i.

Squaring (1) and adding for $i = 1, 2, \ldots, n$, we get

$$\sum_{i=1}^{n} |f_m(i) - f(i)|^2 < \frac{\varepsilon^2}{n} \cdot n = \varepsilon^2$$

or $$d^2(f_m, f) < \varepsilon^2 \text{ for all } m \geq n_0 .$$

Thus $d(f_m, f) < \varepsilon$ for all $m \geq n_0$.

This shows that the Cauchy sequence $< f_m >$ converges to the limit f and consequently $\mathbf{R}^n$ is complete.

Completeness of real Hilbert space l_2:

Recall that l_2 is the set of all real sequences $< s_n >$ such that the series $\sum_{n=1}^{\infty} s_n^2$ is convergent.

Example 54: *The Hilbert space (l_2, d) is complete.*

Solution: Let $< s_n >$ be a Cauchy sequence in l_2 . Since each s_n is itself a sequence, we shall write

$$s_n = < s_1^n, s_2^n, s_3^n, \ldots >,$$

so that s_i^n denotes the ith term of the sequence s_n . Recall that

$$d(s_m, s_n) = \left[\sum_{i=1}^{\infty} \left(s_i^m - s_i^n \right)^2 \right]^{1/2}.$$

Since $< s_n >$ is a Cauchy sequence, for any given $\varepsilon > 0$, there exists a positive integer n_0 such that

$$m, n \geq n_0 \Rightarrow d(s_m, s_n) < \varepsilon$$

$$\Rightarrow \quad d^2(s_m, s_n) < \varepsilon^2 \Rightarrow \sum_{i=1}^{\infty} (s_i^m - s_i^n)^2 < \varepsilon^2$$

$$\Rightarrow \quad (s_i^m - s_i^n)^2 < \varepsilon^2 \ \forall \ i \in \mathbf{N}$$

$$\Rightarrow \quad |s_i^m - s_i^n| < \varepsilon \ \forall \ i \in \mathbf{N}$$

It follows that $< s_i^n >$ is a Cauchy sequence of real numbers for each $i \in \mathbf{N}$ and consequently it must converge to some real number x_i. Let $x = < x_1, x_2, x_3, \ldots >$. We claim that the sequence $< s_n >$ converges to x and that $x \in l_2$. First we prove that

$$\lim_{n \to \infty} s_n = x .$$

For a fixed positive integer L, we have

$$\sum_{i=1}^{L} (s_i^m - s_i^n)^2 < \varepsilon^2$$

for all $m, n \geq m_0$. If we fix n and let $m \to \infty$, we obtain

$$\sum_{i=1}^{L} (x_i - s_i^n)^2 < \varepsilon^2 \qquad \ldots(1)$$

Since (1) holds for every $L \in \mathbf{N}$, we have for $n \geq m_0$

$$\sum_{i=1}^{\infty} (x_i - s_i^n)^2 < \varepsilon^2 \qquad \ldots(2)$$

i.e., $\quad d^2(x, s_n) < \varepsilon^2 \quad$ or $\quad d(x, s_n) < \varepsilon$

whenever $n \geq m_0$. It follows that

$$\lim_{n \to \infty} s_n = x.$$

We now show that $x \in l_2$. We have

$$\begin{aligned} x_i^2 &= (x_i - s_i^{m_0} + s_i^{m_0})^2 \\ &= (x_i - s_i^{m_0})^2 + (s_i^{m_0})^2 + 2 (x_i - s_i^{m_0}) (s_i^{m_0}) \\ &\leq 2 (x_i - s_i^{m_0})^2 + 2 (s_i^{m_0})^2 . \qquad [\because 2ab \leq a^2 + b^2] \end{aligned}$$

Hence $\quad \sum_{i=1}^{\infty} x_i^2 \leq 2 \sum_{i=1}^{\infty} (x_i - s_i^{m_0})^2 + 2 \sum_{i=1}^{\infty} (s_i^{m_0})^2$

$$< 2\varepsilon^2 + 2 \sum_{i=1}^{\infty} (s_i^{m_0})^2 . \qquad \ldots(3)$$

[By (2)]

Also since $s_{m_0} \in l_2$, the series $\sum_{i=1}^{\infty} (s_i^{m_0})^2$ converges so that for some $A > 0$, we have

$$\sum_{i=1}^{\infty} (s_i^{m_0})^2 < A. \qquad \ldots(4)$$

From (3) and (4), we have

$$\sum_{i=1}^{\infty} x_i^2 < 2\varepsilon^2 + 2A$$

so that the series $\sum_{i=1}^{\infty} x_i^2$ is convergent and consequently $x \in l_2$.

Thus we have shown that every Cauchy sequence $< s_n >$ in l_2 converges to a point x in l_2 and so the space (l_2, d) is complete.

Completeness of $C[a, b]$: Let $\mathbf{C}[a, b]$ denote the set of all real-valued bounded continuous functions defined on the closed interval $I = [a, b]$. For $f, g \in C[a, b]$, we define

$$d(f, g) = \sup \{ | f(x) - g(x) | : x \in I \}$$

We have seen earlier that d is a metric for $C[a, b]$. We shall call this space the **space of continuous functions on I.** We now prove that this space is complete.

Example 55: *The space $C[a, b]$ is complete.*

Solution: Let $[a, b] = I$ and let $< f_n >$ be a Cauchy sequence in $C[a, b]$. Let $\varepsilon > 0$ be given. Since $< f_n >$ is a Cauchy sequence, there exists a positive integer m_0 such that

$$m, n \geq m_0 \Rightarrow d(f_m, f_n) < \varepsilon$$

$$\Rightarrow \quad \sup \{ | f_m(x) - f_n(x) | : x \in I \} < \varepsilon.$$

It follows that

$$| f_m(x) - f_n(x) | < \varepsilon$$

for all $m, n \geq m_0$ and all $x \in I$. This is Cauchy's condition for uniform convergence. Thus the sequence $< f_n >$ is a uniformly convergent sequence of continuous functions and so by a well-known result in analysis, it must converge to some continuous function f on I.

Thus we have shown that every Cauchy sequence $< f_n >$ in $C[a, b]$ converges to a point f in $C[a, b]$. Hence this space is complete.

Some Incomplete Metric Spaces

Illustrative Examples

Example 56: *Let $X = [0, 1]$ and let d be the usual metric on X i.e.,*

$$d(x, y) = | x - y |, \forall x, y \in X .$$

The metric space (X, d) is incomplete.

Solution: Consider the sequence $s = < s_n >$ in X, where

$$s_n = \frac{1}{n}, n \in \mathbf{N} .$$

Obviously $0 < \frac{1}{n} \leq 1, \forall n \in \mathbf{N}$ and so $s_n \in X, \forall n \in \mathbf{N}$.

We know that

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{1}{n} = 0.$$

Since $0 \notin X$, therefore the sequence s does not converge to a point of X.

But s is a Cauchy sequence in X as shown below.

Take any given $\varepsilon > 0$.

By Archimedean property of real numbers there exists a positive integer m such that

$$m > \frac{1}{\varepsilon} \quad i.e., \quad \frac{1}{m} < \varepsilon .$$

Now for all $n \geq m$, we have

$$|s_n - s_m| = \left|\frac{1}{n} - \frac{1}{m}\right| = \left|\frac{m-n}{mn}\right| = \frac{n-m}{nm}$$

$$= \frac{n-m}{n} \cdot \frac{1}{m} < \frac{1}{m} \qquad \left[\because 0 \leq \frac{n-m}{n} < 1\right]$$

$$< \varepsilon .$$

Thus for any given $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$d(s_n, s_m) = |s_n - s_m| < \varepsilon , \forall\, n \geq m.$$

$\therefore \quad s = \langle s_n \rangle$ is a Cauchy sequence in X.

Thus $s = \langle s_n \rangle$ is a Cauchy sequence in X which does not converge to a point in X.

Hence, the metric space (X, d) is not complete.

Example 57: *The metric space of rational numbers with the usual metric is incomplete.*

Solution: Let $\mathbf{Q}$ be the set of rational numbers and let d be the metric on $\mathbf{Q}$ defined by setting

$$d(x, y) = |x - y| , \forall\, x, y \in \mathbf{Q}.$$

We shall show that the metric space $(\mathbf{Q}, d)$ is incomplete.

Consider the sequence $s = \langle s_n \rangle$ in $\mathbf{Q}$, where $s_n = \left(1 + \frac{1}{n}\right)^n$, $n \in \mathbf{N}$.

Obviously $\left(1 + \frac{1}{n}\right)^n$ is a rational number for every $n \in \mathbf{N}$.

We know that

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e .$$

Since e is an irrational number, therefore $e \notin \mathbf{Q}$. So the sequence s does not converge to a point of $\mathbf{Q}$.

But s is a Cauchy sequence in $\mathbf{Q}$ as shown below.

Since the sequence s is a convergent sequence in the metric space $(\mathbf{R}, d)$ where d is the usual metric on $\mathbf{R}$, therefore it is a Cauchy sequence in $\mathbf{R}$. Since each term of the sequence s is a rational number, therefore s is a sequence in $\mathbf{Q}$. Hence, s is a Cauchy sequence in $\mathbf{Q}$.

Thus $s = \langle s_n \rangle$ is a Cauchy sequence in $\mathbf{Q}$ which does not converge to a point of $\mathbf{Q}$. Hence, the metric space $(\mathbf{Q}, d)$ is not complete.

Example 58: *Let $C[a, b]$ denote the set of all real-valued continuous functions on $[a, b]$. For $f, g \in C[a, b]$, define*

$$\rho(f, g) = \left(\int_a^b |f(x) - g(x)|^2 \, dx \right)^{1/2}.$$

Show that ρ is a metric for $C[a, b]$. Also show that this space is not complete.

Solution: It is easy to see that ρ is a metric for $C[a, b]$. We now show that the space is not complete. To prove this, consider the sequence of continuous functions $\langle f_n \rangle$ defined on $[-1, 1]$ by

$$f_n(x) = \begin{cases} 0 & \text{if } -1 \le x \le 0 \\ nx & \text{if } 0 \le x \le \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} \le x \le 1. \end{cases}$$

Then for $m > n$, we have

$$d^2(f_m, f_n) = (m - n)^2 \int_0^{1/m} x^2 \, dx + \int_{1/m}^{1/n} (1 - nx)^2 \, dx$$

$$= \frac{(m-n)^2}{3m^3} + \left[\frac{1}{3n} \left(1 - \frac{n}{m} \right)^3 \right] = \frac{(m-n)^2}{3m^3} + \frac{(m-n)^3}{3nm^3} = \frac{(m-n)^2}{3nm^2}$$

$$< \frac{1}{3n} < \varepsilon \quad \text{if} \quad n > \frac{1}{3\varepsilon}.$$

The sequence is therefore a Cauchy sequence.

Suppose, if possible, this Cauchy sequence converges to a continuous function f so that

$$d(f_n, f) = \int_{-1}^{1} |f_n(x) - f(x)|^2 \, dx \to 0.$$

This implies that the integral with any limits between ± 1 also tends to 0. Thus

$$\int_{-1}^{0} |f_n(x) - f(x)|^2 \, dx \to 0.$$

But $f_n(x) = 0$ when $x \le 0$ and consequently this interval is independent of n. Therefore the continuous function f is such that

$$\int_{-1}^{0} |f(x)|^2 \, dx = 0.$$

It follows that $f(x) = 0$ when $x \le 0$. Again, if $c > 0$, then

$$\int_{c}^{1} |f_n(x) - f(x)|^2 \, dx \to 0 \quad \text{as} \quad n \to \infty.$$

If we choose $n > \frac{1}{c}$, we have

$$\int_{c}^{1} |1 - f(x)|^2 \, dx \to 0 \quad \text{as} \quad n \to \infty.$$

As the integral is independent of n, it vanishes, and since $f(x)$ is continuous, we have $f(x) = 1$ for $x \ge c$. But we can choose c as near to zero as we please. Thus there exists a continuous function which vanishes when $x \le 0$ and which is equal to 1 when $x > 0$. Therefore the Cauchy sequence does not converge to a point of $C[a, b]$. Hence the space is not complete.

10.27 Product of Complete Metric Spaces

Theorem 1: *Let (X, d) and (Y, e) be two complete metric spaces. Then the product space $Z = X \times Y$ with metric*

$$\rho(z_1, z_2) = \sqrt{[d^2(x_1, x_2) + e^2(y_1, y_2)]}$$

is complete where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$.

Proof: Recall that ρ is actually a metric for Z. We now show that the product space (Z, ρ) is complete. Let $< z_n >$ be a Cauchy sequence in Z. Then for a given $\varepsilon > 0$, there exists a positive integer m_0 such that

$$\begin{aligned} m, n \ge m_0 &\Rightarrow \rho(z_m, z_n) < \varepsilon \\ &\Rightarrow \rho^2(z_m, z_n) < \varepsilon^2 \\ &\Rightarrow d^2(x_m, x_n) + e^2(y_m, y_n) < \varepsilon^2 \\ &\Rightarrow d^2(x_m, x_n) < \varepsilon^2 \text{ and } e^2(y_m, y_n) < \varepsilon^2 \\ &\Rightarrow d(x_m, x_n) < \varepsilon \text{ and } e(y_m, y_n) < \varepsilon. \end{aligned}$$

It follows that $< x_n >$ and $< y_n >$ are Cauchy sequences in the spaces X and Y respectively. Since these spaces are complete, the sequences $< x_n >$ and $< y_n >$ converge respectively to points $x \in X$ and $y \in Y$. It follows that the sequence $< z_n >$ converges to $z = (x, y) \in Z$ and consequently the product space $Z = X \times Y$ is complete.

Comprehensive Exercise 3

1. Define a complete metric space. Give an example of a complete metric space.
2. Define an incomplete metric space. Give an example of an incomplete metric space.
3. Show that the set $\mathbf{C}^n$ of all ordered n-tuples $z = (z_1, z_2, \ldots, z_n)$ of complex numbers is a complete metric space with respect to the usual metric d defined by

$$d(z, u) = \left[\sum_{i=1}^{n} | z_i - u_i |^2 \right]^{1/2},$$

where $z = (z_1, z_2, \ldots, z_n)$
and $u = (u_1, u_2, \ldots, u_n)$.

4. Show that the space $C[0, 1]$ of all bounded continuous real-valued functions defined on the closed interval $[0, 1]$ with the metric d given by

$$d(f, g) = \max_{0 \le x \le 1} | f(x) - g(x) |$$

is a complete metric space.

5. Show that every convergent sequence in a metric space is a Cauchy sequence. Give an example to show that the converse is not true.
6. Let $X =]0, 1[$ and let d be the usual metric on X i.e.,

$$d(x, y) = | x - y |, \forall\, x, y \in X.$$

Show that the metric space (X, d) is not complete.

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. Let A and B be any two subsets of a metric space X. Then
   (a) $\overline{A \cap B} = \overline{A} \cap \overline{B}$ (b) $(A \cap B)^\circ = A^\circ \cap B^\circ$
   (c) $D(A \cap B) = D(A) \cap D(B)$ (d) $(A \cup B)^\circ = A^\circ \cup B^\circ$.
2. If A and B are any two subsets of a metric space X, then
   (a) $A \subseteq B \Rightarrow \text{ext}(B) \subseteq \text{ext}(A)$ (b) $A \subseteq B \Rightarrow \text{int}(B) \subseteq \text{int}(A)$
   (c) $A \subseteq B \Rightarrow D(B) \subseteq D(A)$ (d) $A \subseteq B \Rightarrow \overline{B} \subseteq \overline{A}$.

3. In any metric space X which of the following statements is correct ?
   (a) Union of an arbitrary collection of closed subsets of X is closed.
   (b) Intersection of an arbitrary collection of open subsets of X is open.
   (c) Every finite subset of X is a closed set.
   (d) If a subset A of X is not an open set, then A is a closed set.
4. Which of the following statements is correct ?
   (a) If $\mathbf{Q}$ is the set of rational numbers and d is the usual metric on $\mathbf{Q}$ *i.e.*, $d(x, y) = |x - y|, \forall\ x, y \in \mathbf{Q}$, then the metric space $(\mathbf{Q}, d)$ is complete.
   (b) A subset A of a metric space X is open iff $\overline{A} = A$.
   (c) If A is any subset of a metric space X, then int $(X - A)$ is a closed set.
   (d) In a metric space X every convergent sequence is a Cauchy sequence.
5. If A is any subset of a metric space X, then
   (a) A is open iff A contains all its limit points.
   (b) A is open iff A is a neighbourhood of each of its points.
   (c) A is closed iff int $(A) = A$.
   (d) The diameter of A is less than the diameter of $\overline{A}$.
6. In the metric space $(\mathbf{R}, d)$, where d is the usual metric on $\mathbf{R}$, if $A =]0, 1]$, then
   (a) $\overline{A} =]0, 1]$ (b) int $(A) =]0, 1]$
   (c) int $(A) =]0, 1[$ (d) $D(A) =]0, 1]$.

Fill In The Blank(s)

Fill in the blanks "....." so that the following statements are complete and correct.

1. In a metric space (X, d), we have $d(x, y) = 0$ if and only if ..... .
2. In a metric space (X, d), if x, y, z are any three points of X, then
   $d(x, z) \leq$ ..... $+ d(y, z)$.
3. In a metric space (X, d), if x, y, z are any three points of X, then
   $|d(x, z) - d(z, y)| \leq$ .....
4. If A and B are any two non-empty subsets of a metric space (X, d), then
   $d(A, B) =$ .....
5. Let (X, d) be a metric space and let A be a non-empty subset of X. The diameter of A, denoted by $d(A)$, is defined by $d(A) =$ .....
6. In the metric space $(\mathbf{R}, d)$, where d is the usual metric on $\mathbf{R}$, we have
   $d(|5, 9|) =$ .....
7. Let $X = \mathbf{R}^n$ denote the set of all ordered n-tuples of real numbers.
   Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$.

The usual metric d on $\mathbf{R}^n$ is defined by $d(x, y) = \ldots\ldots$.

8. In a metric space (X, d), the metric d on X is a function from $X \times X$ into $\ldots\ldots$.

9. In a metric space (X, d), if a is any point of X and r is any positive real number, then the open sphere $S_r(a)$ is defined as the set $S_r(a) = \ldots\ldots$.

10. In a metric space (X, d), if a is any point of X and r is any positive real number, then the closed sphere $S_r[a]$ is defined as the set $S_r[a] = \ldots\ldots$.

11. In the metric space $(\mathbf{R}^2, d)$ where d is the usual metric on $\mathbf{R}^2$, the open sphere with centre (x_0, y_0) and radius r is the interior of the circle $\ldots\ldots$.

12. In the metric space $(\mathbf{R}, d)$ where d is the usual metric on $\mathbf{R}$, the open sphere $S_r(4)$ of radius $r = \frac{1}{2}$ and centre as the point 4 is the open interval $\ldots\ldots$.

13. Let (X, d) be any discrete metric space. If $x_0 \in X$ and r is any positive number ≤ 1, then the open sphere $S(x_0, r) = \ldots\ldots$.

14. Let (X, d) be a metric space. A subset G of X is said to be open if to each $x \in G$ there exists $r > 0$ such that $S(x, r) \ldots\ldots$.

15. In a metric space, the union of an arbitrary collection of open sets is $\ldots\ldots$.

16. In a metric space, the intersection of a finite number of open sets is $\ldots\ldots$.

17. Let (X, d) be a metric space. A subset F of X is said to be closed if the $\ldots\ldots$ of F is open.

18. Let (X, d) be a metric space. A subset F of X is closed if and only if F contains all its $\ldots\ldots$.

19. Let (X, d) be a metric space and let $p \in X$. A subset N of X is said to be a neighbourhood of p if there exists $r > 0$ such that $S(p, r) \ldots\ldots$.

20. A subset in a metric space is open if and only if it is a $\ldots\ldots$ of each of its points.

21. Let (X, d) be a metric space and let A be a subset of X. A point $x \in X$ is called a limit point of A iff every neighbourhood of x contains a point of $A \ldots\ldots$.

22. A subset A of a metric space is closed if and only if $D(A) \ldots\ldots$.

23. If A and B are any subsets of a metric space X, then $D(A \cap B) \subseteq \ldots\ldots$.

24. Let A and B be any subsets of a metric space X, then $D(A \cup B) = \ldots\ldots$.

25. Let A be a subset of a metric space. Then $\overline{A} = A \cup \ldots\ldots$.

26. In the metric space $(\mathbf{R}, d)$ where d is the usual metric on $\mathbf{R}$, $D(]2, 3[) = \ldots\ldots$.

27. In the metric space $(\mathbf{R}, d)$ where d is the usual metric on $\mathbf{R}$, if $A = [0, 1]$, then $\overline{A} = \ldots\ldots$.

28. A subset A of a metric space X is closed if and only if $\overline{A} = \ldots\ldots$.

29. A subset A of a metric space X is open if and only if int (A) = ...
30. If A is any subset of a metric space X, then ..... = int $(X - A)$.
31. A subset A of a metric space X is said to be dense in X if $\overline{A}$ = ...
32. If A and B are any subsets of a metric space X, then $(A \cap B)^\circ$ = ...
33. In the metric space $(\mathbf{R}, d)$, where d is the usual metric for $\mathbf{R}$, $\overline{\mathbf{Q}}$ = ...
34. A metric space (X, d) is said to be complete if and only if every Cauchy sequence in X ..... to a point in X.
35. If a Cauchy sequence in a metric space has a convergent subsequence, then the sequence is ..... .
36. Let (X, d) be a metric space and let $< F_n >$ be a nested sequence of non-empty closed subsets of X such that $\delta(F_n) \to 0$ as $n \to \infty$. Then X is complete if and only if $\bigcap_{n=1}^{\infty} F_n$ consists of ..... .

True or False

Write 'T' for true and 'F' for false statement.

1. The metric space $(\mathbf{R}, d)$, where d is the usual metric on $\mathbf{R}$, is complete.
2. If d is the usual metric on $\mathbf{R}$ and $A = [2, 3[$, $B =]3, 5]$, then $d(A, B) = 1$.
3. In a metric space X every convergent sequence is always a Cauchy sequence.
4. In a metric space X every Cauchy sequence is always a convergent sequence.
5. In the metric space $(\mathbf{R}^n, d)$, where d is the usual metric on $\mathbf{R}$, every Cauchy sequence is a convergent sequence.
6. In a metric space X, every open sphere is an open set.
7. In a metric space X every open sphere is a neighbourhood of each of its points.
8. In a metric space X the intersection of two neighbourhoods of a point $x \in X$ is also a neighbouhood of x.
9. If A is any subset of a metric space X, then ext (A) = int $(X - A)$.
10. In a metric space X the union of an infinite collection of closed sets is necessarily a closed set.
11. If A and B are any subsets of a metric space X, then $(A \cup B)^\circ = A^\circ \cup B^\circ$.
12. If A is any subset of a metric space X, then $D(A) \subseteq A$.
13. In a metric space X the complement of a finite set is an open set.

Answers

Multiple Choice Questions

1. (b) 2. (a) 3. (c) 4. (d) 5. (b)
6. (c)

Fill in the Blank(s)

1. $x = y$ 2. $d(x, y)$ 3. $d(x, y)$
4. $\inf \{d(x, y) : x \in A, y \in B\}$ 5. $\sup \{d(x, y) : x \in A, y \in A\}$
6. 4 7. $\left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2}$ 8. **R**
9. $\{x \in X : d(x, a) < r\}$ 10. $\{x \in X : d(x, a) \leq r\}$
11. $(x - x_0)^2 + (y - y_0)^2 = r^2$ 12. $\left]3\frac{1}{2}, 4\frac{1}{2}\right[$ 13. $\{x_0\}$
14. $\subseteq G$ 15. open 16. open
17. complement 18. limit points 19. $\subseteq N$
20. neighbourhood 21. distinct from x 22. $\subseteq A$.
23. $D(A) \cap D(B)$ 24. $D(A) \cup D(B)$. 25. $D(A)$.
26. $[2, 3]$ 27. $[0, 1]$. 28. A
29. A 30. ext (A) 31. X
32. $A^\circ \cap B^\circ$ 33. **R** 34. converges
35. convergent 36. exactly one point

True or False

1. T 2. F 3. T 4. F 5. T
6. T 7. T 8. T 9. T 10. F
11. F 12. F 13. T

□

Notes

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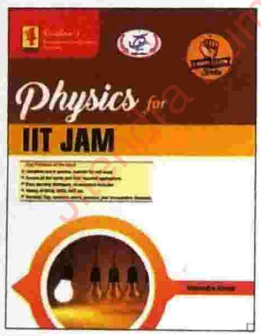

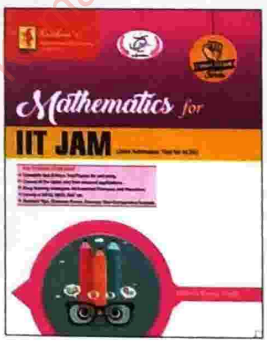

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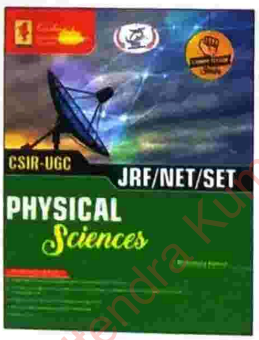

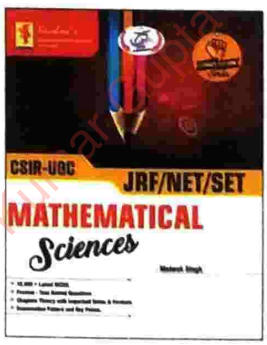

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